Combinatorial Applications to the Stokes Phenomenon

Tomás Inácio
Department of Mathematics
Faculty of Sciences, University of Lisbon

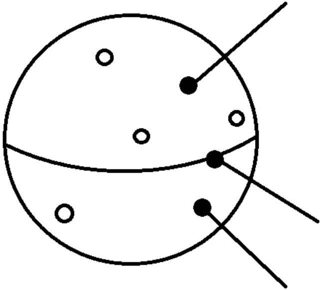
March 24, 2025

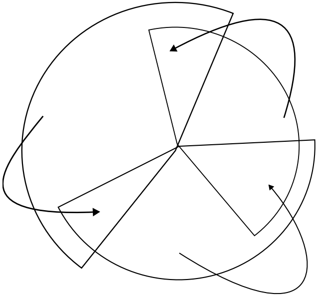
Fundação para a Ciência e a Tecnologia
CEMS.UL CENTER FOR MATHEMATICAL STUDIES
UID/04561/2025
Ciências ULisboa

Isomonodromic deformations

We start off with an ODE on $\mathbb{P}^1 \backslash \{a_0, a_1, a_2, \ldots, a_p\}$.

$$\frac{d}{dz} Y(z) = A(z) Y(z).$$





$$Y_{m+1}(z) = Y_m(z) S_m.$$

Riemann-Hilbert-Birkhoff Problem

Suppose $Y(z) \in GL(N, \mathbb{C})$

$$\frac{dY}{dz}(z) = \left(\frac{U}{z^2} + \frac{V}{z}\right) Y(z), \quad U = \mathrm{diag}(u_1, u_2, \ldots, u_N),$$

and V is off-diagonal and holomorphic.

RIemann-Hilbert-Birkhoff Problem:

$$\boxed{\text{Stokes Factors } S_l} \longleftrightarrow \boxed{\text{Matrix } V}$$

Assumption: (Isomonodromy)

$$dV_\alpha = \sum_{\beta+\gamma=\alpha} [V_\beta, V_\gamma] d \log U(\gamma).$$

Power Series

Theorem

For each Stokes ray l, the Stokes factor is given by,

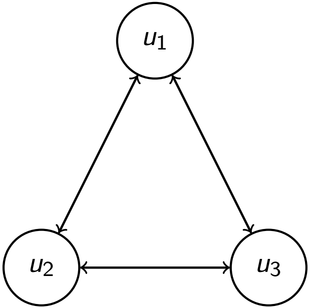
$$S_l = I_N + 2\pi i \sum_{m\geq 1} \sum_{\substack{U(\alpha_1),\dots,U(\alpha_m)\neq 0 \\ U(\alpha_1+\cdots+\alpha_m)\in l}} M_m(U(\alpha_1),\dots,U(\alpha_m)) V_{\alpha_1}\dots V_{\alpha_m},$$

Examples

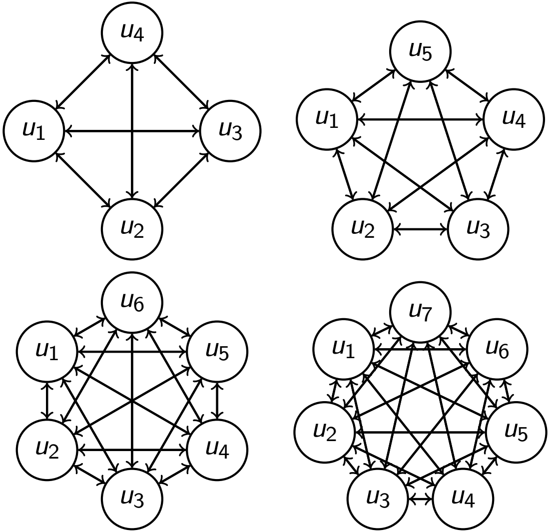
$\alpha_1 = r_1 - r_2 \Rightarrow \mathrm{U}(\alpha_1) = u_1 - u_2$ and $V_{\alpha_1} = V_{12}$
$\alpha_2 = r_2 - r_3 \Rightarrow \mathrm{U}(\alpha_1 + \alpha_2) = u_1 - u_3$ and $V_{\alpha_1+\alpha_2} = V_{13}$.

