

Chow polynomials

of uniform matroids¹

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Matroids

•) A MATROID $\mathcal{M}$ is a pair $(E, \mathcal{B})$ such that

ground set, finite

(1) $\mathcal{B} \neq \emptyset$

set of bases

$$\mathcal{B} \subseteq \mathcal{P}(E)$$

(2) $\mathcal{B}_1, \mathcal{B}_2 \in \mathcal{B}$ and $x \in \mathcal{B}_1 \setminus \mathcal{B}_2$.

basis exchange property

Then, there exist a $y \in \mathcal{B}_2 \setminus \mathcal{B}_1$ s.t. $(\mathcal{B}_1 \setminus \{x\}) \cup \{y\} \in \mathcal{B}$.

•) $I \subseteq E$ is **INDEPENDENT** if $I \subseteq \mathcal{B}$ for some basis $\mathcal{B} \in \mathcal{B}$.

•) All bases have the same size, called the **RANK** $\text{rk}(\mathcal{M})$ of $\mathcal{M}$.

•) The rank of a subset $A \subseteq E$ is

$$\text{rk}_{\mathcal{M}}(A) := \max(\{ \#I \mid I \subseteq A \text{ independent} \})$$

•) $A \subseteq E$ is a **FLAT** if $\text{rk}_{\mathcal{M}}(A \cup \{x\}) = \text{rk}_{\mathcal{M}}(A)$ for all $x \in E \setminus A$

•) $\mathcal{L}(\mathcal{M}) = (\{\text{flats}\}, \subseteq)$ the **LATTICE OF FLATS** is a geometric lattice of rank $\text{rk}(\mathcal{M})$.

Examples

0) $E = \{v_1, \dots, v_m\} \subseteq \mathbb{R}^n$

$\mathcal{B} = \{ \{v_{i_1}, \dots, v_{i_k}\} \mid \text{basis of } \langle v_{i_1}, \dots, v_{i_k} \rangle \}$

flats: $\{ \langle A \rangle \mid A \subseteq E \}$

(central)

1) Hyperspace Arrangements

lattice of intersections $\longleftrightarrow$ lattice of flats of hyperplanes

Matroids

Examples

2) Uniform matroid $U_{k,n}$ ^{rank} $\nwarrow$ size of groundset

$$E = [n] = \{1, \dots, n\}$$

$$\mathcal{B} = \binom{[n]}{k} = \{B \in [n] \mid |B| = k\}$$

$$\text{rk}_{U_{k,n}}(A) = \min\{\#A, k\}$$

$$\mathcal{L}(U_{k,n}) = (\{A \subseteq E \mid \#A \leq k\} \cup \{E\}, \subseteq)$$

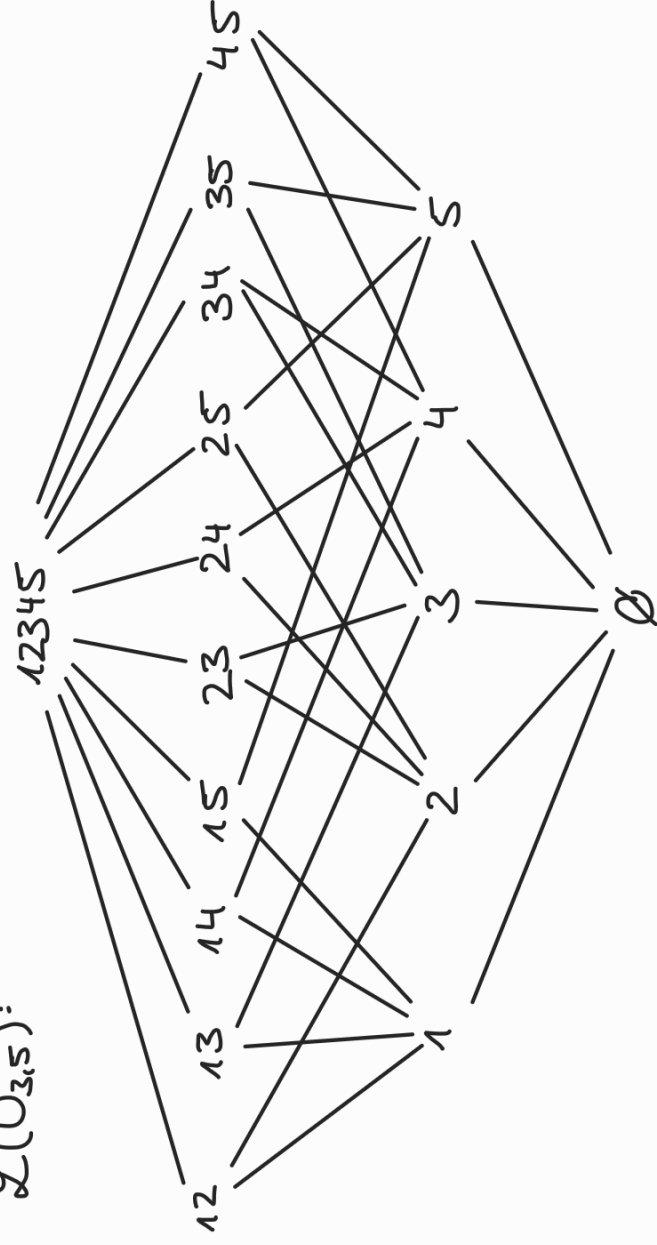
Example

$U_{3,5}$, lattice of flats $\mathcal{L}(U_{3,5})$:

rank 3 $\nwarrow$ on groundset $\{1, \dots, 5\}$

The set of bases is

$$\{B \subset \{1, \dots, 5\} \mid \#B = 3\}$$



Chow ring

(Fukutera - Yuzvinsky, 2004)

