Young–Fibonacci Character Tables

Soichi OKADA (Nagoya University)

93rd Séminaire Lotharingien de Combinatoire
Pocinho, March 24, 2025

Motivation

- Stanley introduced the notion of differential posets, as a generalization of Young's lattice of integer partitions.
- Young's lattice $\mathbb{Y}$ and the Young–Fibonacci lattice $\mathbb{YF}$ are the extreme examples of differential posets.
- Young's lattice $\mathbb{Y}$ is associated with
  - symmetric groups,
  - ring of symmetric functions,
  - character rings of symmetric groups.

Question Are there similar algebraic structures associated with the Young–Fibonacci lattice $\mathbb{YF}$?

Plan

1. Differential posets
2. Properties of character table of symmetric groups
3. Young–Fibonacci algebras and $\mathbb{YF}$-analogue of the ring of symmetric functions
4. $\mathbb{YF}$-character tables
5. $\mathbb{YF}$-character rings

Notations

Let $(P, \leq)$ be a partially ordered set (poset for short).

- Let x, $y \in P$. We say that x is covered by y, denoted by $x \lessdot y$, if $x < y$ and there is no $z \in P$ such that $x < z < y$.
- For $x \in P$, we put

$$C^{+}(x) = \{y \in P : x \lessdot y\}, \qquad C^{-}(x) = \{y \in P : y \lessdot x\}.$$

Differential posets

Definition (Stanley) A poset P is called differential if it satisfies the following three conditions:

(1) P is a graded poset with $\#P_n < \infty$ for $n = 0, 1, 2, \ldots,$ and has the minimum element $\widehat{0}$.

(2) If $x \neq y$ in P, then $\#\big(C^+(x) \cap C^+(y)\big) = \#\big(C^-(x) \cap C^-(y)\big)$.

(3) If $x \in P$, then $\#C^+(x) = 1 + \#C^-(x)$.

Let P be a graded poset satisfying Condition (1) above. Let $\mathbb{C}P$ be the complex vector space with a basis P and define linear maps $U, D : \mathbb{C}P \to \mathbb{C}P$ by

$$Ux = \sum_{y \in C^+(x)} y, \quad Dx = \sum_{z \in C^-(x)} z.$$

Proposition (Stanley)

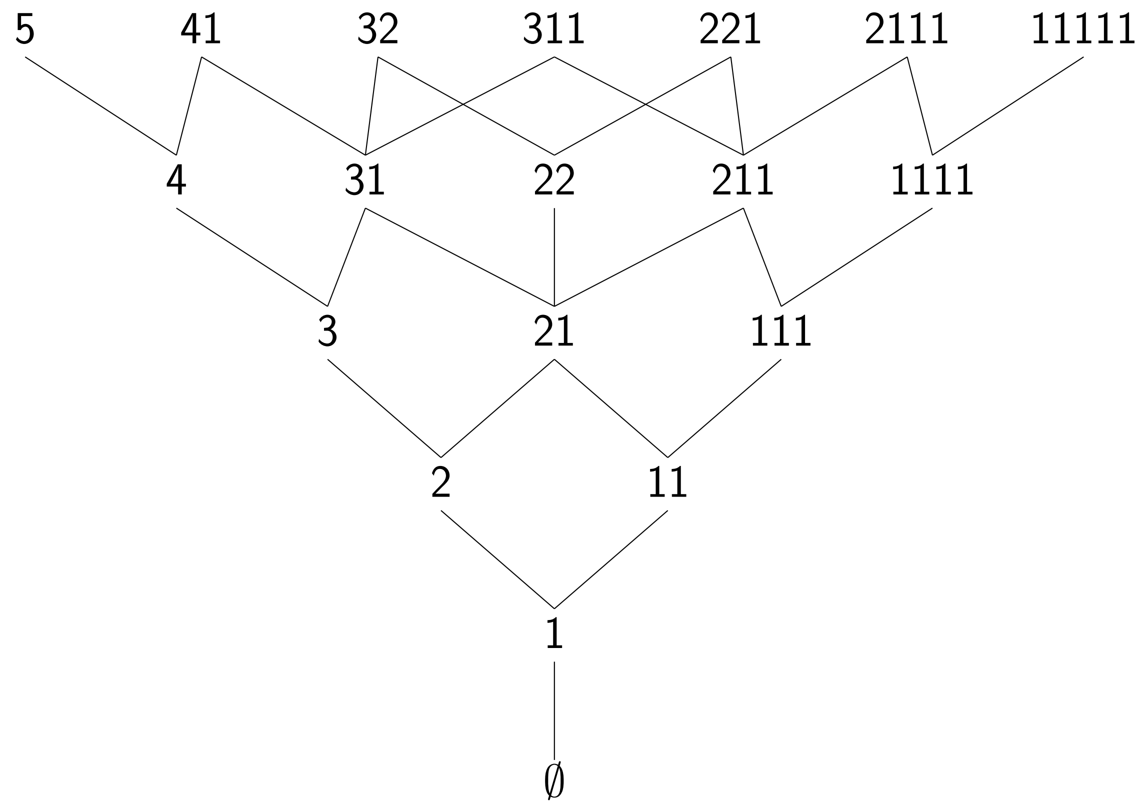
$$P \text{ is differential} \iff DU - UD = I.$$