Number of terms and paths on graphs

m	2	3	4	5	6	7	8	9	10
number of terms	1	2	5	10	21	42	85	170	341



Higher dimensions



Multiple Polylogarithms

Definition

Set $M_1(z) = 1$ and, for $n \geq 2$, define the function $M_n : (\mathbb{C}^*)^n \to \mathbb{C}$ by the iterated integral

$$M_n(z_1, \dots, z_n) = \int_C \frac{dt}{t - s_1} \circ \cdots \circ \frac{dt}{t - s_{n-1}},$$

where $s_i = z_1 + \cdots + z_i$, $1 \leq i \leq n$ and the path of integration C is the line segment $]0, s_n[$ perturbed if necessary to avoid any point $s_i \in [0, s_n]$ by clockwise arcs.

$$\int_0^1 \underbrace{\frac{dt}{t - (x_1 \dots x_m)^{-1}} \circ \frac{dt}{t}}_{n_1\text{-times}} \circ \cdots \circ \frac{dt}{t} \circ \cdots \circ \underbrace{\frac{dt}{t - x_1^{-1}} \circ \frac{dt}{t} \circ \cdots \circ \frac{dt}{t}}_{n_m\text{-times}} =$$

$$=(\text{-}1)^n Li_{n_1, n_2, \dots, n_m}(x_1, x_2, \dots, x_m) = \sum_{0 < k_1 < \cdots < k_m} \frac{x_1^{k_1} \dots x_m^{k_m}}{k_1^{n_1} \dots k_m^{n_m}}.$$

Example: $N = 2$ case

$$V = \begin{bmatrix} 0 & v_{12} \\ v_{21} & 0 \end{bmatrix}, \quad M_{2m-1} = \int_0^1 \underbrace{\frac{dt}{1-t} \circ \frac{dt}{t} \dots \frac{dt}{1-t} \circ \frac{dt}{t}}_{(m-1)\text{-times}} = (-1)^{m-1} \mathrm{Li}_{2,\dots,2}(1,\dots,1).$$

Using

$$\mathrm{Li}_{\underbrace{2,\dots,2}_{(m-1)\text{-times}}}(1,\dots,1) = \zeta\left(\{2\}^{m-1}\right) := \sum_{0<l_1<l_2<\cdots<l_{m-1}} \frac{1}{l_1^2 l_2^2 \cdots l_{m-1}^2} = \frac{\pi^{2m-2}}{(2m-1)!}.$$

We obtain,

$$s_{l_{21}} = \begin{bmatrix} 1 & 0 \\ 2\pi i v_{21} \frac{\sin(\sqrt{v_{12} v_{21}}\pi)}{\sqrt{v_{12} v_{21}}\pi} & 1 \end{bmatrix}, \quad s_{l_{12}} = \begin{bmatrix} 1 & 2\pi i v_{12} \frac{\sin(\sqrt{v_{12} v_{21}}\pi)}{\sqrt{v_{12} v_{21}}\pi} \\ 0 & 1 \end{bmatrix}$$

Inverting the power series

Theorem

The inverse power series is given in the variables s_α by

$$V_\alpha = \sum_{m\geq 1} \sum_{\substack{U(\alpha_i)\neq 0 \\ \alpha_1+\cdots+\alpha_m=\alpha}} F_m(U(\alpha_1),\ldots,U(\alpha_m)) s_{\alpha_1} s_{\alpha_2} \ldots s_{\alpha_m},$$

Special function

Definition

The function $L_n : (\mathbb{C}^*)^n \to \mathbb{C}$ is given by $L_1 = 2\pi i$ and, for $n \geq 2$,

$$L_n(z_1, \dots, z_n) = \sum_{k=1}^{n} \sum_{\substack{0=i_0<\cdots<i_k=n \\ s_{i_j}-s_{i_{j-1}} \in \mathbb{R}_{>0} . s_n}} \frac{(-1)^{k-1}}{k} (2\pi i)^k \prod_{j=0}^{k-1} M_{i_{j+1}-i_j}(z_{i_j+1}, \dots, z_{i_{j+1}}),$$

