Quantum positroids in the quantum grassmannian

Séminaire Lotharingien de Combinatoire

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Quantum 2×2 matrices

The coordinate ring of quantum 2×2 matrices

$$\mathcal{O}_q(\mathcal{M}_2(\mathbb{C})) := \mathbb{C}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is generated by four indeterminates a, b, c, d subject to the following rules:

$$\boxed{\begin{array}{cc} ab = qba, & cd = qdc \\ ac = qca, & bd = qdb \\ bc = cb, & ad - da = (q - q^{-1})cb. \end{array}}$$

The **quantum determinant** $ad - qbc$ is a central element

The algebra of $m \times p$ quantum matrices.

$$R = O_q\left(\mathcal{M}_{m,p}(\mathbb{C})\right) := \mathbb{C}\begin{bmatrix} Y_{1,1} & \dots & Y_{1,p} \\ \vdots & & \vdots \\ Y_{m,1} & \dots & Y_{m,p} \end{bmatrix},$$

where each 2×2 sub-matrix is a copy of $O_q\left(\mathcal{M}_2(\mathbb{C})\right)$.

$O_q\left(\mathcal{M}_{m,p}(\mathbb{C})\right)$ is an iterated Ore extension with the indeterminates $Y_{i,\alpha}$ adjoined in the lexicographic order and so is a noetherian integral domain.

In the square case $(m = p = n)$

$$D_q = \sum_{\sigma \in S_n} (-q)^{l(\sigma)} Y_{1,\sigma(1)} \cdots Y_{n,\sigma(n)}$$

is the **quantum determinant**. D_q is a central element.

Quantum minors of $R = \mathcal{O}_q(\mathcal{M}_{m,p}(\mathbb{C}))$

They are the quantum determinants of square sub-matrices of $\mathcal{O}_q(\mathcal{M}_{m,p}(\mathbb{C}))$.

More precisely, if $I \subseteq [\![1, m]\!]$ and $\Lambda \subseteq [\![1, p]\!]$ with $\mid I \mid = \mid \Lambda \mid$, the **quantum minor** associated with the rows I and columns Λ is

$$[I \mid \Lambda] := D_q(\mathcal{O}_q(M_{I,\Lambda}(\mathbb{C}))).$$

For example, $[12|23] = Y_{1,2}Y_{2,3} - qY_{1,3}Y_{2,2}$ is the quantum minor of R associated with the rows $1, 2$, and the columns $2, 3$.

The quantum grassmannian $\mathcal{G}_q(k,n)$

The quantum grassmannian $\mathcal{G}_q(k,n)$ is the subalgebra of $\mathcal{O}_q(\mathcal{M}(k,n))$ generated by the maximal $k \times k$ quantum minors

Denote by $[I]$ the quantum minor $[1 \ldots k|I]$.

Example $\mathcal{G}_q(2,4)$ is generated by the six quantum minors $[12],[13],[14],[23],[24],[34]$.

Most minors $q^\bullet$-commute, for example, $[12]\,[34] = q^2\,[34]\,[12]$, however, $[13]\,[24] = [24]\,[13] + (q - q^{-1})\,[14]\,[23]$ and there is a quantum Plücker relation

$$[12]\,[34] - q\,[13]\,[24] + q^2\,[14]\,[23] = 0.$$

Noncommutative dehomogenisation:

$$\mathcal{G}_q(k,n)[[12...k]^{-1}] \simeq \mathcal{O}_q(\mathcal{M}(k,n-k))[Z^{\pm 1};\sigma]$$

The prime spectrum of $\mathcal{O}_q(\mathcal{M}_{m,p}(\mathbb{C}))$

Assume that $q \in \mathbb{C}^*$ **is not a root of unity**, and set $R := \mathcal{O}_q(\mathcal{M}_{m,p}(\mathbb{C}))$.

- **Goodearl-Letzter** Prime ideals of R are (completely) prime.

The torus $\mathcal{H} := (\mathbb{C}^*)^{m+p}$ acts by automorphisms on R via :

$$(a_1, \ldots, a_m, b_1, \ldots, b_p).Y_{i,\alpha} = a_i b_\alpha Y_{i,\alpha}.$$

This action of $\mathcal{H}$ on R induces an action of $\mathcal{H}$ on Spec(R). We denote by $\mathcal{H}$-Spec(R) the set of those prime ideals in R which are $\mathcal{H}$-invariant.

- **Goodearl-Letzter** R has at most 2^{mp} $\mathcal{H}$-primes.

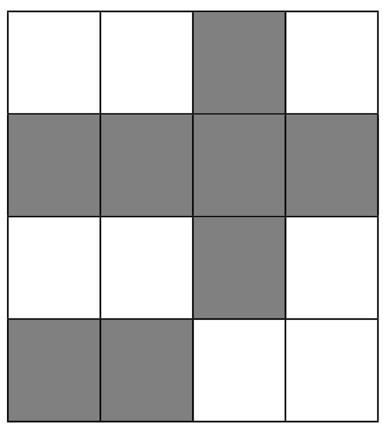
Note that 0 is always an $\mathcal{H}$-prime ideal.

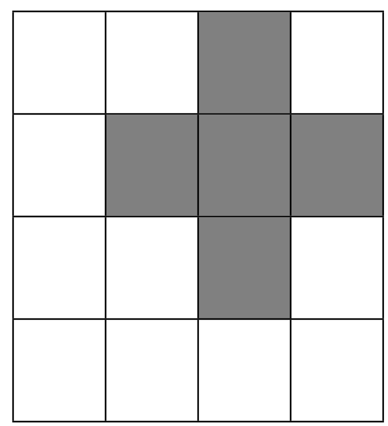
By the Stratification Theorem of Goodearl-Letzter, $\mathcal{H}$-Spec(R) "controls" Spec(R).

Cauchon diagrams

A **Cauchon diagram** on an $m \times p$ array is an $m \times p$ array of squares coloured either black or white such that for any square that is coloured black the following holds:
Either each square strictly to its left is coloured black, or each square strictly above is coloured black.