Young's lattice $\mathbb{Y}$

We define a partital ordering $\geq$ on the set $\mathbb{Y}$ of all partitions by

$$\lambda \geq \mu \iff \lambda_i \geq \mu_i \, (i = 1, 2, \dots) \iff D(\lambda) \supset D(\mu).$$

The resulting poset $\mathbb{Y}$, called Young's lattice, is a differential poset.



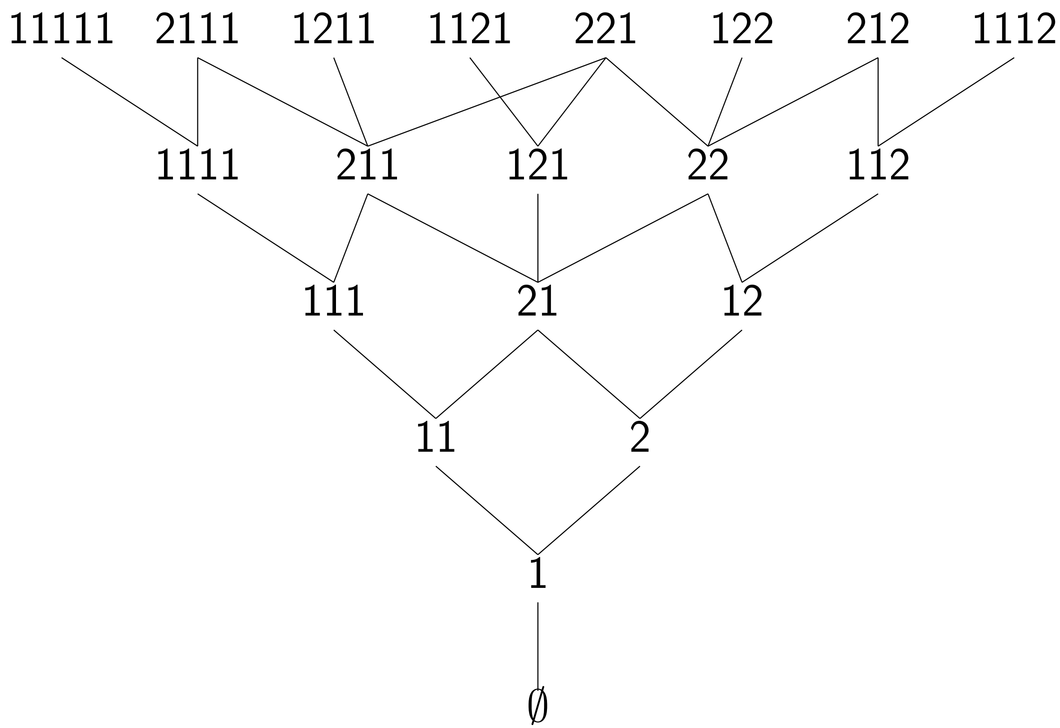
Young–Fibonacci lattice $\mathbb{YF}$

The Young–Fibonacci lattice is the set $\mathbb{YF}$ of all finite words of 1 and 2, with the partial order given by the following covering relations:

$$C^-(1v) = \{v\}, \quad C^-(2v) = C^+(v).$$

Then $\mathbb{YF}$ is a differential poset.

Remark $\#\mathbb{YF}_n$ is the nth Fibonacci number.