The Chow Ring of a matroid $\mathcal{U}$ is

$$A = A(\mathcal{U}) = S / (I+J)$$

- where
- $S = \mathbb{Z}[x_F]_{F \in \mathcal{U} \setminus \{\emptyset\}}$
 - $I = (x_F x_{F'} \mid F, F' \in \mathcal{U} \text{ are incomparable})$
 - $J = (\sum_{\substack{0 \neq F \in \mathcal{U} \\ i \in F}} x_F \mid i \in E)$

[Fukutera - Yuzvinsky, 2004]

$$A = A^0 \oplus A^1 \oplus \dots \oplus A^\Gamma$$

with $\dim_{\mathbb{Q}}(A^k) = a_k$

has Hilbert series

$$H_{\mathcal{U}}(x) = \sum_{k=0}^{\Gamma} a_k \cdot x^k$$

$\Gamma+1 = r^k(\mathcal{U})$

graded $\mathbb{Z}$ -algebra

CHOW POLYNOMIAL

The Chow ring is defined for

- any finite lattice $\mathcal{L}$
- any building set on $\mathcal{L}$

The AUGMENTED CHOW RING of $\mathcal{U}$

$$\tilde{A} = \tilde{A}(\mathcal{U}) = \tilde{S} / (\tilde{I}_1 + \tilde{I}_2 + \tilde{J})$$

satisfies similar properties and has Hilbert series $\text{augH}_{\mathcal{U}}(x)$

AUGMENTED
CHOW POLYNOMIAL

Chow polynomial : Combinatorial properties

The CHOW POLYNOMIAL of a matroid $\mathcal{M}$ of rank $\text{rk}(\mathcal{M}) = r+1$ is the Hilbert series of its Chow ring $A = \bigoplus_{k \geq 0} A^k$,

$$H_{\mathcal{M}}(x) = \sum_{k \geq 0} \dim_{\mathbb{Q}}(A^k) \cdot x^k = \sum_{k=0}^r a_k \cdot x^k = \sum_{i=0}^{\lfloor \frac{r+1}{2} \rfloor} f_i x^i (1+x)^{r-2i}$$

? What do we know about $a_0, \dots, a_r$? What are $a_0, \dots, a_r$ and $f_0, \dots, f_{\lfloor \frac{r+1}{2} \rfloor}$?

Chow polynomial : Combinatorial properties

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? What do we know about $a_0, \dots, a_r$? What are $a_0, \dots, a_r$ and $\gamma_0, \dots, \gamma_{\lfloor \frac{r+1}{2} \rfloor}$?

Combinatorial properties of $H_{\mathcal{M}}(x)$:

[Adiprasito - 2018] [Huh-Katz, 2018] $\left\{ \begin{array}{l} \checkmark \text{ unimodal} \\ \checkmark \text{ palindromic} \end{array} \right.$ $a_0 \leq a_1 \leq \dots \leq a_{\lfloor \frac{r}{2} \rfloor} = a_{\lceil \frac{r}{2} \rceil} \geq \dots \geq a_r$
 $a_k = a_{r-k}$ for $k \leq \frac{r}{2}$

? $\log$ -concave $a_k^2 \geq a_{k-1} \cdot a_{k+1}$ for $1 \leq k \leq r$

[Feroni-Mattheus - 2024] [Sturmfels - Vecchi, 2024] $\checkmark$ γ -positive $H_{\mathcal{M}}(x) = \sum_{i=0}^{\lfloor \frac{r+1}{2} \rfloor} \gamma_i x^i (1+x)^{r-2i}$, $\gamma_i > 0$

? real-rooted

Chow polynomial : Combinatorial properties

The CHOW POLYNOMIAL of a matroid $\mathcal{M}$ of rank $\text{rk}(\mathcal{M}) = r+1$ is the Hilbert series of its Chow ring $A = \bigoplus_{k \geq 0} A^k$,

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Combinatorial properties of $H_{\mathcal{M}}(x)$:

[Adiprasito - , 2018] ✓ unimodal $a_0 \leq a_1 \leq \dots \leq a_{\lfloor \frac{r}{2} \rfloor} = a_{\lceil \frac{r}{2} \rceil} \geq \dots \geq a_r$
✓ palindromic $a_k = a_{r-k}$ for $k \leq \frac{r}{2}$

? ✓ log-concave $a_k^2 \geq a_{k-1} \cdot a_{k+1}$ for $1 \leq k \leq r$

[Feroni - Mattheus - , 2024] ✓ γ -positive $H_{\mathcal{M}}(x) = \sum_{i=0}^{\lfloor \frac{r+1}{2} \rfloor} \gamma_i x^i (1+x)^{r-2i}$, $\gamma_i > 0$
? real-rooted

Chain polynomial : Combinatorial formulas

$$\mathcal{L} = \mathcal{L}(\mathcal{M})$$

Thm: [Feroni-Mathieu-Sturms-Vecchi, 2024]

$$H_{\mathcal{M}}(x) = \sum_{\substack{e = \{\tilde{0} = c_1 < \dots < c_{k-1} = \hat{1}\} \\ \text{chain in } \mathcal{L}}} \overline{\chi}_{\mathcal{L}, e}(x)$$

$$\text{avg } H_{\mathcal{M}}(x) = \sum_{\substack{e = \{c_1 < \dots < c_{k-1} = \hat{1}\} \\ \text{chain in } \mathcal{L}}} x^{\text{rt}(c_1)} \cdot \overline{\chi}_{\mathcal{L}, e}(x)$$

$$\chi_{\mathcal{L}}(t) = \sum_{F \in \mathcal{L}} \mu(\tilde{0}, F) \cdot t^{\text{rank}(\mathcal{L}) - \text{rank}(F)}$$

$$\overline{\chi}_{\mathcal{L}}(t) = \frac{\chi_{\mathcal{L}}(t)}{(t-1)}$$

$$\overline{\chi}_{\mathcal{L}, e}(t) = \frac{t}{\prod_{i=1}^k} \overline{\chi}_{[c_i, c_{i+1}]}(t)$$

Thm: [Stump, 2024+]