Here are an example and a non-example





Parametrisation of $\mathcal{H}$-Spec$(\mathcal{O}_q(\mathcal{M}_{m,p}(\mathbb{C})))$

• **Cauchon (2003)** There is a bijection between Cauchon diagrams on an $m \times p$ array and $\mathcal{H} -$ Spec$(\mathcal{O}_q(\mathcal{M}_{m,p}(\mathbb{C})))$.

If C is a Cauchon diagram, then we denote by J_C the unique $\mathcal{H}$-prime associated to C.

• **L., Yakimov, Casteels** $\mathcal{H}$-primes are generated by quantum minors.

• Dimensions of $\mathcal{H}$-strata and the poset of $\mathcal{H}$-primes are described through another parametrization.

The quantum grassmannian $\mathcal{G}_q(k,n)$

The quantum grassmannian $\mathcal{G}_q(k,n)$ is the subalgebra of $\mathcal{O}_q(\mathcal{M}(k,n))$ generated by the maximal $k \times k$ quantum minors

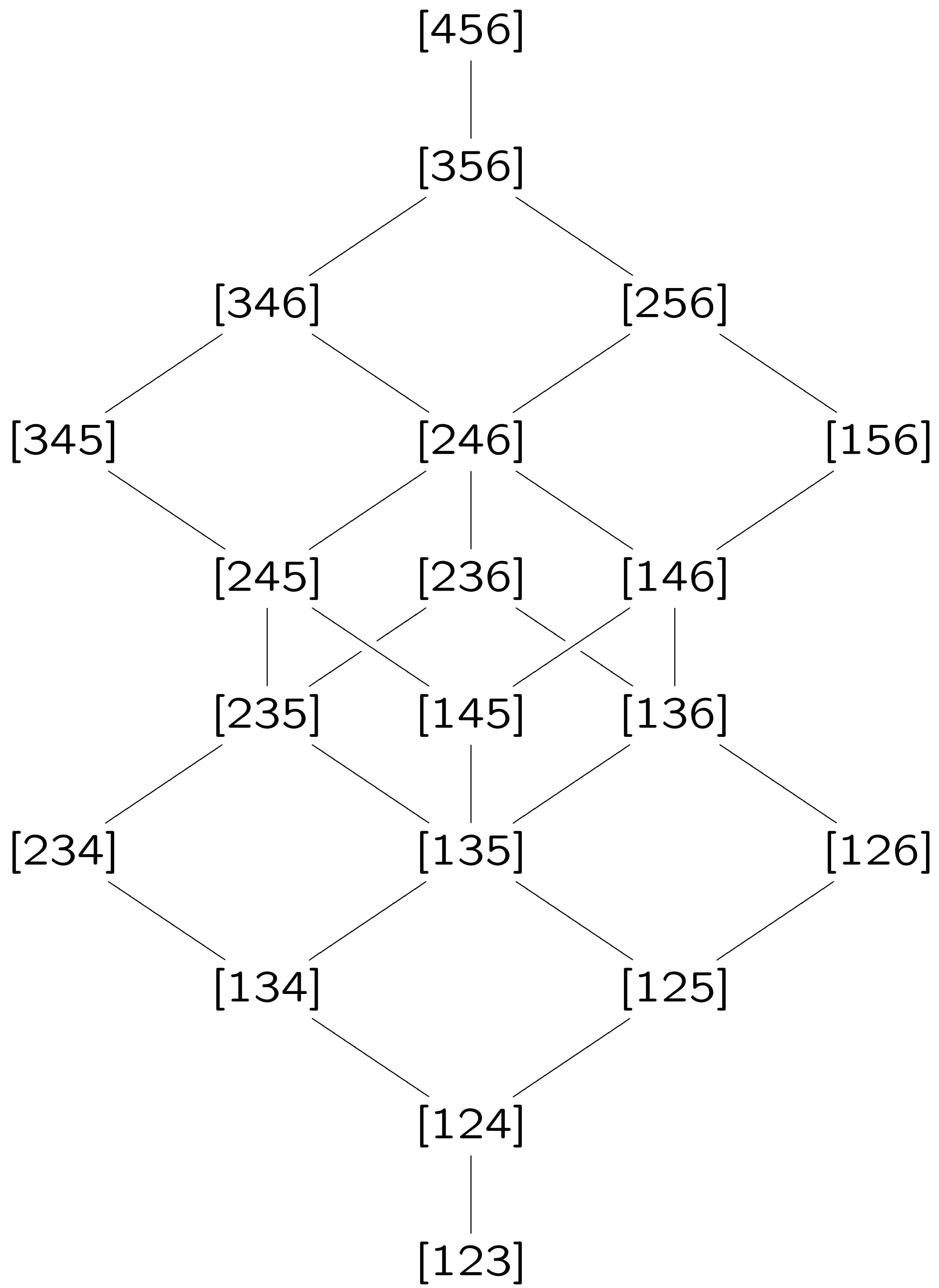
Denote by $[I]$ the quantum minor $[1 \dots k|I]$. There is a torus action of $\mathcal{H} = (\mathbb{C}^*)^n$ given by column multiplication. There are finitely many $\mathcal{H}$-primes.

Example $\mathcal{G}_q(2,4)$ is generated by the six quantum minors $[12],[13],[14],[23],[24],[34]$.