Representation theory of symmetric groups

Let $\mathfrak{S}_n$ be the symmetric group of n letters $\{1, 2, \ldots, n\}$. It is classically known that

- The irreducible representations of $\mathfrak{S}_n$ over $\mathbb{C}$ are indexed by partitions of n:
$$\mathrm{Irr}(\mathfrak{S}_n) \longleftrightarrow \mathbb{Y}_n, \qquad [S^\lambda] \longleftrightarrow \lambda,$$
where $\mathrm{Irr}(\mathfrak{S}_n)$ is the set of isomorphism classes of irrducible repsentations of $\mathfrak{S}_n$, and $\mathbb{Y}_n$ is the set of partitions of n.
- $\dim S^\lambda = f^\lambda =$ number of standard tableaux of shape λ.
- The restriction of S^λ to the subgroup $\mathfrak{S}_{n-1} = \{\sigma \in \mathfrak{S}_n : \sigma(n) = n\}$ is decomposed as
$$\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} S^\lambda \cong \bigoplus_{\mu \lessdot \lambda} S^\mu.$$

Ring of symmetric functions

Let $R(\mathfrak{S}_n)$ be the Grothendieck group of representations of $\mathfrak{S}_n$:

$$R(\mathfrak{S}_n) = \bigoplus_{\lambda \in \mathbb{Y}_n} \mathbb{Z}[S^\lambda].$$

The direct sum of the Grothendieck groups

$$R(\mathfrak{S}_\bullet) = \bigoplus_{n \geq 0} R(\mathfrak{S}_n)$$

has a structure of commutative, associative, graded ring with respect to

$$[V] \circ [W] = \left[\mathrm{Ind}_{\mathfrak{S}_m \times \mathfrak{S}_n}^{\mathfrak{S}_{m+n}} (V \boxtimes W) \right],$$

where V and W are representations of $\mathfrak{S}_m$ and $\mathfrak{S}_n$ respectively. Moreover, the graded ring $R(\mathfrak{S}_\bullet)$ is isomorphic to the ring Λ of symmetric functions via the correspondence $[S^\lambda] \mapsto s_\lambda$ (Schur function).

$$R(\mathfrak{S}_\bullet) \cong \Lambda, \qquad [S^\lambda] \longleftrightarrow s_\lambda.$$

Character table of symmetric groups

The character table $\left(\chi_{\alpha}^{\lambda}\right)_{\alpha,\lambda\in\mathbb{Y}_n}$ is the transition matrix between Schur functions $\{s_\lambda\}_{\lambda\in\mathbb{Y}_n}$ and power-sum functions $\{p_\alpha\}_{\alpha\in\mathbb{Y}_n}$:

$$p_\alpha = \sum_{\lambda\in\mathbb{Y}_n} \chi_{\alpha}^{\lambda} s_\lambda.$$

If we define U and $D : \Lambda \to \Lambda$ by

$$U s_\lambda = \sum_{\mu \gtrdot \lambda} s_\mu, \quad D s_\lambda = \sum_{\nu \lessdot \lambda} s_\nu,$$

and $m_1(\alpha)$ denotes the multiplicity of 1 in a partition α, then

$$U p_\alpha = p_{\alpha\cup(1)}, \quad D p_\alpha = m_1(\alpha) p_{\alpha\setminus(1)}.$$

Moreover, we have

$$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda,\mu}, \quad \langle p_\alpha, p_\beta \rangle = \delta_{\alpha,\beta} z_\alpha,$$

where $z_\alpha = \prod_{i\geq 1} i^{m_i} m_i!$ for $\alpha = 1^{m_1} 2^{m_2} \cdots$.

Properties of character table of symmetric groups

Proposition We have

$$\sum_{\alpha\in\mathbb{Y}_n}\frac{1}{z_\alpha}\chi^\lambda_\alpha\chi^\mu_\alpha=\delta_{\lambda,\mu},\quad \sum_{\lambda\in\mathbb{Y}_n}\chi^\lambda_\alpha\chi^\lambda_\beta=\delta_{\alpha,\beta}z_\alpha,$$

and

$$\frac{n!}{z_\alpha}\in\mathbb{N},\quad \sum_{\alpha\in\mathbb{Y}_n}\frac{n!}{z_\alpha}=n!,$$

where $\mathbb{N}$ is the set of nonnegative integers.

Proposition

$$g^\nu_{\lambda,\mu}=\sum_{\alpha\in Y_n}\frac{1}{z_\alpha}\chi^\lambda_\alpha\chi^\mu_\alpha\chi^\nu_\alpha\in\mathbb{N},$$

and the degree n part Λ_n has a structure of fusion algebra at algebraic level with structure constants $g^\nu_{\lambda,\mu}$ with respect to Schur functions $\{s_\lambda\}_{\lambda\in\mathbb{Y}_n}$.