Let λ be an R -labeling on $\mathcal{L}(\mathcal{M})$. Then,

$$H_{\mathcal{M}}(x) = \sum_e x^{\text{des}(\lambda_e)} \cdot (x+1)^{\text{rt}(\mathcal{M}) - 1 - 2 \cdot \text{des}(\lambda_e)}$$

some maximal chains in $\mathcal{L}(\mathcal{M})$

λ R -labeling

edge-labeling, s.t. every interval has unique max. chain w/ increasing labels

One proof of the χ -positivity

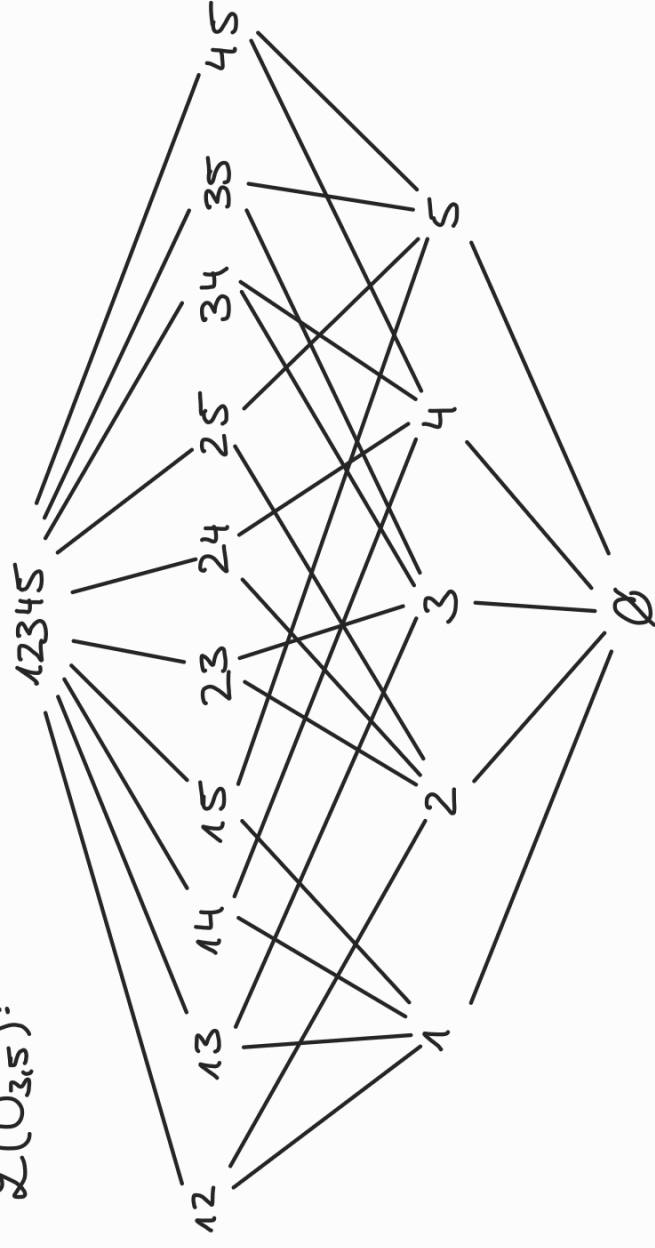
Let's start "simple" - with uniform matroids

Example $U_{3,5}$, lattice of flats $\mathcal{L}(U_{3,5})$:

rank 3 $\nwarrow$ on groundset $\{1, \dots, 5\}$

The set of bases is

$$\{B \subset \{1, \dots, 5\} \mid \#B = 3\}$$



Thm: [Feroni-Kathlene-Skurus-Vecchi, 2024]

Uniform matroids maximize Chow polynomials coefficient-wise:

$$H_{U_{k,n}}(x) - H_{\mathcal{M}}(x) \in \mathbb{N}[x]$$

$$\text{aug } H_{U_{k,n}}(x) - \text{aug } H_{\mathcal{M}}(x) \in \mathbb{N}[x]$$

for $\mathcal{M}$ matroid of rank k on a ground set of size n .

Chow polynomials of uniform matroids

Thm [H., 2024+]

$$H_{U_{k,n}}(x) = \sum_{m=0}^{k-1} \# \left\{ \begin{array}{l} \text{loopless Schubert matroids} \\ \text{on ground set } [n] \text{ of rank } m+1 \\ \text{and } \text{cogirth} > n-k \end{array} \right\} \cdot x^m$$

cogirth of μ
size of the smallest
circuit of μ^*

$$I = \{i_1 < i_2 < \dots < i_k\} \subseteq [n]$$

$$\pi \in \mathcal{G}_n$$

SCHUBERT MATROID $S_{I,\pi}$ on the ground set $[n]$
has bases $\{ \pi^{-1}(J) \mid J = \{j_1 < j_2 < \dots < j_k\} \subseteq [n] \mid i_1 \leq j_1, i_2 \leq j_2, \dots, i_k \leq j_k \}$

$$= \sum_{\substack{I \subseteq \{1, \dots, k\} \\ 1 \in I}} \binom{n}{\Delta I} \cdot x^{|I|-1}$$

multinomial coefficient

$$= \sum_{\substack{D \subseteq \{2, \dots, k-1\} \\ i \in D \Rightarrow i+1 \notin D}} E(n, D) \cdot x^{\#D} \cdot (1+x)^{k-1-2 \cdot \#D}$$

Eulerian number
 $E(n, D) = \# \{ w \in \mathcal{G}_n \mid \text{Des}(w) = D \}$

$$= \sum_{\substack{\sigma \in \mathcal{G}_k \\ \sigma(1) < \sigma(2) \\ \sigma(i) > \sigma(i+1) \Rightarrow \sigma(i+1) < \sigma(i+2)}}$$

$$\binom{n - \sigma(k)}{k - \sigma(k)} \cdot x^{\text{des}(\sigma)} \cdot (1+x)^{k-1-2 \cdot \text{des}(\sigma)}$$