Most minors $q^\bullet$-commute, for example, $[12]\,[34] = q^2\,[34]\,[12]$, however, $[13]\,[24] = [24]\,[13] + (q-q^{-1})\,[14]\,[23]$ and there is a quantum Plücker relation

$$[12]\,[34] - q\,[13]\,[24] + q^2\,[14]\,[23] = 0.$$

Partial order:
$[i_1 < \cdots < i_k] \leq [j_1 < \cdots < j_k]$ whenever $i_s \leq j_s$ for all s.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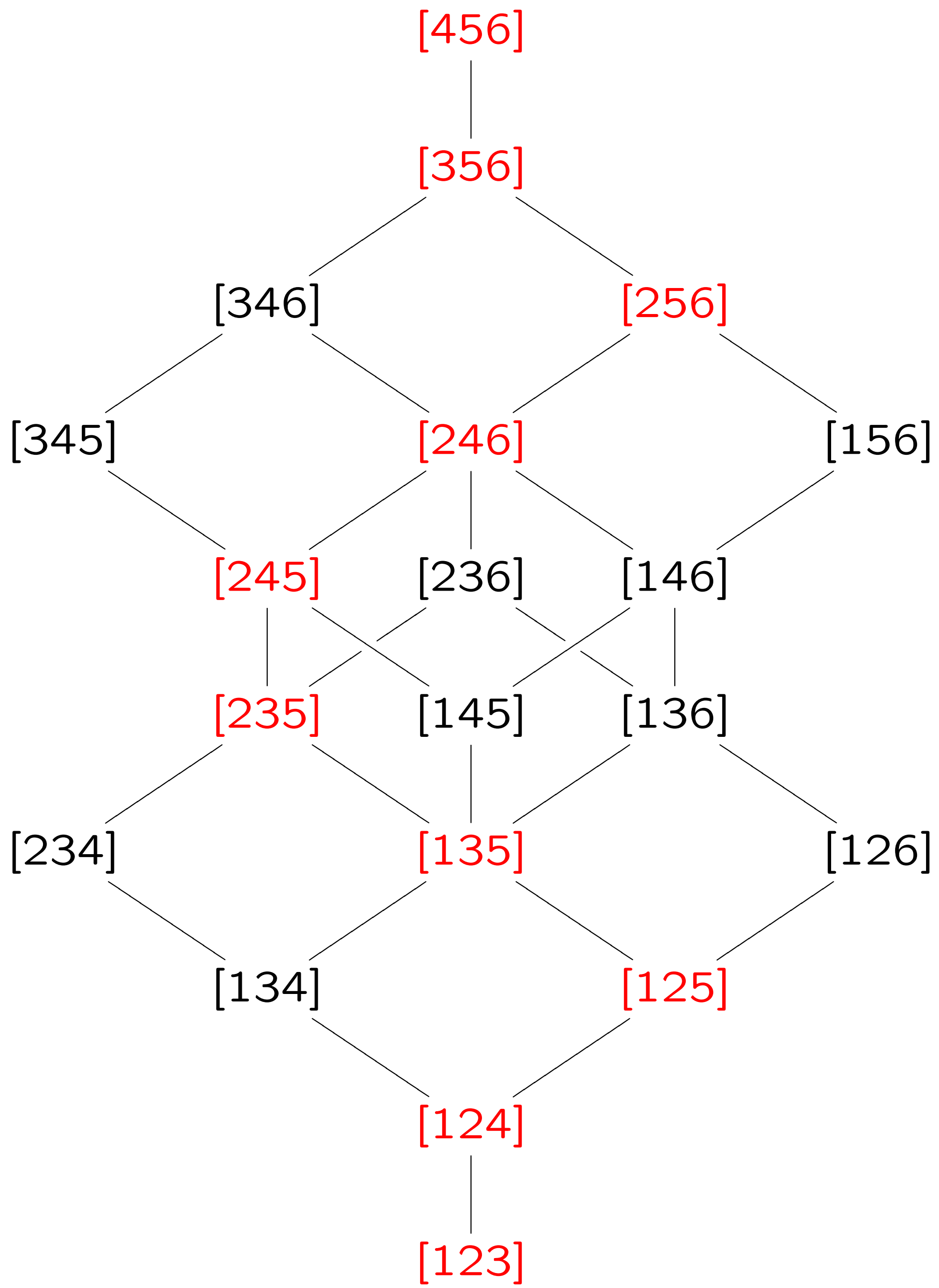
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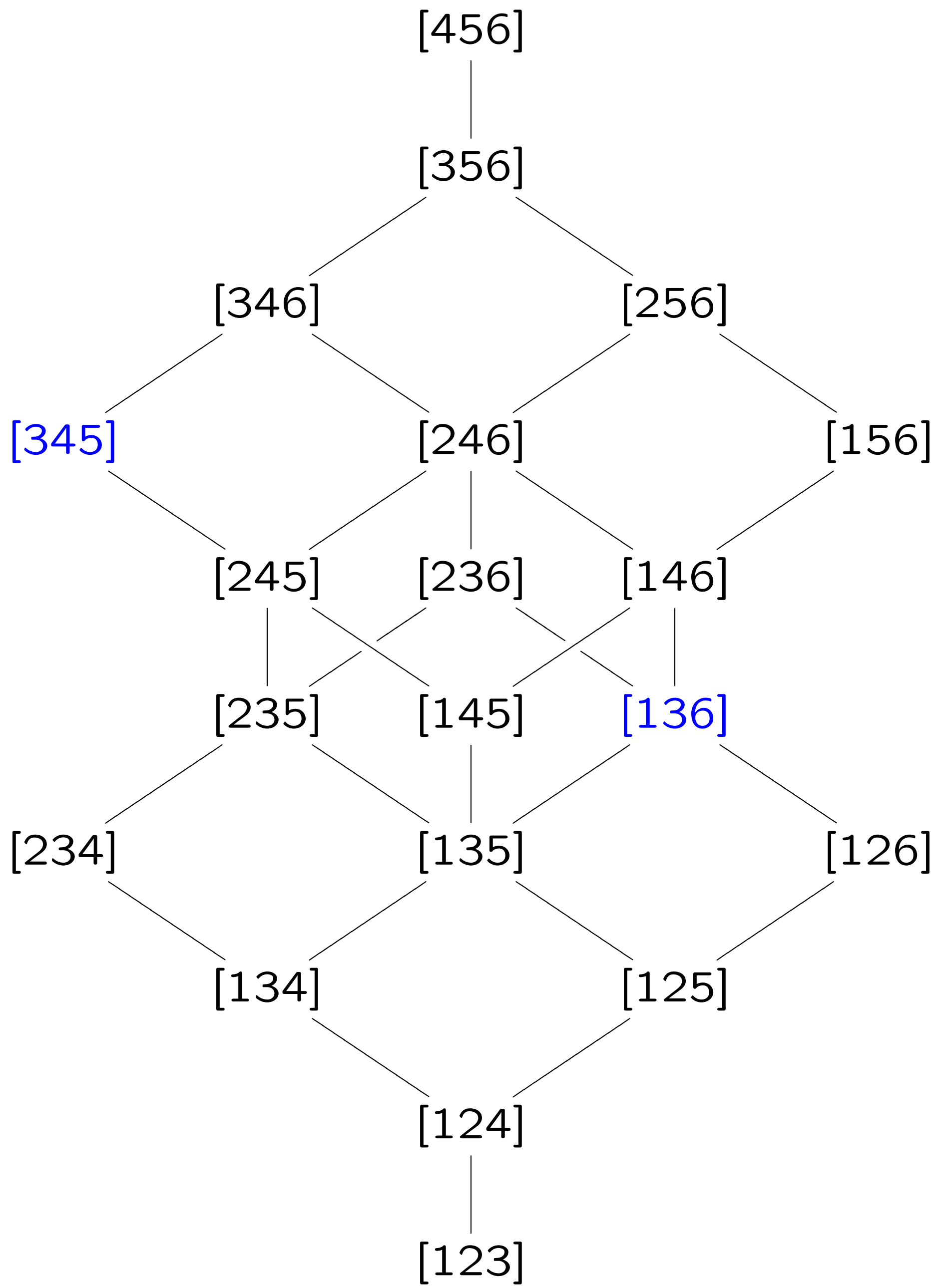
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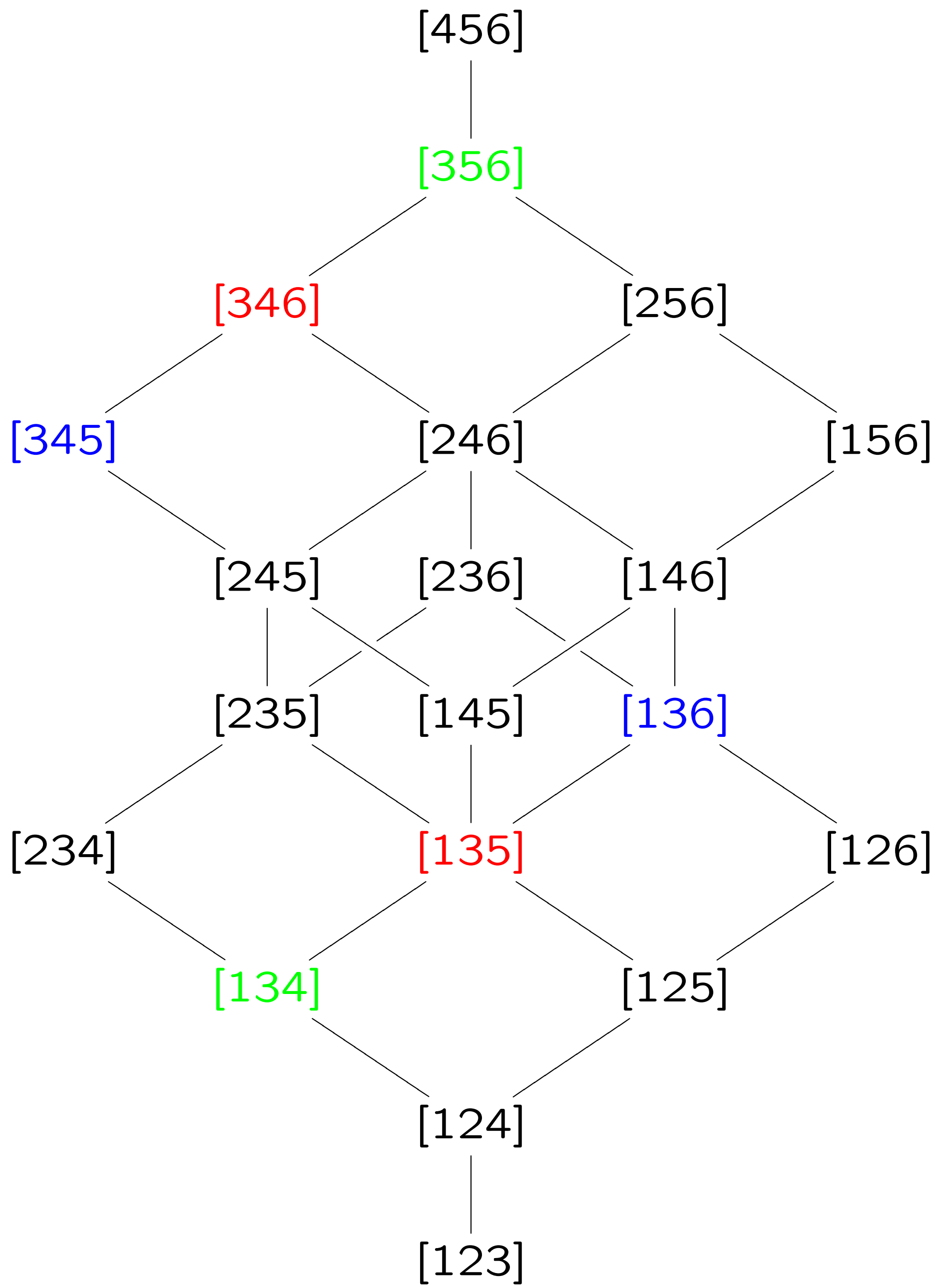
