Quasisymmetric polynomials revisited

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For $n = 2, f = x_1^2 x_2$.
For $n = 3, f = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3$.

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Motivation(s)

- Introduced in Stanley's thesis (1970), explicitly identified by Gessel (1984). They are the natural setting for certain generating functions for posets.
- Terminal object in a certain category of Hopf algebras.
- Relation to symmetric polynomials: create bases that refine symmetric bases, expand (quasi)symmetric polynomials in these bases,...

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Today: a new approach via operators
and its consequences

Trimming operators

We define operators that "detect quasisymmetry".

Definition. For $f \in \mathrm{Pol}_n$ and $i < n$, define

$$\mathsf{R}_i(f(x_1, \ldots, x_n)) := f(x_1, \ldots, x_{i-1}, 0, x_i, x_{i+1}, \ldots, x_{n-1})$$

This is an algebra morphism $\mathrm{Pol}_n \to \mathrm{Pol}_{n-1}$.

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Lemma. $f \in \mathrm{QSym}_n$ if and only if $\mathrm{R}_1(f) = \mathrm{R}_2(f) = \cdots = \mathrm{R}_n(f)$.

This characterization is related to (Hivert, 2000).

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We now define the main "trimming" operators T_i.

Definition. For $f \in \mathrm{Pol}_n$ and $i < n$,

$$\mathrm{T}_i = \frac{\mathrm{R}_{i+1} - \mathrm{R}_i}{x_i}$$

We get $f \in \mathrm{QSym}_n$ if and only if $\mathrm{T}_1 f = \mathrm{T}_2 f = \cdots = \mathrm{T}_{n-1} f = 0$.

Trimming operators

Explicitly,

$$\mathrm{T}_i(f) = \frac{f(x_1, \dots, x_{i-1}, x_i, 0, x_{i+1}, \dots, x_{n-1}) - f(x_1, \dots, x_{i-1}, 0, x_i, x_{i+1}, \dots, x_{n-1})}{x_i}$$

$$T_1(x_1^a x_2^b) = \begin{cases} 0 & \text{if } ab > 0 \text{ or } a = b = 0 \\ x_1^{a-1} & \text{if } a > 0 \text{ and } b = 0 \\ -x_1^{b-1} & \text{if } b > 0 \text{ and } a = 0. \end{cases}$$

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Let $\mathrm{Pol} := \mathbb{Q}[x_1, x_2, \dots,] = \lim \mathrm{Pol}_n$ and consider $\mathrm{T}_i : \mathrm{Pol} \to \mathrm{Pol}$.

The T_i satisfy the relations of the Thompson monoid

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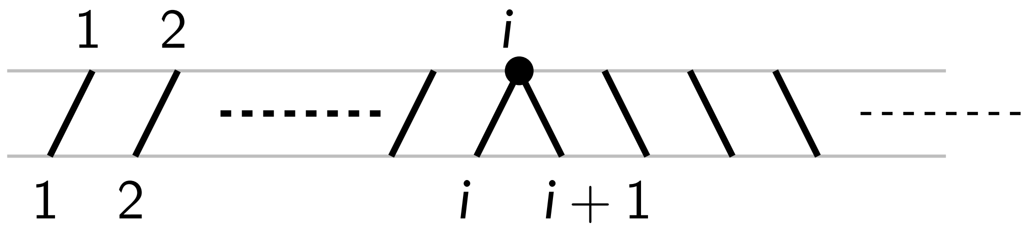
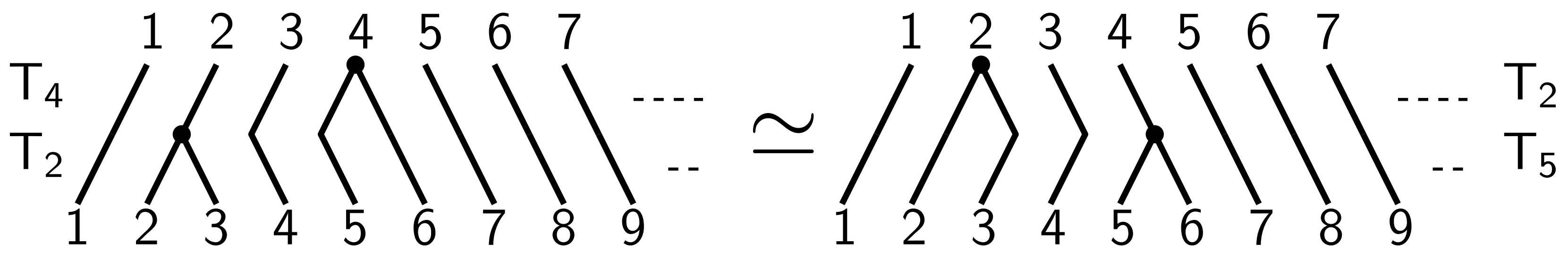
To study the combinatorics, associate to T_i the elementary diagram