Chow polynomials of uniform matroids

Thm [H., 2024+]

$$H_{U_{k,n}}(x) = \sum_{m=0}^{k-1} \# \left\{ \begin{array}{l} \text{loopless submatroids} \\ \text{on ground set } [n] \text{ of rank } m+1 \\ \text{and corank} > n-k \end{array} \right\} \cdot x^m \quad (1)$$

$$= \sum_{\substack{I \subseteq \{1, \dots, k\} \\ \lambda \in I}} \binom{n}{\Delta I} \cdot x^{|I|-1} \quad (2)$$

$$= \sum_{\substack{D \subseteq \{2, \dots, k-1\} \\ i \in D \Rightarrow i+1 \notin D}} E(n, D) \cdot x^{\#D} \cdot (1+x)^{k-1-2 \cdot \#D} \quad (3)$$

$$= \sum_{\substack{\sigma \in \mathcal{C}_k \\ \sigma(1) < \sigma(2) \\ \sigma(i) > \sigma(i+1) \Rightarrow \sigma(i+1) < \sigma(i+2)}} \binom{n - \sigma(k)}{k - \sigma(k)} \cdot x^{\text{des}(\sigma)} \cdot (1+x)^{k-1-2 \cdot \text{des}(\sigma)} \quad (4)$$

Overview: (1) was a conjecture by Luis Ferraci

(1) = (2), thanks

$$\left\{ \begin{array}{l} I \longleftrightarrow \text{set of submatroids} \\ \lambda \in I \longleftrightarrow \text{loopless} \\ I \subseteq [k] \longleftrightarrow \text{corank} > n-k \end{array} \right\}$$

(3), (4) follow from f -expansion by Stump

(2) = (3) by comparing coefficients

alternative proof known, ongoing work w/ Lorenzo Vecchi

Augmented Chow polynomials of uniform matroids

Thm [H., 2024+]

$$\text{aug } H_{U_{k,n}}(x) = \sum_{m=0}^{\cancel{k-1}} \# \left\{ \begin{array}{l} \text{loopless} \\ \text{on ground set } [n] \text{ of rank } m+1 \\ \text{and corank} > n-k \end{array} \right\} \cdot x^m \quad (1)$$

$$= \sum_{\substack{I \subseteq \{1, \dots, k\} \\ \cancel{1 \notin I}}} \binom{n}{\Delta I} \cdot x^{|\Delta I| - \cancel{1}} \quad (2)$$

$$= \sum_{\substack{D \subseteq \{2, \dots, k-1\} \\ i \in D \Rightarrow i+1 \notin D}} E(n, D) \cdot x^{\#D} \cdot (1+x)^{\cancel{k-1-2 \cdot \#D}} \quad (3)$$

$$= \sum_{\substack{\sigma \in \tilde{S}_k \\ \sigma(1) < \sigma(2) \\ \sigma(i) > \sigma(i+1) \Rightarrow \sigma(i+1) < \sigma(i+2)}} \binom{n - \sigma(k)}{k - \sigma(k)} \cdot x^{\text{des}(\sigma)} \cdot (1+x)^{\cancel{k-1-2 \cdot \text{des}(\sigma)}} \quad (4)$$

Overview: (1) was a conjecture by Luis Ferraci

(3), (4) follow from f -expansion by Stump

(1) = (2), thereby

(2) = (3) by comparing coefficients

$$\left\{ \begin{array}{l} I \longleftrightarrow \text{set of loopless matroids} \\ 1 \in I \longleftrightarrow \text{loopless} \\ I \subseteq [k] \longleftrightarrow \text{corank} > n-k \end{array} \right\}$$

Chow polynomials of uniform matroids

Tlum [H., 2024+]

$$H_{U_{k,n}}(x) = \sum_{m=0}^{k-1} \# \left\{ \begin{array}{l} \text{loopless} \\ \text{on ground set } [n] \\ \text{and } \text{cogirth} > n-k \end{array} \right. \text{Subset matroids of rank } m+1 \Big\} \cdot x^m \quad (1)$$

$$\begin{aligned} H_{U_{3,5}}(x) &= 1 + 11x + x^2 \\ &= \sum_{\substack{I \subseteq \{1, \dots, k\} \\ 1 \in I}} \binom{n}{\Delta I} \cdot x^{|I|-1} \end{aligned} \quad (2)$$

$$\begin{aligned} &= 1 + x + 10x + x^2 \\ &= \sum_{\substack{D \subseteq \{2, \dots, k-1\} \\ i \in D \Rightarrow i+1 \notin D}} E(n, D) \cdot x^{\#D} \cdot (1+x)^{k-1-2 \cdot \#D} \end{aligned} \quad (3)$$

$$\begin{aligned} &= (1+x)^2 + 9x \\ &= \sum_{\substack{\sigma \in \mathcal{C}_k \\ \sigma(1) < \sigma(2) \\ \sigma(i) > \sigma(i+1) \Rightarrow \sigma(i+1) < \sigma(i+2)}} \binom{n-\sigma(k)}{k-\sigma(k)} \cdot x^{\text{des}(\sigma)} \cdot (1+x)^{k-1-2 \cdot \text{des}(\sigma)} \end{aligned} \quad (4)$$

$$= (1+x)^2 + 3x + 6x$$

Chow polynomial of $U_{3,5}$

$$(2) \quad H_{U_{k,n}}(x) = \sum_{\substack{I \subseteq \{1, \dots, k\} \\ \lambda \in I}} \binom{n}{\Delta I} \cdot x^{|I|-1}$$