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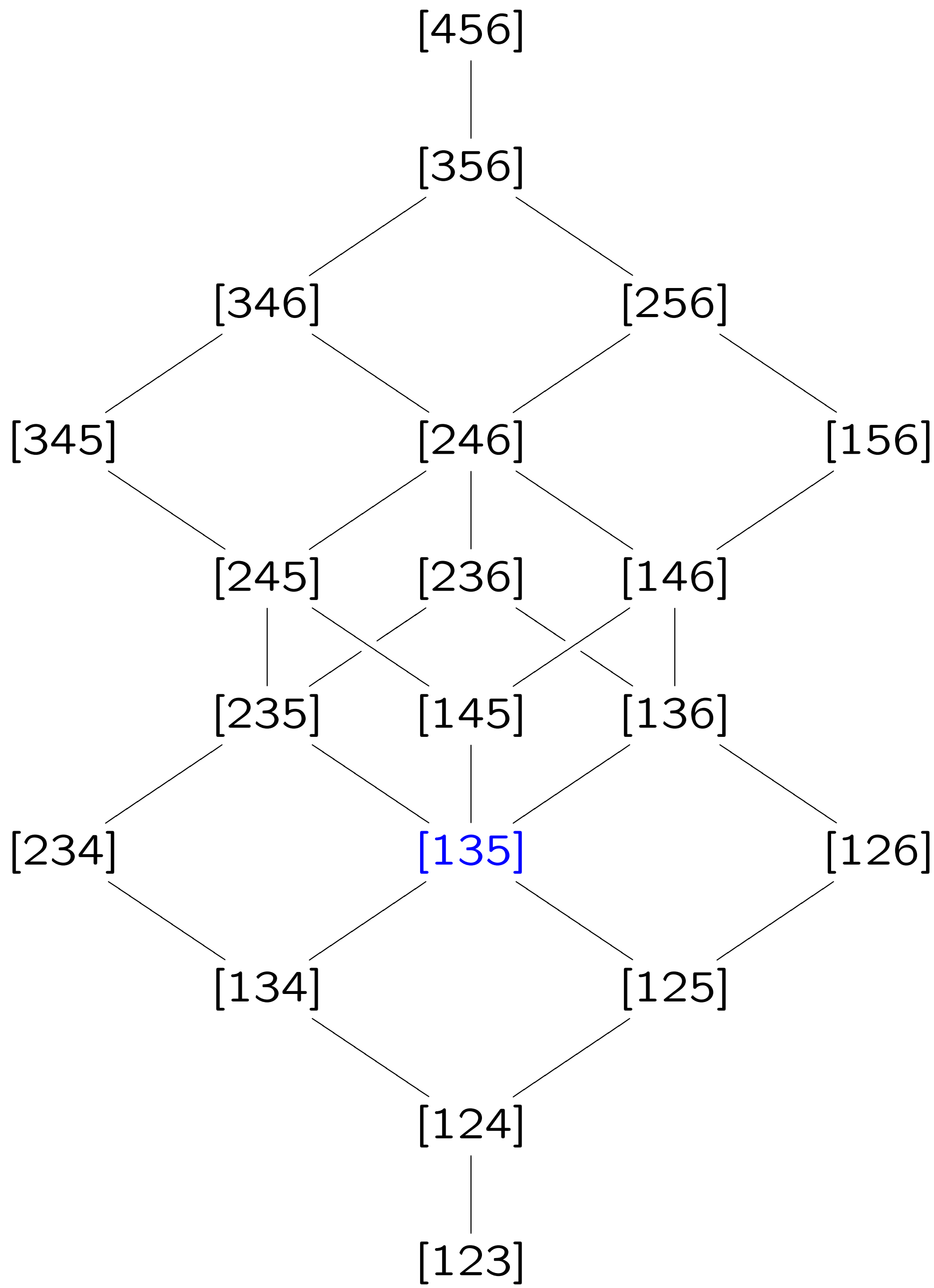
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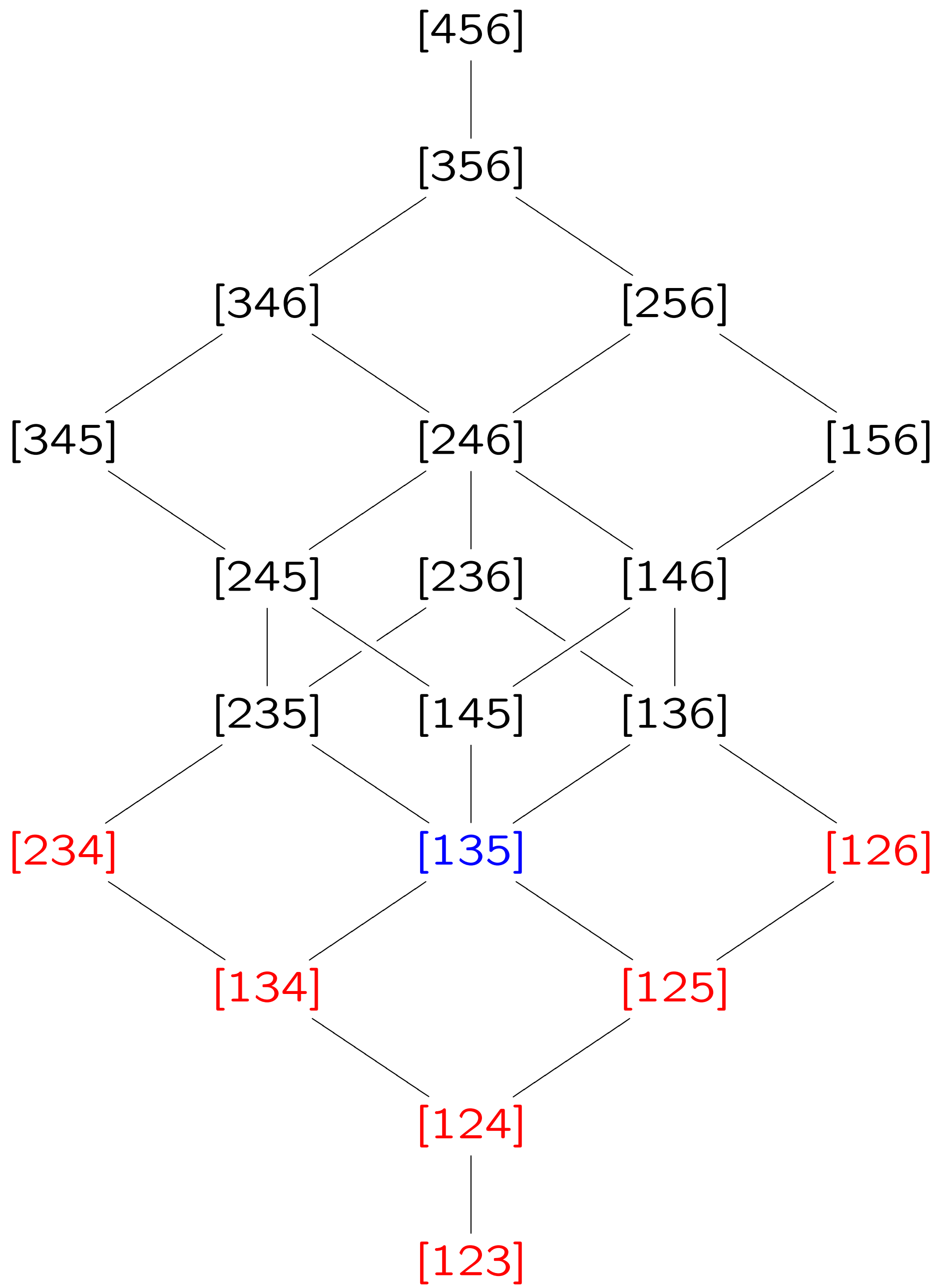
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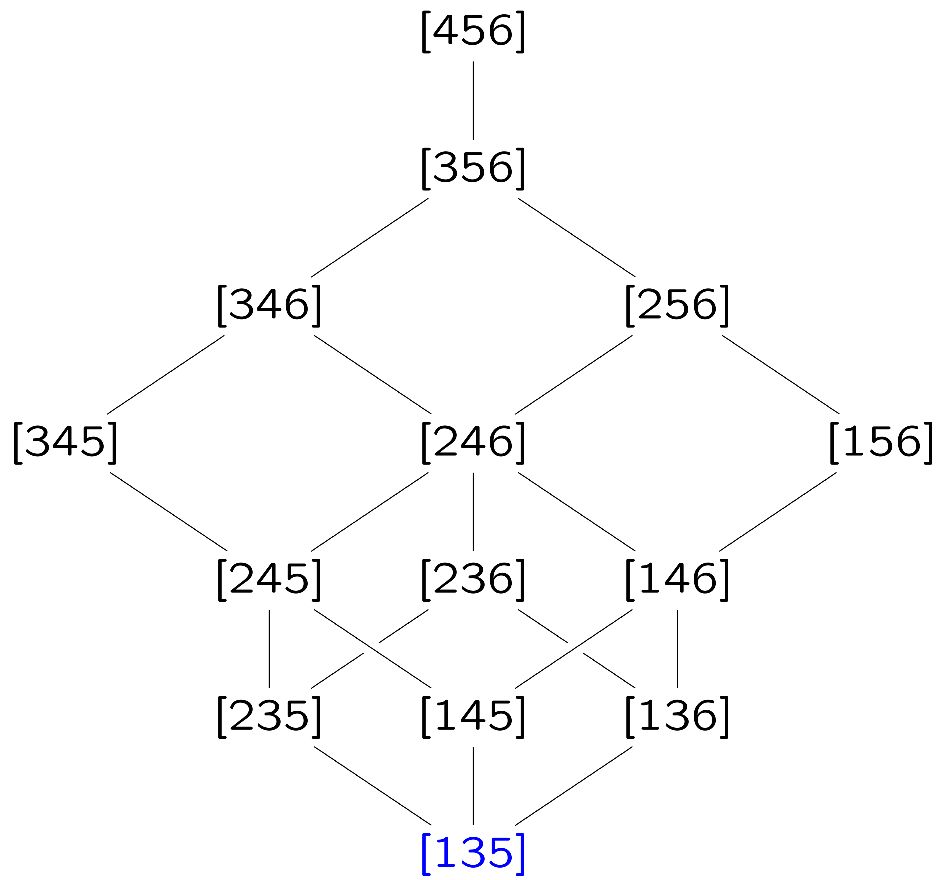
Quantum Schubert variety corresp to [135]