Equivalence classes are certain forests

We can illustrate $T_4 T_2 = T_2 T_5$ as



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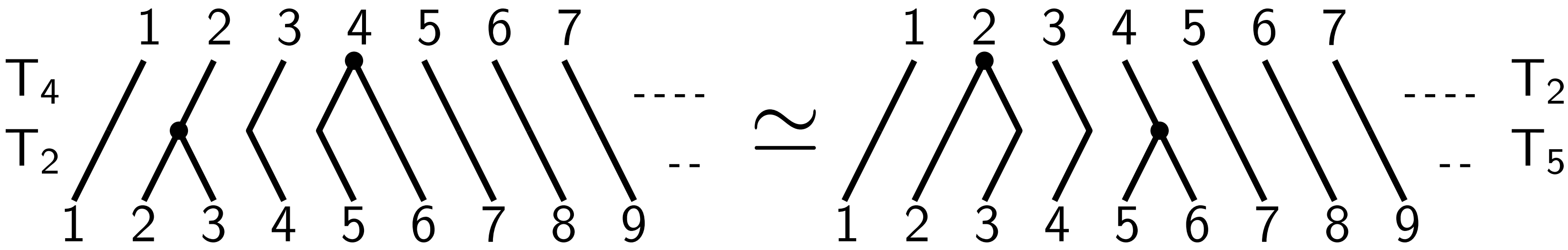
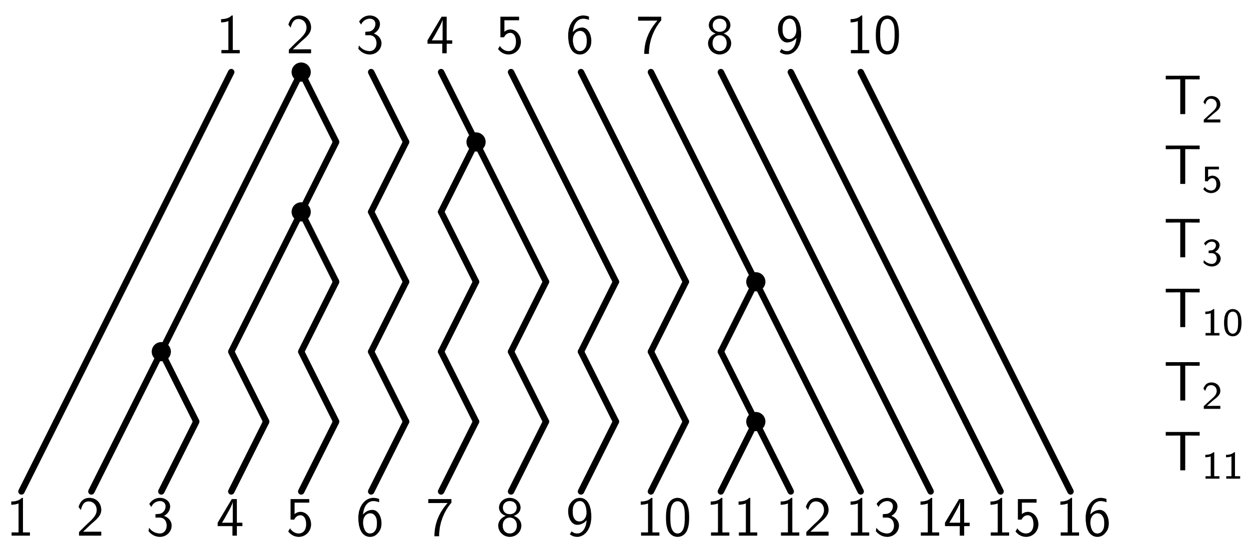
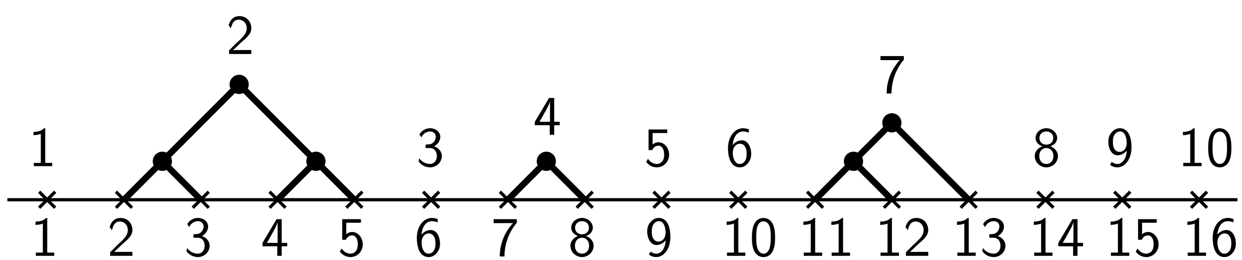