Let $I = I_1 \cup I_2 \cup \dots \cup I_s$ be the set partition

•) into maximal consecutive subsets

•) s.t. $\min I_i < \min I_{i+1}$

$$\binom{n}{\Delta I} := \frac{n!}{(\min I_1 - 1)! (\min I_2 - \min I_1)! \dots (\min I_s - \min I_{s-1})! (n+1 - \min I_s)!}$$

$$H_{U_{3,5}}(x) = \sum_{\substack{I \subseteq \{1,2,3\} \\ \lambda \in I}} \binom{5}{\Delta I} \cdot x^{|I|-1}$$

$$= x^0 + (1+10)x^1 + x^2$$

$$= 1 + 11x + x^2$$

$\begin{matrix} I \subseteq \{1,2,3\} \\ \lambda \in I \end{matrix}$	$\binom{5}{\Delta I}$
$\{1, \underline{1}\}$	$\frac{5!}{0! 5!} = 1$
$\{1, \underline{2}\}$	$\frac{5!}{0! 5!} = 1$
$\{1, \underline{3}\}$	$\frac{5!}{0! 2! 3!} = 10$
$\{1, \underline{2}, \underline{3}\}$	$\frac{5!}{0! 5!} = 1$

Chow polynomial of $U_{3,5}$

$$(3) \quad H_{U_{k,n}}(x) = \sum_{\substack{D \subseteq \{2, \dots, k-1\} \\ i \in D \Rightarrow i+1 \notin D}} E(n, D) \cdot x^{\#D} \cdot (1+x)^{k-1-2 \cdot \#D}$$

Eulerian number

$$E(n, D) = \#\{\omega \in \mathcal{G}_n \mid \text{Des}(\omega) = D\}$$

$$H_{U_{3,5}}(x) = \sum_{\substack{D \subseteq \{2\} \\ i \in D \Rightarrow i+1 \notin D}} E(5, D) \cdot x^{\#D} \cdot (1+x)^{2-2 \cdot \#D}$$

$$= x^0 \cdot (1+x)^2 + 9 \cdot x^1 \cdot (1+x)^0$$

$$= 1 + 2x + x^2 + 9x$$

$D \subseteq \{2\}$ $i \in D \Rightarrow i+1 \notin D$	$E(5, D)$
$\emptyset$	1
$\{2\}$	$\binom{5}{2} - 1 = 9$

$$(3) \quad H_{U_{k,n}}(x) = \sum_{\substack{\sigma \in \mathcal{G}_k \\ \sigma(1) < \sigma(2)}} \binom{n - \sigma(k)}{k - \sigma(k)} \cdot x^{\text{des}(\sigma)} \cdot (1+x)^{k-1-2 \cdot \text{des}(\sigma)}$$

$\sigma(i) > \sigma(i+1) \Rightarrow \sigma(i+1) < \sigma(i+2)$

σ	$\binom{5 - \sigma(3)}{3 - \sigma(3)}$	$\text{des}(\sigma)$	$\binom{5 - \sigma(3)}{3 - \sigma(3)} \cdot x^{\text{des}(\sigma)} \cdot (1+x)^{2-2 \cdot \text{des}(\sigma)}$
$[1, 2, 3]$	1	0	$(1+x)^2$
$[1, 3, 2]$	$\binom{3}{1} = 3$	1	$3 \cdot x$
$[2, 3, 1]$	$\binom{4}{2} = 6$	1	$6 \cdot x$

$$H_{U_{3,5}}(x) = (1+x)^2 + 3x + 6x$$

Proof of (3) : $H_{U_{k,n}}(x) = \sum_{\substack{D \subseteq \{2, \dots, k-1\} \\ i \in D \Rightarrow i+1 \notin D}} E(n, D) \cdot x^{\#D} \cdot (1+x)^{k-1-2 \cdot \#D}$

λ R-labeling on $\mathcal{L}(U_k)$

For $\mathcal{C} = \{C_0 < C_1 < \dots < C_k\}$ maximal chain in $\mathcal{L}(U_k)$

$\lambda_{\mathcal{C}} = (\lambda_1, \dots, \lambda_k)$ where $\lambda_i = \lambda(C_{i-1} < C_i)$

[Stump 2024+]

$H_{U_k}(x) = \sum_{\mathcal{C}} x^{\text{des}(\lambda_{\mathcal{C}})} (1+x)^{r_{\mathcal{C}}(U_k)-1-2 \cdot \text{des}(\lambda_{\mathcal{C}})}$

Sum ranges over

maximal chains $\mathcal{C}$ in $\mathcal{L}(U_k)$

s.t. for $\lambda_{\mathcal{C}} = (\lambda_1, \lambda_2, \dots, \lambda_k)$

•) $\lambda_1 < \lambda_2$

•) $\lambda_{i-1} > \lambda_i \Rightarrow \lambda_i < \lambda_{i+1}$

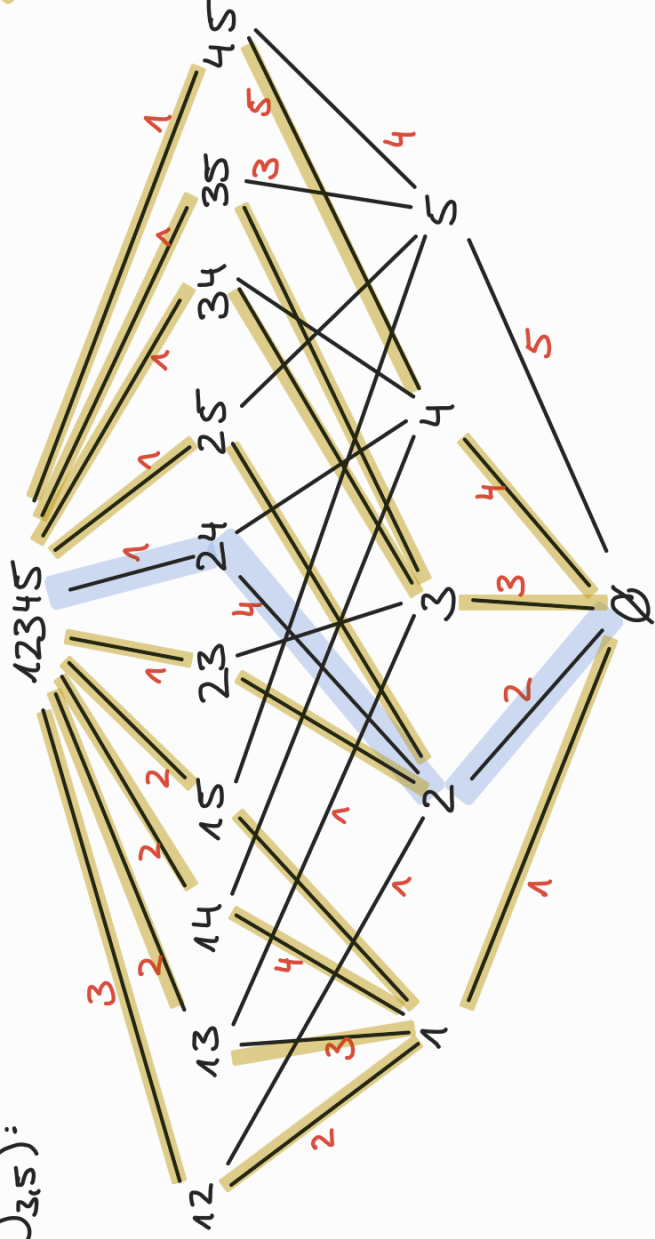
$\text{des}(\lambda_{\mathcal{C}}) = \#\{i \in [k-1] \mid \lambda_i > \lambda_{i+1}\}$