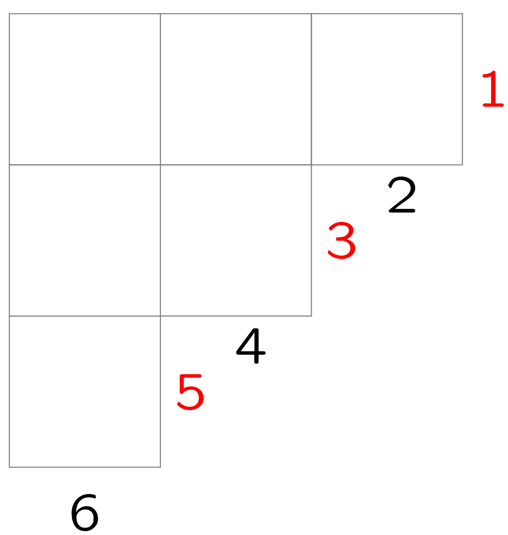


Schubert cell: use noncommutative dehomogenisation at [135]

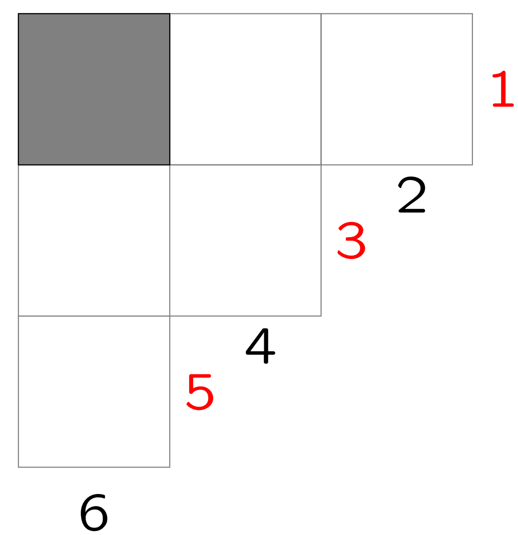
L, Lenagan and Rigal (2008) There is a bijection between $\mathcal{H}-\mathrm{Spec}(\mathcal{G}_q(k,n))$ (ignoring the irrelevant ideal) and Cauchon-Le diagrams on Young diagrams that fit inside a $k\times(n-k)$ array