Fusion algebra at algebraic level (1/2)

Definition (Bannai) Let $\mathcal{A}$ be an commutative associative algebra over $\mathbb{C}$ with basis $a_1, \ldots, a_r$ and structure constants $A_{i,j}^k$ defined by

$$a_i a_j = \sum_{k=1}^{r} A_{i,j}^k a_k.$$

We call $\mathcal{A}$ a fusion algebra at algebraic level if the following four conditions are satisfied:

(a) $A_{i,j}^k \in \mathbb{R}$;

(b) There exists an involutive bijection $[r] \ni i \mapsto \widehat{i} \in [r]$ such that

$$A_{\widehat{i},\widehat{j}}^{\widehat{k}} = A_{i,j}^k, \quad A_{i,j}^{\widehat{k}} = A_{k,j}^{\widehat{i}};$$

(c) a_1 is the identity element of $\mathcal{A}$;

(d) There exists a linear representation $\Delta : \mathcal{A} \to \mathbb{C}$ such that $\Delta(a_i)$ are positive real number for all $i \in [r]$.

Fusion algebra at algebraic level (2/2)

A fusion algebra $\mathcal{A}$ is called strictly integral if

$$A_{i,j}^{k} \in \mathbb{N}, \quad \Delta(a_i) \in \mathbb{N}.$$

Let $\mathcal{A} = \langle a_1, \ldots, a_r \rangle$ and $\mathcal{B} = \langle b_1, \ldots, b_s \rangle$ be two fusion algebras. An algebra homomorphism $F : \mathcal{A} \to \mathcal{B}$ is called a homomorphism of fusion algebras if

- If $F(a_j) = \sum_{i=1}^{s} F_{i,j} b_i$, then $F_{i,j} \in \mathbb{R}$ and $F_{\widehat{i},\widehat{j}} = F_{i,j}$;
- $\Delta_{\mathcal{B}} \circ F = \Delta_{\mathcal{A}}$.

We say that F is integral if $\mathcal{A}$ and $\mathcal{B}$ are strictly integral and $F_{i,j} \in \mathbb{N}$ for all i and j.

Character rings of finite groups

For a finite group G, let $R_{\mathbb{C}}(G)$ be the (complexified) Grothendieck group of finite-dimensional representations of G. Then $R_{\mathbb{C}}(G)$ is a strictly integral fusion algebra with respect to

- the product given by $[V] * [W] = [V \otimes_{\mathbb{C}} W]$;
- the basis consisting of the classes of irreducible representations of G;
- the homomorphism Δ given by $\Delta([V]) = \dim V$.

The structure constants are given by

$$A_{[V],[W]}^{[U]} = \frac{1}{\#G} \sum_{g \in G} \chi_V(g) \chi_W(g) \chi_U(g^{-1}).$$

Moreover, if H is a subgroup of G, then the restriction map

$$\mathrm{Res} : R_{\mathbb{C}}(G) \to R_{\mathbb{C}}(H), \quad [V] \mapsto \left[\mathrm{Res}_H^G V \right]$$

is an integral homomorphism of fusion algebras.

Young–Fibonacci algebra

Theorem (Okada) Let $\mathcal{F}_n$ be the associative algebra over $\mathbb{C}$ defined by the following presentation:

$$\begin{array}{ll} \text{generators}: & E_1, E_2, \ldots, E_{n-1}, \\ \text{relations}: & E_i^2 = x_i E_i \ (i = 1, \ldots, n-1), \\ & E_{i+1} E_i E_{i+1} = E_{i+1} \ (i = 1, \ldots, n-2), \\ & E_i E_j = E_j E_i \ (\text{if } |i-j| \geq 2). \end{array}$$

where $x = (x_1, x_2, \ldots)$ are complex parameters. If the parameters x are generic, then we have

(1) The algebra $\mathcal{F}_n$ is a semisimple algebra of dimension $n!$, and its irreducible representations are indexed by words in $\mathbb{YF}_n$:

$$\begin{array}{ccc} \mathrm{Irr}(\mathcal{F}_n) & \longleftrightarrow & \mathbb{YF}_n \\ [V_v] & \longleftrightarrow & v \end{array}$$

(2) For the inclusion $\mathcal{F}_{n-1} = \langle E_1, \ldots, E_{n-2} \rangle \subset \mathcal{F}_n$, we have

$$\mathrm{Res}^{\mathcal{F}_n}_{\mathcal{F}_{n-1}} V_v \cong \bigoplus_{u \lessdot v} V_u.$$