The equivalence class of

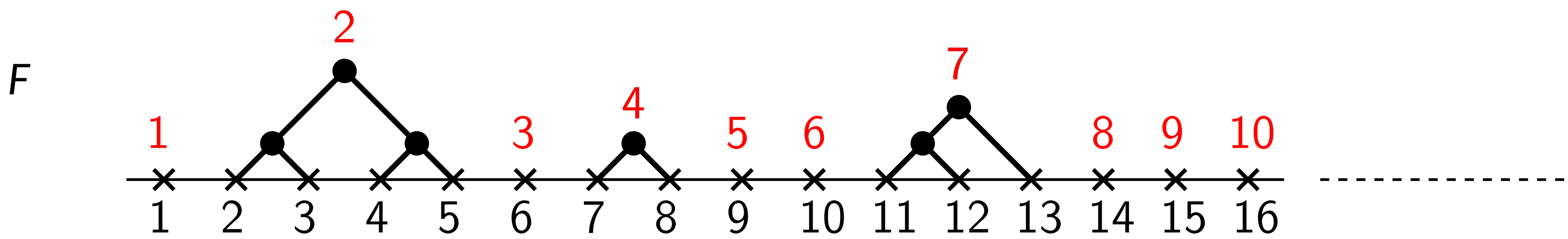


can be represented by



Combinatorics

Definition. An indexed forest F is a sequence of plane binary trees, eventually trivial.