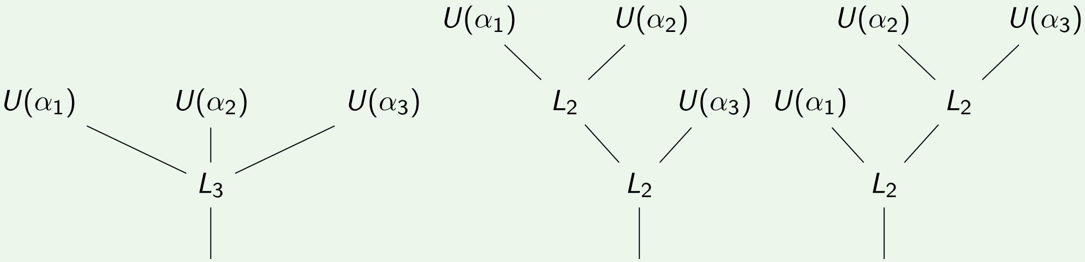
where $s_j = z_1 + \cdots + z_j$.

The F functions

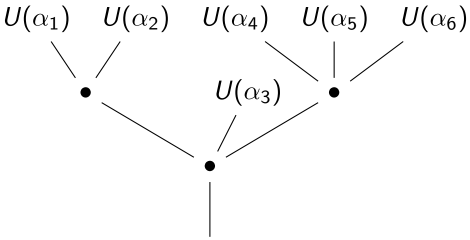
For each m, the function F_m will be a sum over trees with m leaves.

Example

For $m = 3$:

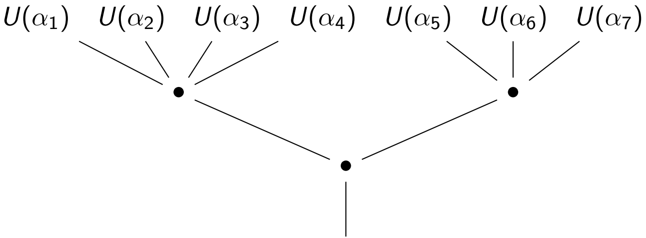


Example



$$\propto -L_3(U(\alpha_1)+U(\alpha_2), U(\alpha_3), U(\alpha_4)+U(\alpha_5)+U(\alpha_6))L_2(U(\alpha_1), U(\alpha_2))L_3(U(\alpha_4), U(\alpha_5), U(\alpha_6))$$

Example



$$\propto -L_2(U(\alpha_1)+U(\alpha_2)+U(\alpha_3)+U(\alpha_4), U(\alpha_5)+U(\alpha_6)+U(\alpha_7))\cdot$$

$$\mathrm{L}_4(U(\alpha_1), U(\alpha_2), U(\alpha_3), U(\alpha_4))L_3(U(\alpha_5), U(\alpha_6), U(\alpha_7))$$

Example: $N = 2$ case

We have two Stokes factors:

$$s_{12} = 2\pi i v_{12} \frac{\sin(\sqrt{v_{12}v_{21}}\pi)}{\sqrt{v_{12}v_{21}}\pi}, \quad s_{21} = 2\pi i v_{21} \frac{\sin(\sqrt{v_{12}v_{21}}\pi)}{\sqrt{v_{12}v_{21}}\pi}.$$

Recall that since M does not depend on u_1 nor u_2, both the L and F functions also do not depend on u_1 and u_2. The equation becomes:

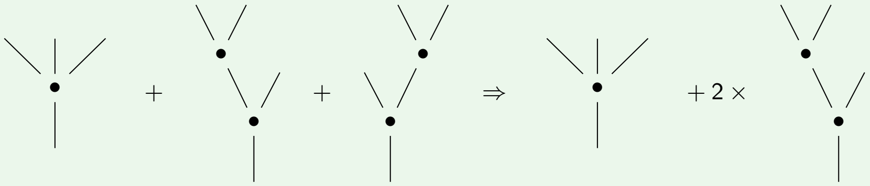
$$V_{12} = \sum_{m \geq 1} F_{2m-1} (s_{21}s_{12})^{m-1} s_{12}.$$

The F functions

Since the functions do not depend on the variables u_1 and u_2, we only need to look at trees up to permutations of their subtrees.