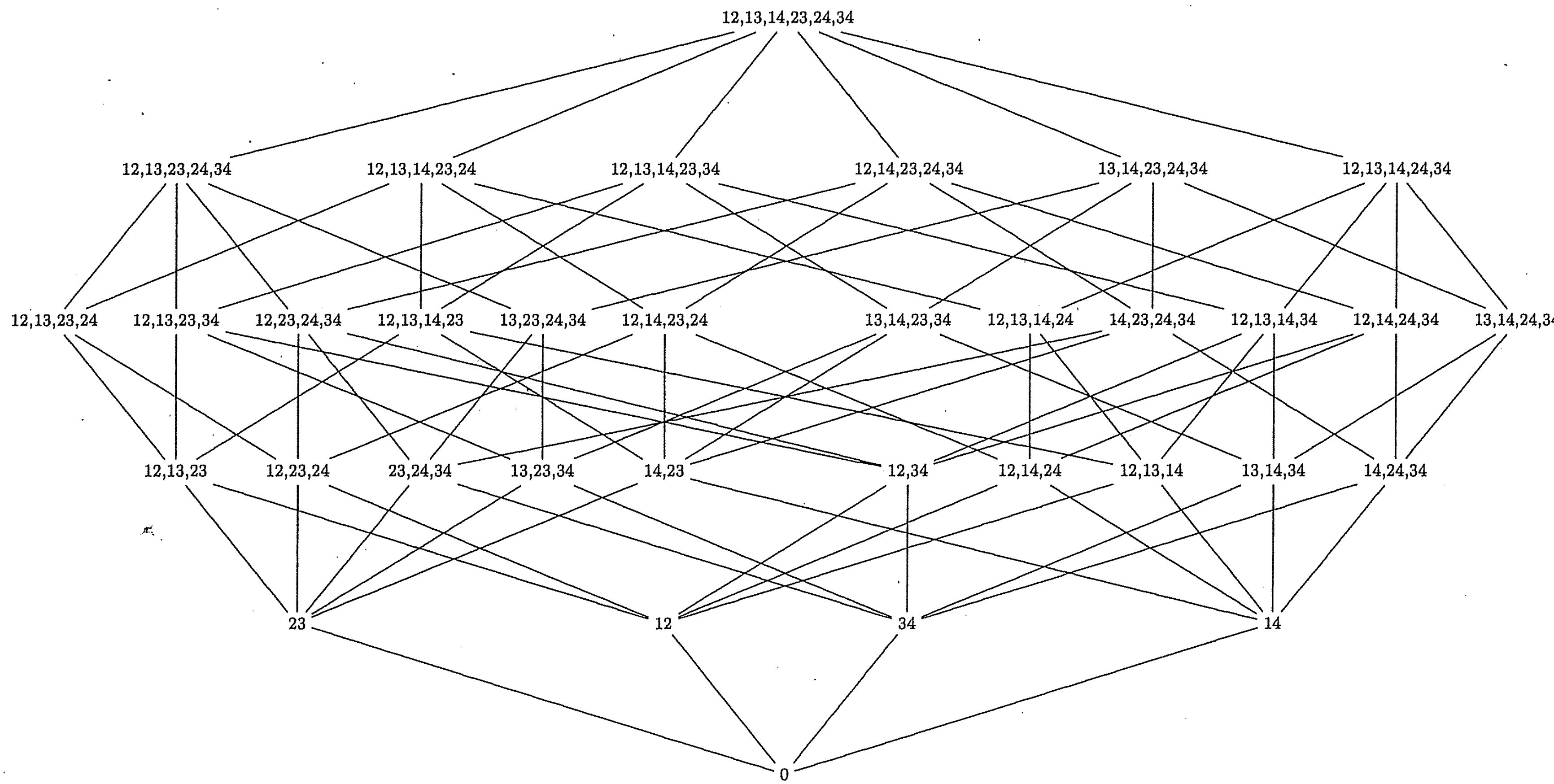
The theorem is proved by defining quantum algebras with a straightening law, quantum Schubert varieties, quantum Schubert cells, partition subalgebras of quantum matrices and using a non-commutative version of dehomogenisation.

Schubert cell for [135]



$\mathcal{H}$-prime in Schubert cell [135]





$\mathcal{H} - \mathrm{Spec}(\mathcal{O}_q(G_{2,4}(\Bbbk)))$

A few questions to consider:

Questions:

1. Can we specify the quantum Plücker coordinates in a given H-prime?

2. Are $\mathcal{H}$-primes generated by quantum Plücker coordinates?

3. Can we describe the poset of $\mathcal{H}$-primes in $\mathcal{G}_q(k,n)$? (Yakimov's conjecture)

The nonnegative world

TNN grassmannian

A point P in the grassmannian $\mathcal{G}_{kn}(\mathbb{R})$ is **totally nonnegative** if its Plücker coordinates can be represented by the $k \times k$ minors of a $k \times n$ matrix A such that each of these $k \times k$ minors are nonnegative.

Cells are specified by stating precisely which Plücker coordinates are zero. If Z is a subset of Plücker coordinates then S_Z° is the cell where minors in Z are zero (and those not in Z are nonzero, so positive).

• **Postnikov (arXiv:math/0609764)** There is a bijection between Le-diagrams (=Cauchon diagrams) on partitions that fit into a $k \times (n-k)$ array and non-empty cells S_Z° in $\mathcal{G}_{kn}^{\text{tnn}}$.

(A Young diagram representing a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ fits inside a $k \times (n-k)$ array provided that $(n-k) \geq \lambda_1 \geq \lambda_2 \geq \lambda_k \geq 0$.)