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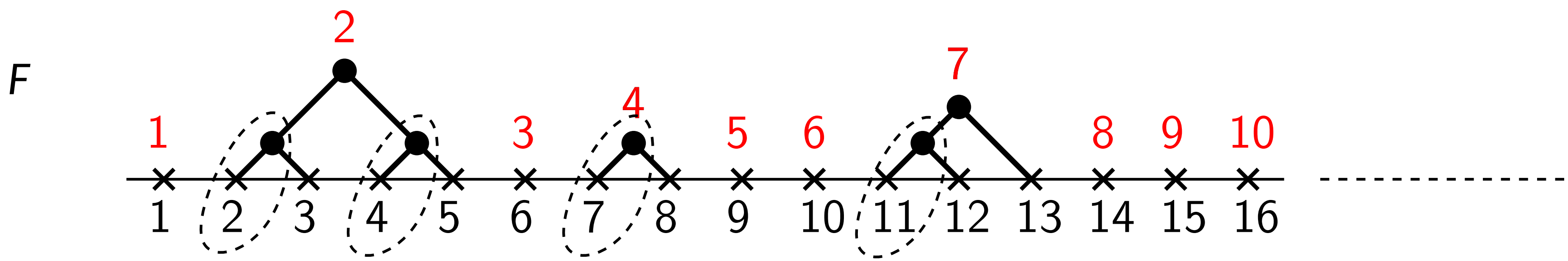
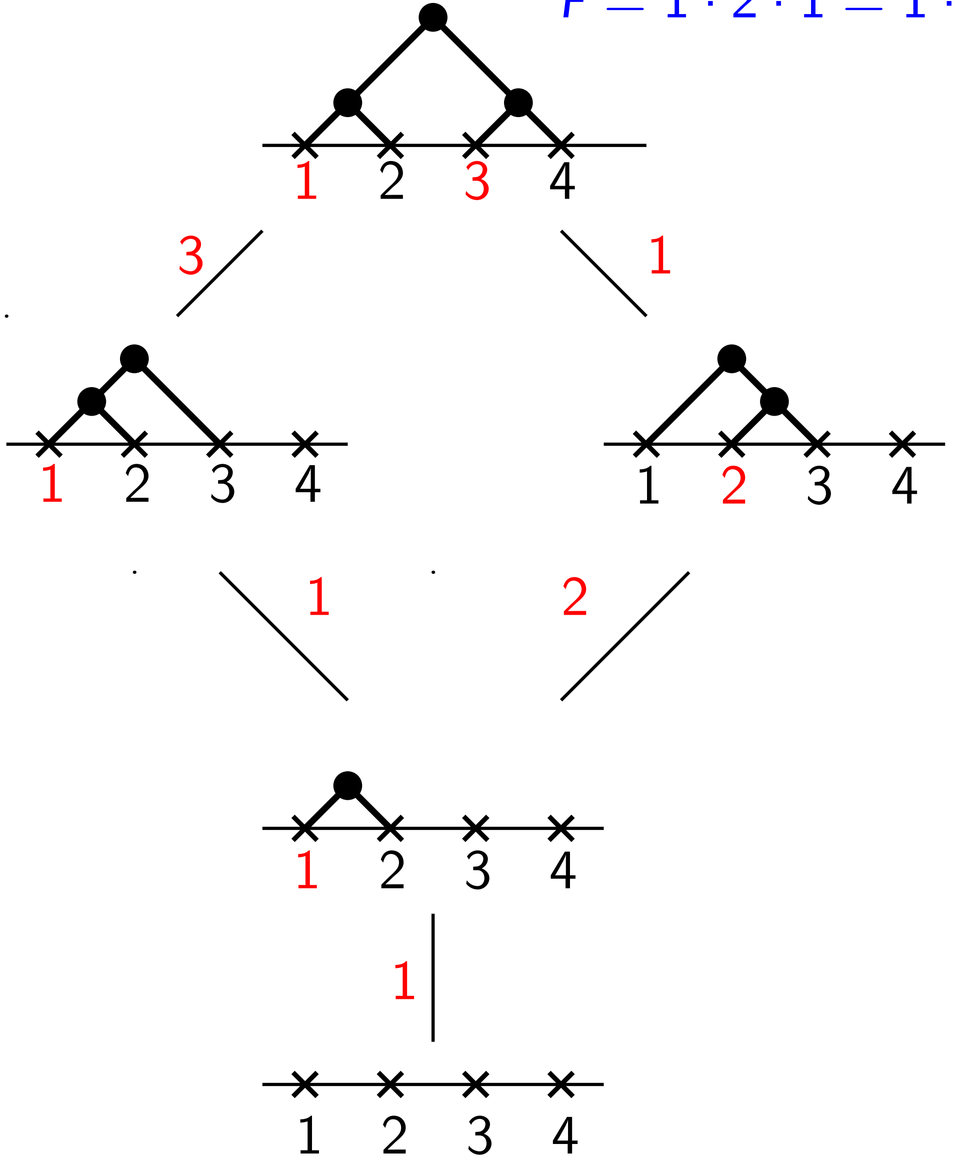


Let For be the set of indexed forests.