What to do?

- 1) Define R-labeling on $\mathcal{L}(U_{k,n})$
- 2) Map the sequences $\lambda_{\mathcal{C}} \mapsto \sigma$ to permutations

Chow polynomial of $U_{3,5}$

$\mathcal{L}(U_{3,5})$:



R-labeling $\lambda: \mathcal{E}(\mathcal{L}(U_{3,5})) \rightarrow \mathbb{Z}_{>0}$
 $(S \subset T) \mapsto \min(T \setminus S)$

$\mathcal{C} = \{ \emptyset \subset 2 \subset 24 \subset 1234 \}$

$\lambda_{\mathcal{C}} = (2, 4, 1, 1)$

$\downarrow$
 $\sigma = [2, 4, 1, 3, 5] \in \mathcal{C}_5$

$\mathcal{L}(\mathcal{M})$ geometric lattice, i.e.

$\text{FE}\mathcal{L}(\mathcal{M})$ is a join of atoms

Let $\alpha_1 < \alpha_2 < \dots < \alpha_s$ total ordering of the atoms in $\mathcal{L}(\mathcal{M})$.

$\lambda(s < t) = \min \{ i \mid s \vee \alpha_i = t \}$

defines R-labeling on $\mathcal{L}(\mathcal{M})$.

$$H_{U_{3,5}}(x) = \sum_{\text{chains}} x^{\text{des}(\cdot)} \cdot (1+x)^{2-2 \cdot \text{des}(\cdot)}$$

$$= \sum_{\substack{\sigma \in \mathcal{C}_5 \\ \sigma(1) < \sigma(2) \\ \sigma(3) < \sigma(4) < \sigma(5)}} x^{\text{des}(\sigma)} (1+x)^{2-2 \cdot \text{des}(\sigma)}$$

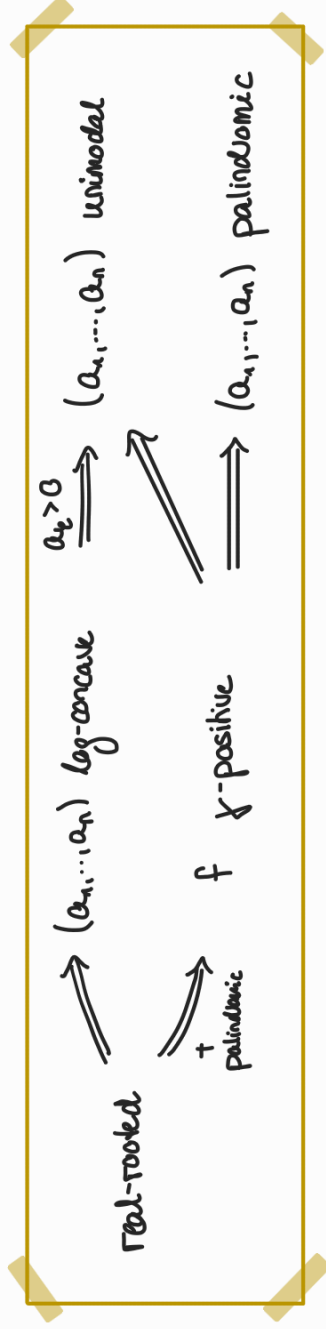
$$= \sum_{\mathcal{D} \subseteq \{2\}} E(S, \mathcal{D}) \cdot x^{|\mathcal{D}|} \cdot (1+x)^{2-2 \cdot |\mathcal{D}|}$$

$$= x^0 \cdot (1+x)^2 + \left[\binom{5}{2} - 1 \right] \cdot x^1 \cdot (1+x)^0$$

Future directions

The Chow polynomial ...

- ... and its coefficients
-) log-concavity
-) real-rootedness



... and its friends

-) $H_P(x)$ defined for P finite, graded poset
-) $H_P(x)$ as evaluation of $\text{ex } \Psi_P(\gamma; a, b) \in \mathbb{N}[\gamma] \langle a, b \rangle$

$$H_P(x) = \sum_{\substack{c = \{\beta = c_1 < \dots < c_{\text{ht}(P)} = \lambda\} \\ \text{chain in } P}} \bar{\chi}_{P,c}(x)$$

$$H_P(x) = \sum_{\substack{c \text{ max.} \\ \text{s.t. } \dots}} x^{\text{des}(\lambda_c)} \cdot (x+1)^{\text{rt}(\lambda_c) - 1 - 2 \cdot \text{des}(\lambda_c)}$$

$$\text{aug } H_P(x) = H_{\text{aug}(P)}(x)$$

... for large classes of matroids / posets

-) find a nice restriction of $E(n, D)$

Future directions (1/4)

•) More general settings:

Ferroni-Kathlene-Vecchi recently defined the k -Chow function $H_{st}(x)$ for any interval $[s, t]$ of a poset P weakly ranked, locally finite

$\rightarrow k = \chi$, then $H_{\mu}(x) = H_{\hat{0}, \hat{\lambda}}(x)$ for $P = \mathcal{L}(\mu)$

$H_{st}(x)$ unimodal

$H_{st}(x)$ χ -positive for P Colver-Macaulay

$\rightarrow \chi$ -positivity statements and real-rootedness conjectures

Combinatorial formulas:

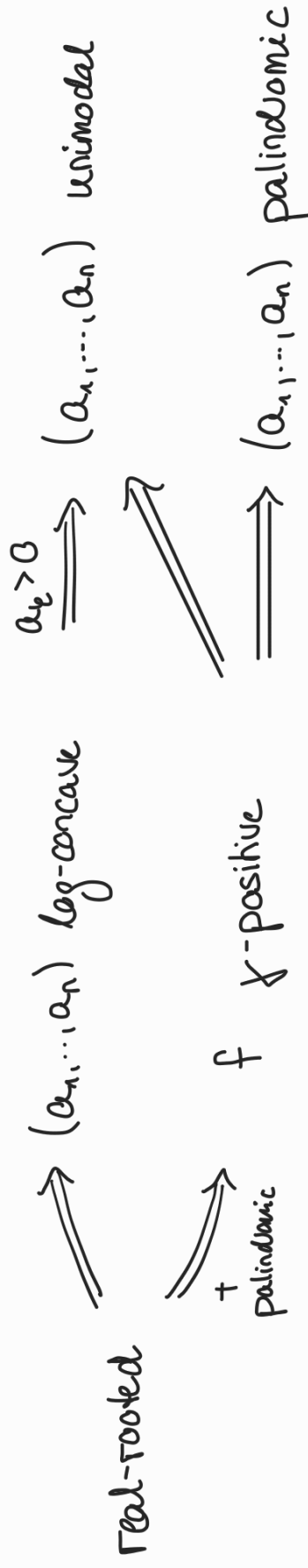
$$H_P(x) = \sum_{\substack{e = \{\hat{0} = c_1 < \dots < c_{kr_A} = \hat{\lambda}\} \\ \text{chain in } P}} \bar{\chi}_{P,e}(x)$$

$$H_P(x) = \sum_{\substack{e \text{ max.} \\ \text{s.t. } \dots}} x^{\text{des}(\lambda_e)} \cdot (x+1)^{r^e(\mu) - 1 - 2 \cdot \text{des}(\lambda_e)}$$

$$\text{aug } H_P(x) = H_{\text{aug}(P)}(x)$$

Future directions (2/4)

•) Real-rootedness : Let $f(x) = \sum_{i=0}^n a_i x^i \in \mathbb{R}_{\geq 0}[x]$



Conjecture : $[$ Ferroni-Mattheo-Skurus-Vecchi, 2024 $]$ $H_{\mathcal{U}}(x), \text{aug} H_{\mathcal{U}}(x)$ real-rooted for every matroid $\mathcal{U}$

$\rightarrow$ Prove for $\mathcal{U} = U_{e,n}$ [Brändén-Vecchi, 2025]

$\rightarrow$ What makes it difficult? $\leadsto$ recursion formulas for Chow polynomials depend on both Chow polynomial & characteristic polynomial

$\rightarrow$ What makes it true?

Matroid theory?

Lattice theory?

Poset theory?

Future directions (3/4)

•) Poincaré-extended ab-index

$$\text{ex}\Psi_P(\gamma; a, b) = \sum_{e \text{ chain in } P} \text{Poinc}_e(\gamma) \cdot \text{wte}(a, b) \in \mathbb{N}[\gamma] \langle a, b \rangle$$

delete first letter in ab-words

Poincaré chain polynomial

$$= w_0 w_1 \dots w_{\text{rk}(P)-1}$$

$$\text{where } w_i = \begin{cases} b & , i \in \text{rk}(e) \\ a-b & , i \notin \text{rk}(e) \end{cases}$$

$$\tilde{\text{ex}}\Psi_P(\gamma; a, b)$$

Some evaluations: [Dipalu-Bary-Maglione-Stump, 2023] [Stump, 2024+]

Conjecture to be real-rooted:

$$\cdot) \text{ex}\tilde{\Psi}_{y(\mu)}(-x, 1, x) = (1-x)^{\text{rk}(\mu)} \cdot H_\mu(x)$$

$$\cdot) \text{ex}\Psi_{y(\mu)}(-x, 1, x) = (1-x)^{\text{rk}(\mu)} \cdot \text{aug}H_\mu(x)$$

coarse flag HP series [Maglione-Voll, 23]

$$\cdot) \text{ex}\tilde{\Psi}_P(\gamma, 1, t) = (1-x)^{\text{rk}(P)} \cdot \text{cfHP}_P(\gamma, t) \text{ for } \gamma=1$$

$$\cdot) \text{ex}\tilde{\Psi}_{y(\mu)}(0, 1, x) = (1-x)^{\text{rk}(\mu)} \cdot P_{\mu(\mu)}\left(\frac{x}{1-x}\right)$$

chain polynomial

[Athanasiadis-Kalampania-Evangelina, 2023]

[Ferroni-Schröter, 2024]

[Skovs, 2021]

[Kühne-Maglione, 2023]

?