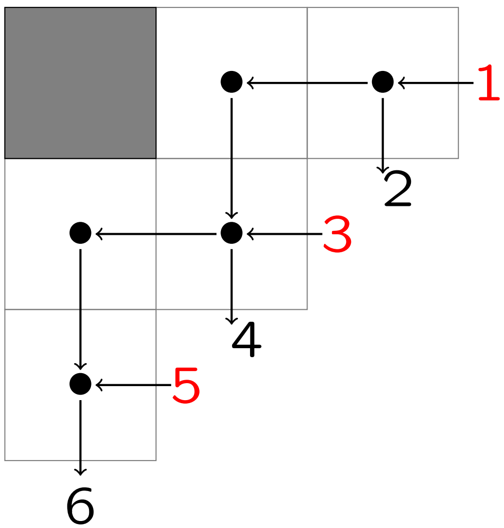
Quantum Plücker coordinates in a given $\mathcal{H}$-prime

L-Lenagan-Nolan Let $\mathcal{F}$ be a family of Plücker coordinates and $\mathcal{F}_q$ the corresponding family of quantum Plücker coordinates. TFAE

- The totally nonnegative cell associated to $\mathcal{F}$ in $\mathcal{G}_{kn}^{\text{tnn}}$ is non-empty.
- $\mathcal{F}_q$ is the set of all quantum minors that belong to torus-invariant prime in $\mathcal{G}_q(k,n)$.

Strategy: Let C be a Cauchon-Le diagram of shape λ. Then we prove that $[I] \in J_C$ iff there are no vertex disjoint set of paths from $\lambda \setminus I$ to $I \setminus \lambda$ in the Postnikov graph. The case of quantum matrices was already known thanks to work of Casteels.

Consequence: We have an explicit description of these families thanks to work of Oh.



There is a vertex disjoint set of paths from $\{1, 3\}$ to $\{2, 4\}$ so [245] is not in the prime.

There is no vertex disjoint set of paths from $\{1, 3\}$ to $\{4, 6\}$ so [456] is in the prime.

Quantum Plücker coordinates generates $\mathcal{H}$-primes

L-Lenagan-Nolan Assume q is transcendental. Then torus-invariant primes in $\mathcal{G}_q(k,n)$ are generated by quantum Plücker coordinates.

In the proof, we make use of the above techniques as well as Gröbner basis techniques. In particular, we use a theorem of Casteels that asserts that torus-invariant primes in quantum matrices are generated by quantum minors and these quantum minors form a GB of the torus-invariant prime that they generate.

Quantum positroids in the quantum grassmannians: these are the quotients of $\mathcal{G}_q(k,n)$ by torus-invariant primes.

Applications

* We can explicitly described the quantum Plücker coordinates that generate $\mathcal{H}$-primes.

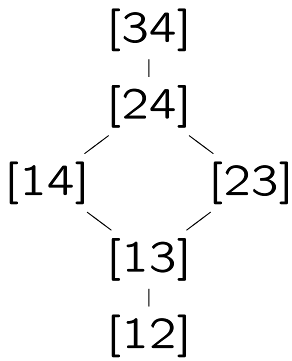
* Grassmann necklace: we can put different orders on the set of quantum Plücker coordinates. For each i with $1 \leq i \leq n$ we can define the i-*ordering*, denoted by $<_i$. In this ordering, we have

$$i <_i i+1 <_i i+2 <_i \cdots <_i n <_i 1 <_i \cdots <_i i-1.$$

There is then an induced partial ordering on the quantum Plücker coordinates given by

$$[a_1 <_i \cdots <_i a_m] \leq_i [b_1 <_i \cdots <_i b_m]$$

if and only if $a_j \leq_i b_j$ for each $1 \leq j \leq m$.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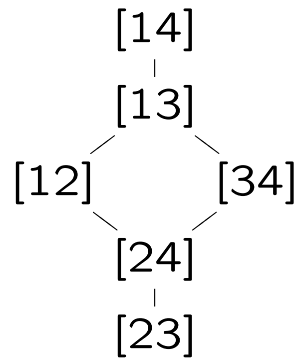
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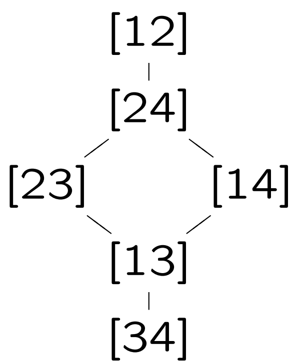
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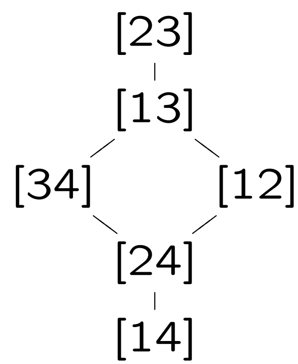
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Grassmann necklace

Consider the QGASL structure on $\mathcal{G}_q(k,n)$ determined by the poset Π_i. Let P be an $\mathcal{H}$-prime ideal of $\mathcal{G}_q(k,n)$. Then there is a unique quantum Plücker coordinate $[I_i]$ in Π_i with the property that $[I_i] \notin P$ but $[J] \in P$ for all $J \ngeq_i I_i$.

The sequence Neck(P) := $([I_1], \ldots, [I_n])$ of quantum Plücker coordinates is called the Grassmann necklace of P.

Let P and Q are $\mathcal{H}$-prime ideals of $\mathcal{G}_q(k,n)$. Then

$Q \subseteq P$ if and only if Neck$(Q) \leq$ Neck(P) (i.e. iff $J_i \leq_i I_i$ for each $i = 1, \ldots, n$.)

Yakimov conjecture

The following posets are isomorphic:

1. $\mathcal{H}$-Spec$\mathcal{G}_q(k,n)$ (endowed with inclusion);

2. the set of torus orbits of symplectic leaves in the grassmannian $SL_{n+1}(\mathbb{C})/P_I$ (ordered by closure);

3. $S_{W,I}$, where $W = S_{n+1}$ and W_I is the subgroup generated by $s_1, \ldots, s_{m-1}, s_{m+1}, \ldots, s_m$, and where the order $(w,v) < (w',v')$ in $S_{W,I}$ is defined by

   $$(w,v) < (w',v') \text{ iff}$$

   there exists $z \in W_I$ such that $w \geq w'z$ and $v \leq v'z$.