- $\mathrm{LTer}(F)$ is the set of left leaves i of a terminal node of F.
  Example $\mathrm{LTer}(F) = \{2, 4, 7, 11\}$ above
- $F \cdot i$ is given by adding a terminal node with left leaf i.
- F/i is the reverse of the above, only defined if $i \in \mathrm{LTer}(F)$.

Example

$F = 1 \cdot 2 \cdot 1 = 1 \cdot 1 \cdot 3.$



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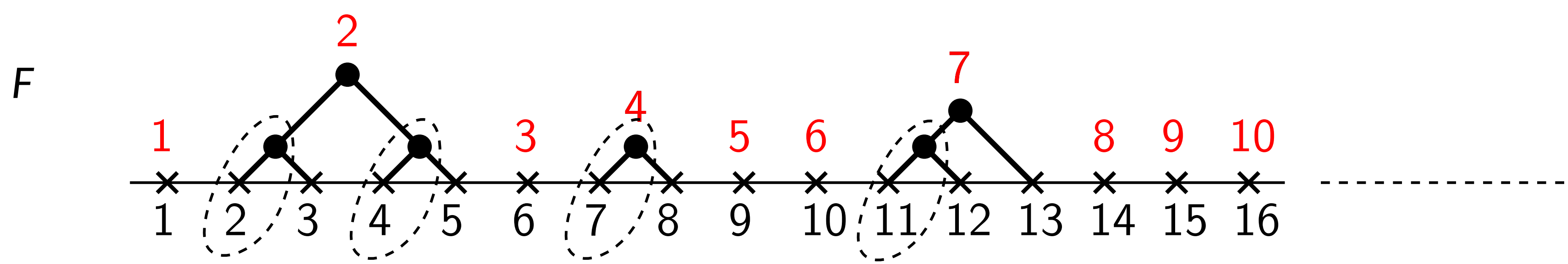

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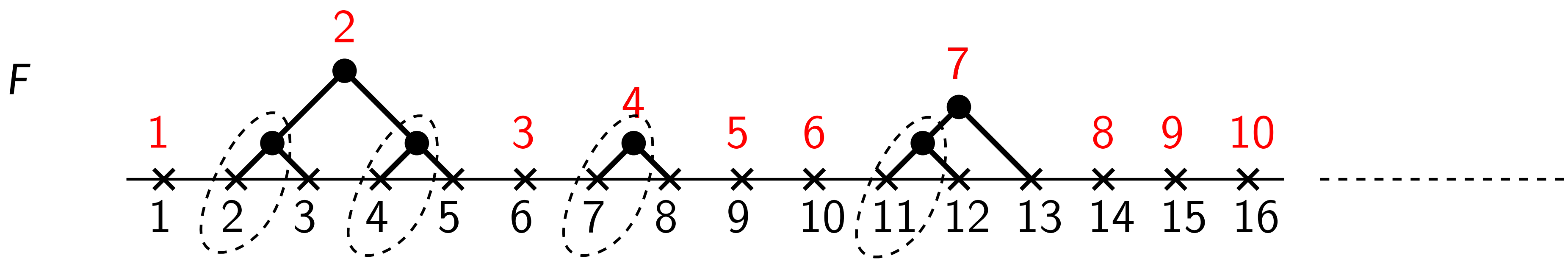

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Proposition. Define $F \cdot G =$ the forest H obtained by identifying the leaves of F with the roots of G. Then For $\simeq$ Thompson monoid.

$\Rightarrow$ We can define $\mathrm{T}_F = T_{i_1} \cdots T_{i_k}$ by taking any decomposition $F = i_1 \cdots i_k$.