While evaluations of $\text{ex}\Psi_P$ and $\text{ex}\tilde{\Psi}$ are real-rooted? [Stump, 2024+]

Future directions (4/4)

- Learning from Chow polynomials of uniform matroids

$$\begin{aligned}
 H_{U_{k,n}}(x) &= \sum_{m=0}^{k-1} \# \left\{ \begin{array}{l} \text{loopless Submatroids} \\ \text{on ground set } [n] \text{ of rank } m+1 \\ \text{and } \text{cogirth} > n-k \end{array} \right\} \cdot x^m \\
 &= \sum_{\substack{D \subseteq \{2, \dots, k-1\} \\ i \in D \Rightarrow i+1 \notin D}} E(n, D) \cdot x^{\#D} \cdot (1+x)^{k-1-2 \cdot \#D}
 \end{aligned}$$

?

$$H_{\mu}(x) = \sum_{m \geq 0} \# \left\{ \begin{array}{l} \text{matroids on } [n] \text{ of rank } m+1 \\ \text{with } \text{Some restriction} \end{array} \right\} \cdot x^m$$

Can we write

$$H_{\mu}(x) = \sum_D \# \left\{ \omega \in \mathcal{G}_n \mid \text{Des}(\omega) = D \text{ and } \text{Some restriction} \right\} \cdot x^{|D|} \cdot (1+x)^{r^E(\mu)-1-2 \cdot |D|}$$

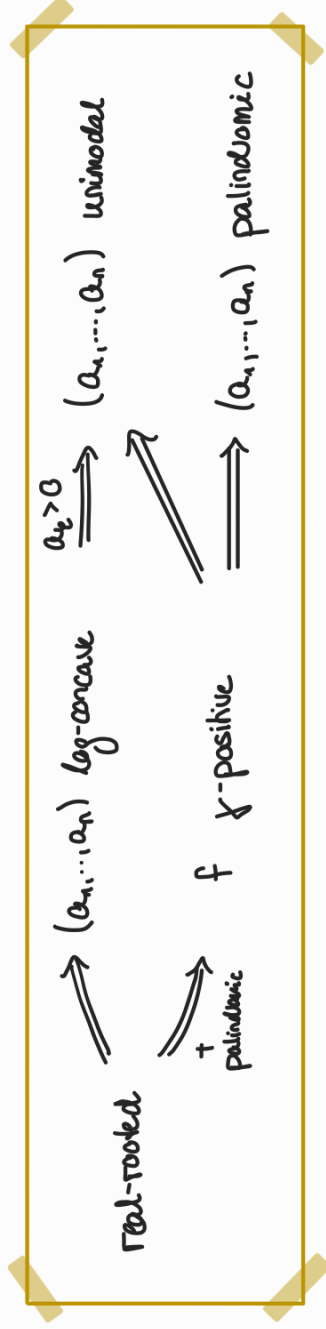
Ex: Chow polynomials for large classes of matroids:

- Submat matroids
- Lattice path matroids

Future directions

The Chow polynomial ...

- ... and its coefficients
-) log-concavity
-) real-rootedness



... and its friends

-) $H_P(x)$ defined for P finite, graded poset
-) $H_P(x)$ as evaluation of $\text{ex } \Psi_P(\gamma; a, b) \in \mathbb{N}[\gamma] \langle a, b \rangle$

$$H_P(x) = \sum_{\substack{c = \{\emptyset = c_1 < \dots < c_{\text{rank}} = \hat{\lambda}\} \\ \text{chain in } P}} \bar{\chi}_{P,c}(x)$$

$$H_P(x) = \sum_{\substack{e \text{ max.} \\ \text{s.t.} \dots}} x^{\text{des}(\lambda_e)} \cdot (x+1)^{\text{rt}(\lambda_e) - 1 - 2 \cdot \text{des}(\lambda_e)}$$

$$\text{aug } H_P(x) = H_{\text{aug}(P)}(x)$$

... for large classes of matroids / posets

-) find a nice restriction of $E(n, D)$

Remark 1.5 (Three existing real-rootedness conjectures). We show in Theorem 2.6 that the Chow polynomial is an evaluation of the extended **ab**-index. This evaluation is conjectured to be real-rooted [FS23, Conjecture 8.18]. It is shown in [DMS23, Corollary 2.23] that the numerator polynomial of the coarse flag Hilbert-Poincaré series is an evaluation of the extended **ab**-index and asked in [KM23, Question 1.6] for which matroids a further evaluation is real-rooted. Also the chain polynomial of the lattice of flats is yet another evaluation of the extended **ab**-index which is conjectured in [AKE23, Conjecture 1.2] to be real-rooted. This suggests the following question.

KE23:

KE23:

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2.4. Evaluating the coarse flag Hilbert-Poincaré series. The coarse flag Hilbert-Poincaré series $\text{cfHP}_P(y, t)$ is originally defined and studied in [MV23] as an algebraic invariant in geometric group theory. It is shown in [DMS23, Corollary 2.22] that this rational function can be expressed as

$$\text{cfHP}_P(y, t) = (1 - t)^{-n} \text{ex} \tilde{\Psi}_P(y, 1, t) .$$

$$h_{\mathcal{L}}(x) = (1 - x)^n p_{\mathcal{L}}\left(\frac{x}{1 - x}\right) ,$$

$$s \quad h_{\mathcal{L}}(x) = \Psi_{\mathcal{L}}(1, x) .$$

$$\uparrow$$

$$ab\text{-index} : \text{ex} \Psi(0, a, b)$$

$$h_{\mathcal{L}}(x) = \text{ex} \Psi_{\mathcal{L}}(0, 1, x) = (1 - x)^n p_{\mathcal{L}}\left(\frac{x}{1 - x}\right)$$

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Theorem 1.4. *There is a unique way to assign to each loopless matroid $\mathbf{M}$ a polynomial $\underline{H}_{\mathbf{M}}(x) \in \mathbb{Z}[x]$ such that the following conditions hold:*

- (i) *If $\text{rk}(\mathbf{M}) = 0$, then $\underline{H}_{\mathbf{M}}(x) = 1$.*
- (ii) *For every matroid $\mathbf{M}$, the following recursion holds:*

$$\underline{H}_{\mathbf{M}}(x) = \sum_{\substack{F \in \mathcal{F}(\mathbf{M}) \\ F \neq \emptyset}} \bar{\chi}_{\mathbf{M}|_F}(x) \underline{H}_{\mathbf{M}/F}(x).$$