Example

For $m = 3$:

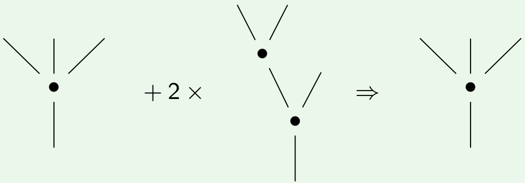


The F functions

$L_m = 2\pi i M_m$. Hence, in particular, we only need to look at trees with even valency greater than 2.

Example

For $m = 3$:



The F functions

$$(2\pi i)^3 F_3 = -M_3 = \frac{\pi^2}{6};$$

$$(2\pi i)^5 F_5 = -M_5 + 3(M_3)^2 = -\frac{\pi^4}{120} + 3\frac{\pi^4}{36} = \frac{3\pi^4}{40}.$$

$$(2\pi i)^7 F_7 = -M_7 + 5M_5M_3 - 3(M_3)^3 - 9(M_3)^3 + 3M_3M_5 = \frac{5\pi^6}{112}.$$

Theorem

There is an identity,

$$F_{2m-1} = \frac{\pi^{2m-2}}{(2\pi i)^{2m-1}} \frac{(2m-2)!}{2^{2m-2}((m-1)!)^2(2m-1)}.$$

Proof

$$F_{2m-1} = \sum_{\substack{k_{2i-1}\in\mathbb{N} \\ k_1+3k_3+\cdots+(2m-3)k_{2m-3}=2m-1}} \frac{k!}{k_1!\ldots k_{2m-3}!}(-1)^k M_k \prod_{i=1}^{m-1} F_{2i-1}^{k_{2i-1}},$$

Plugging in the values of M_k and the induction hypothesis. And after some algebra...

$$\frac{i}{2}\sum\prod_{j=1}^{m}\left[\frac{2i(2j-2)!}{(j-1)!^2(2j-1)}\right]^{k_{2j-1}}\frac{1}{k_{2j-1}!} = 0$$

But these are the odd coefficients of the Taylor series of the exponential function:

$$\to \quad e^{i\arcsin 2t} \quad !!$$

Inverted power series

$$v_{12} = \frac{v_{12}}{\sqrt{v_{12}v_{21}}\pi} \sum_{m\geq 1} \frac{(2m-2)!}{2^{2m-2}((m-1)!)^2(2m-1)} \sin^{2m-1}(\sqrt{v_{12}v_{21}}\pi).$$

Now, we have,

$$\arcsin(x) = \sum_{m\geq 1} \frac{(2m-2)!}{2^{2m-2}((m-1)!)^2(2m-1)} x^{2m-1},$$

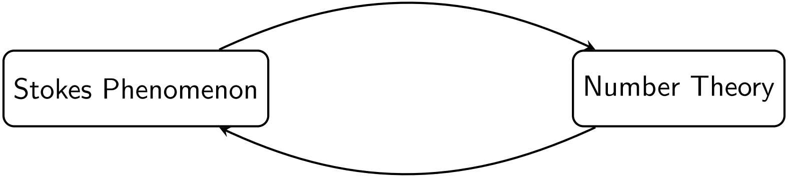
so that,

$$v_{12} = \frac{v_{12}}{\sqrt{v_{12}v_{21}}\pi} \arcsin(\sin(\sqrt{v_{12}v_{21}}\pi)).$$

We conclude as desired

$$V = \begin{bmatrix} 0 & v_{12} \\ v_{21} & 0 \end{bmatrix}.$$

Conclusion



HIGHER DIMENSIONS!!

THANK YOU!