Forest polynomials

Definition-Theorem[N.-Spink-Tewari '24] The forest polynomials $\mathfrak{P}_F$ for $F \in \mathrm{For}$ are the unique family of homogeneous polynomials such that $\mathfrak{P}_\emptyset = 1$ and

$$\mathsf{T}_i(\mathfrak{P}_F) = \begin{cases} \mathfrak{P}_{F/i} & \text{if } i \in \mathrm{LTer}(F) \\ 0 & \text{otherwise.} \end{cases}$$

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By iteration one gets:

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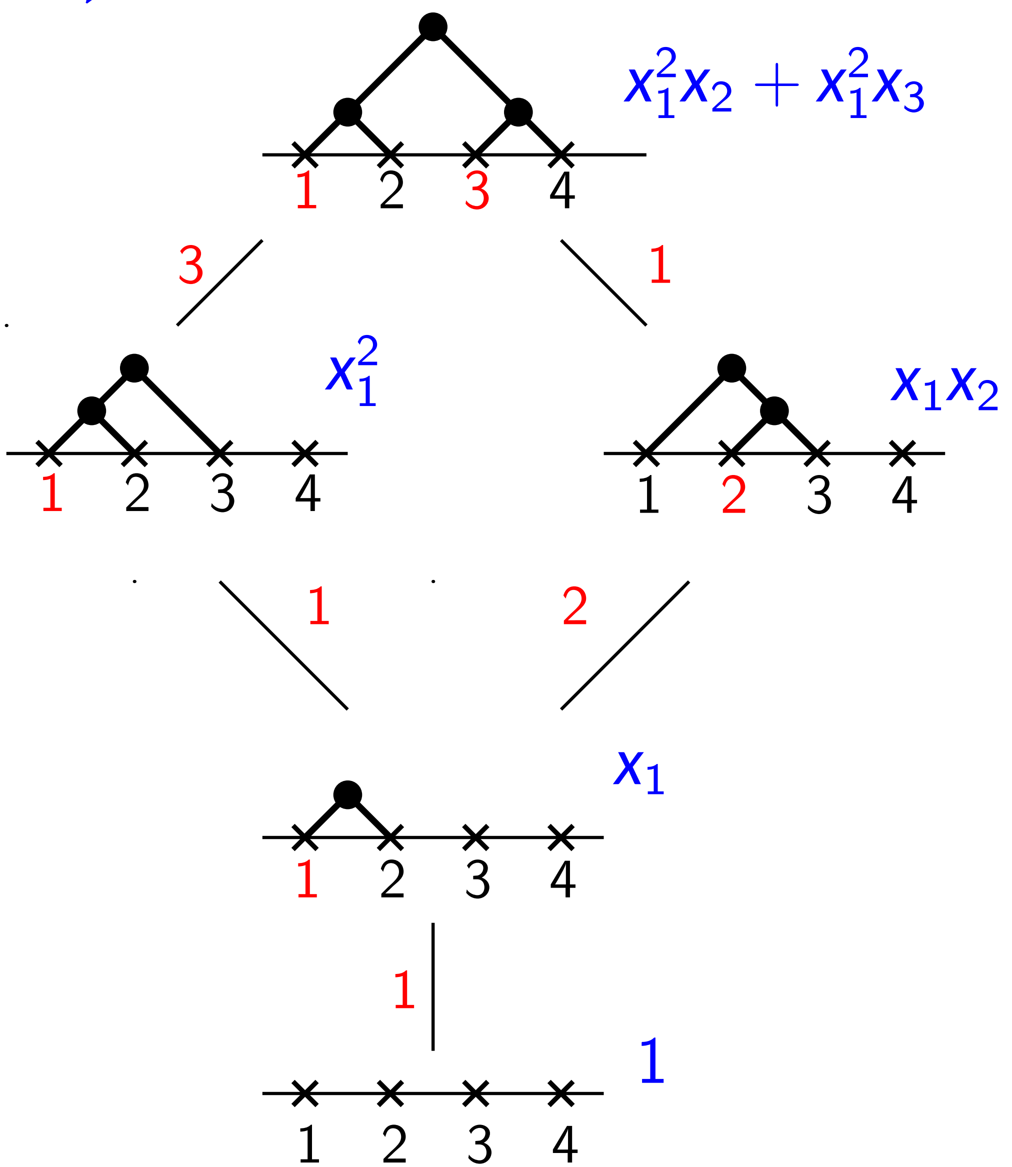
Explicit construction of $\mathfrak{P}_F$: Let $\phi_F(v) =$ label of the leaf at the end of its left branch of $v \in \mathrm{IN}(F)$, an internal node. Then