Direct sum of Grothendieck groups for $\mathbb{YF}$-algebras

Let $R(\mathcal{F}_n)$ be the Grothendieck group of representations of $\mathcal{F}_n$:

$$R(\mathcal{F}_n) = \bigoplus_{v \in \mathbb{YF}_n} \mathbb{Z}[V_v].$$

Then the direct sum of the Grothendieck groups

$$R(\mathcal{F}_\bullet) = \bigoplus_{n \geq 0} R(\mathcal{F}_n)$$

has a structure of associative graded ring with respect to

$$[V_u] \circ [V_v] = \left[\operatorname{Ind}_{\mathcal{F}_m \otimes \mathcal{F}'_n}^{\mathcal{F}_{m+n}} (V_u \boxtimes V'_v) \right],$$

where V_u and V_v are irreducible representations of $\mathcal{F}_m$ and $\mathcal{F}_n$ respectively, and V'_v is the corresponding irreducible representation of $\mathcal{F}'_n = \langle E_{m+1}, \ldots, E_{m+n-1} \rangle$.

$\mathbb{YF}$-analogue of Schur functions

Let $R = \mathbb{Z}\langle X, Y\rangle$ be the noncommutative polynomial ring in two variables X and Y, which is equipped with a graded ring structure $R = \bigoplus_{n\geq 0} R_n$ given by $\deg X = 1$ and $\deg Y = 2$.

We define $\mathbb{YF}$-Schur functions $s_v \in R$ inductively as follows:

$$s_\emptyset = 1, \quad s_{1v} = s_v X - \Big(\sum_{z \lessdot v} s_z\Big) Y, \quad s_{2v} = s_v Y.$$

Then $\{s_v : v \in \mathbb{YF}\}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}\langle X, Y\rangle$,

Theorem (Okada) The correspondence $[V_v] \mapsto s_v$ $(v \in \mathbb{YF})$ gives a graded ring isomorphism $\varphi : R(\mathcal{F}_\bullet) \to R = \mathbb{Z}\langle X, Y\rangle$:

$$\begin{array}{ccc} R(\mathcal{F}_\bullet) & \cong & \mathbb{Z}\langle X, Y\rangle \\ [V_v] & \longmapsto & s_v \end{array}$$

Note that $\varphi([V_1]) = X$ and $\varphi([V_2]) = Y$.

$\mathbb{YF}$-analogue of power-sum functions

We define $\mathbb{YF}$-power-sum functions $p_v \in R$ inductively as follows:

$$p_\emptyset = 1, \quad p_{1v} = p_v X, \quad p_{2v} = p_v(X^2 - (m(v)+2)Y),$$

where $m(v)$ is the number of 1's at the head of v.

Proposition We have

$$Up_v = p_{1v}, \quad Dp_{1v} = m(1v)p_v, \quad Dp_{2v} = 0,$$

where U, $D : R \to R$ are linear maps defined by

$$Us_v = \sum_{w \gtrdot v} s_w, \quad Ds_v = \sum_{u \lessdot v} s_u.$$

Proposition If we define a bilinear form $\langle \quad , \quad \rangle$ on R by $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda,\mu}$, then we have

$$\langle p_\alpha, p_\beta \rangle = \delta_{\alpha,\beta} z_\alpha,$$

where

$$z_\alpha = m_1!(m_2+2)m_2! \cdots (m_{r+1}+2)m_{r+1}!$$

for $\alpha = 1^{m_1}21^{m_2}2 \cdots 21^{m_{r+1}}$.

$\mathbb{YF}$-analogue of the character table

We define a "$\mathbb{YF}$-character table" $X_n = \left(\chi_\alpha^v\right)_{\alpha,v\in\mathbb{YF}_n}$ by

$$p_\alpha = \sum_{v\in\mathbb{YF}_n} \chi_\alpha^v s_v.$$

Example If $n = 5$, then

$$\begin{array}{c|c}
 & \begin{array}{cccccccc} 221 & 212 & 2111 & 122 & 1211 & 1121 & 1112 & 11111 \end{array} \\
\begin{array}{c} 221 \\ 212 \\ 2111 \\ 122 \\ 1211 \\ 1121 \\ 1112 \\ 11111 \end{array} &
\begin{pmatrix}
1 & -1 & -1 & 0 & 0 & -1 & 1 & 1 \\
0 & 1 & -1 & -1 & 1 & 0 & -1 & 1 \\
-2 & -1 & -1 & 3 & 3 & 2 & 1 & 1 \\
0 & 0 & 0 & 1 & -1 & 0 & -1 & 1 \\
0 & 0 & 0 & -1 & -1 & 2 & 1 & 1 \\
-1 & 1 & 1 & 0 & 0 & -1 & 1 & 1 \\
0 & -2 & 2 & -1 & 1 & 0 & -1 & 1 \\
8 & 4 & 4 & 3 & 3 & 2 & 1 & 1
\end{pmatrix}
\end{array}$$