$$\mathfrak{P}_F := \sum_{f:\mathrm{IN}(F)\to\mathbb{Z}_{>0}} \prod_{v\in \mathrm{IN}(F)} x_{f(v)}$$

where the sum is over all f whose values are weakly increasing down left edges, strictly increasing down right edges, and such that $f(v) \leq \phi_F(v)$ for all v.

Back to Example

Some polynomials $\mathfrak{P}_F$



What do we get ?

$\rightarrow$ **Nice bases of various spaces:**

- $\mathfrak{P}_F$ is quasisymmetric in $x_1, \dots, x_n$ if and only F has a unique terminal node at $i = n$.

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Proposition. The $\mathfrak{P}_F$ for $F \in \mathrm{For}_n$ project to a basis of the coinvariant space $\mathrm{Pol}_n/\mathrm{QSym}_n{}^+$.

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$\rightarrow$ **Positivity results**

- By their combinatorial definition, the $\mathfrak{P}_F$ have positive coefficients.

- The structure constants $\mathfrak{P}_F\mathfrak{P}_G = \sum_H d_{FG}^H \mathfrak{P}_H$ are positive.
  This can be proved combinatorially.

(**Key**: Leibniz rule $\mathrm{T}_i(fg) = \mathrm{T}_i(f)\mathrm{R}_{i+1}(g) + \mathrm{R}_i(f)\mathrm{T}_i(g)$.)

Schubert polynomials

The Schubert polynomials $\mathfrak{S}_w$ form a basis in $\mathbb{Z}[x_1, x_2, ...]$, indexed by permutations.

Ex $(w \in S_3)$

$$\mathfrak{S}_{123} = 1 \qquad \mathfrak{S}_{213} = x_1 \qquad \mathfrak{S}_{132} = x_1 + x_2 \qquad \mathfrak{S}_{321} = x_1^2 x_2 \qquad \mathfrak{S}_{312} = x_1^2 \qquad \mathfrak{S}_{231} = x_1 x_2$$

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$S_\infty = \lim_n S_n$ = { Permutations w of $\{1, 2, \ldots\}$ such that $w(i) = i$ for i large enough}.

Define the *divided difference* $\partial_i = \dfrac{\mathrm{id} - s_i}{x_i - x_{i+1}}$ on Pol.

Definition-Theorem. The Schubert polynomials $\mathfrak{S}_w$ for $w \in S_\infty$, are the unique family of homogenous polynomials in Pol such that $\mathfrak{S}_{\mathrm{id}} = 1$ and

$$\partial_i \mathfrak{S}_w = \begin{cases} \mathfrak{S}_{ws_i} & \text{if } i \in \mathrm{Des}(w), \\ 0 & \text{otherwise.} \end{cases}$$

Positivity of Schubert polynomials

A direct check shows:

$$\mathrm{T}_i = \mathrm{R}_i \partial_i$$

Now for $f \in \mathrm{Pol}$ with $f(0) = 0$,

$$f = \sum_{i=1}^{\infty} (\mathrm{R}_{i+1}(f) - \mathrm{R}_i(f)) + \mathrm{R}_1(f)$$

$$= \sum_{i=1}^{\infty} x_i \mathrm{T}_i(f) + \mathrm{R}_1(f) = \sum_{i=1}^{\infty} x_i \mathrm{R}_i \partial_i(f) + \mathrm{R}_1(f)$$

Choose $f = \mathfrak{S}_w$ with $w \neq \mathrm{id}$

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- This is a **new recurrence**.
- Probably the simplest proof that $\mathfrak{S}_w$ **has positive coefficients.**
- Can be interpreted combinatorially on pipe dreams.