Note that this "character table" does not come from any finite groups.

Properties of $\mathbb{YF}$-character table

Proposition The $\mathbb{YF}$-character table X_n satisfies the orthogonality relations:

$$\sum_{\alpha\in\mathbb{YF}_n} \frac{1}{z_\alpha}\chi^v_\alpha\chi^w_\alpha = \delta_{v,w}, \qquad \sum_{v\in\mathbb{YF}_n} \chi^v_\alpha\chi^v_\beta = \delta_{\alpha,\beta}z_\alpha.$$

Moreover we have

$$\frac{n!}{z_\alpha} \in \mathbb{Z}, \qquad \sum_{\alpha\in\mathbb{YF}_n} \frac{n!}{z_\alpha} = n!.$$

Remark We define $F(\sigma) \in \mathbb{YF}_n$ ($\sigma \in \mathfrak{S}_n$) inductively as follows. Let $F(e) = \emptyset$ for $e \in \mathfrak{S}_0$, $F(e) = 1$ for $e \in \mathfrak{S}_1$, and

$$F(\sigma_1\sigma_2\dots\sigma_n) = \begin{cases} F(\sigma_2\dots\sigma_n)1 & \text{if } \sigma_1 < \sigma_2, \\ F(\sigma_3\dots\sigma_n)2 & \text{if } \sigma_1 > \sigma_2. \end{cases}$$

Then we have

$$\#\{\sigma \in \mathfrak{S}_n : F(\sigma) = \alpha\} = \frac{n!}{z_\alpha} \quad (\alpha \in \mathbb{YF}_n).$$

$\mathbb{YF}$-fusion ring ($\mathbb{YF}$-analogue of the character ring)

Using the $\mathbb{YF}$-charcter table, we define $\mathbb{YF}$-Kronecker coefficients $g_{u,v}^w$ by

$$g_{u,v}^w = \sum_{\alpha \in \mathbb{YF}_n} \frac{1}{z_\alpha} \chi_\alpha^u \chi_\alpha^v \chi_\alpha^w.$$

Theorem (Okada) The degree n part R_n of $R_{\mathbb{C}} = \mathbb{C}\langle X, Y \rangle$ admits a structure of strictly integral fusion algebra with product

$$s_u * s_v = \sum_{w \in \mathbb{YF}_n} g_{u,v}^w s_w.$$

In particular, $\mathbb{YF}$-Kronecker coefficients $g_{u,v}^w$ are nonnegative integers. Moreover,

$$D : R_n \to R_{n-1}, \quad s_v \mapsto \sum_{u \lessdot v} s_u$$

is a surjective integral homomorphism of fusion algebras.

Reflection-extension of fusion algebras

Theorem (Okada) Let $\mathcal{A} = \langle a_1, \ldots, a_r \rangle$ and $\mathcal{B} = \langle b_1, \ldots, b_s \rangle$ be fusion algebras, and $F : \mathcal{A} \to \mathcal{B}$ be a surjective homomorphism of fusion algebras. Then the direct sum

$$\mathcal{C} = \mathcal{A} \oplus \mathcal{B}$$

admits a structure of fusion algebra with product defined by

$$(a, 0) \cdot (a', 0) = (aa', 0), \quad (a, 0) \cdot (0, b) = (0, F(a)b),$$
$$(0, b) \cdot (0, b') = (F^*(bb'), (FF^* - I)(bb')),$$

where $F^* : \mathcal{B} \to \mathcal{A}$ is the linear map given by $F^*(b_i) = \sum_{j=1}^{r} F_{i,j} a_j$ with $F(a_j) = \sum_{i=1}^{s} F_{i,j} b_i$. Moreover, the map

$$G : \mathcal{C} \to \mathcal{A}, \quad G(a, b) = a + F^*(b)$$

is a surjective homomorphism of fusion algebras.