A Littlewood-Type Identity for Robbins Polynomials

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Outline

- Littlewood identities, Robbins polynomials & main results
- Result I: Littlewood-type identity for Robbins polynomials
- Result II: Connection to the partition function Z_{DSASM} of diagonally symmetric alternating sign matrices
- Result III: Coefficient in the polynomial expansion of Z_{DSASM}

The classical Littlewood identities

Theorem (Schur, Littlewood)

- $$\sum_{\lambda} s_\lambda(x_1,\ldots,x_n) = \prod_{i=1}^{n} \frac{1}{1-x_i} \prod_{1\leqslant i<j\leqslant n} \frac{1}{1-x_i x_j}$$
- $$\sum_{\lambda \text{ even}} s_\lambda(x_1,\ldots,x_n) = \prod_{i=1}^{n} \frac{1}{1-x_i^2} \prod_{1\leqslant i<j\leqslant n} \frac{1}{1-x_i x_j}$$
- $$\sum_{\lambda' \text{ even}} s_\lambda(x_1,\ldots,x_n) = \prod_{1\leqslant i<j\leqslant n} \frac{1}{1-x_i x_j}$$

$\longrightarrow$ Proofs by Robinson-Schensted-Knuth correspondence

Semistandard Young tableaux

Schur functions s_λ are generating functions of semistandard Young tableaux (SSYT) of shape $\lambda = (\lambda_1 \geqslant \cdots \geqslant \lambda_n \geqslant 0)$:

$\leqslant$

$\wedge$

1	1	1	4	6
2	3	3	5	
4	5	7		
6	6	8		
7				

$$\lambda = (5, 4, 3, 3, 1)$$

`weight:` $\prod_{i \geqslant 1} x_i^{\#i} = x_1^3 x_2 x_3^2 x_4^2 x_5^2 x_6^3 x_7^2 x_8$

Combinatorial interpretation of the Littlewood identity

$$\sum_{\lambda} s_\lambda(x_1, \ldots, x_n) = \prod_{i=1}^{n} \frac{1}{1 - x_i} \prod_{1 \leqslant i < j \leqslant n} \frac{1}{1 - x_i x_j}$$

Schur functions $s_\lambda(x_1, \ldots, x_n)$: generating function of semistandard Young tableaux of shape λ

generating function of symmetric matrices $A = (a_{i,j})_{1 \leqslant i,j \leqslant n}$ with non-negative integer entries via $\frac{1}{1-x_i} = \sum_{a_{i,i} \geqslant 0} x_i^{a_{i,i}}$ and $\frac{1}{1-x_i x_j} = \sum_{a_{i,j} \geqslant 0} (x_i x_j)^{a_{i,j}}$

Robinson–Schensted–Knuth correspondence

pairs (P, Q) of SSYT of the same shape $\overset{\mathrm{RSK}}{\longleftrightarrow}$ matrices with non-negative integer entries

Symmetry of RSK:

$$(P, Q) \overset{\mathrm{RSK}}{\longleftrightarrow} A \iff (Q, P) \overset{\mathrm{RSK}}{\longleftrightarrow} A^{\top}.$$

RSK on symmetric matrices A implies the Littlewood identity:

$$\sum_{\lambda} s_{\lambda}(x_1, \ldots, x_n) = \prod_{i=1}^{n} \frac{1}{1 - x_i} \prod_{1 \leqslant i < j \leqslant n} \frac{1}{1 - x_i x_j}.$$

Gelfand–Tsetlin patterns

Semistandard Young tableau

1	1	1	4	6
2	3	3	5	
4	5	7		
6	6	8		
7				

$\longleftrightarrow$

Gelfand–Tsetlin pattern

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 3 & & & & & & & \\
 & & & & & & 1 & & 3 & & & & & & \\
 & & & & & 0 & & 3 & & 3 & & & & & \\
 & & & & 0 & & 1 & & 3 & & 4 & & & & \\
 & & & 0 & & 0 & & 2 & & 4 & & 4 & & & \\
 & & 0 & & 0 & & 2 & & 2 & & 4 & & 5 & & \\
 & 0 & & 0 & & 1 & & 2 & & 3 & & 4 & & 5 & \\
0 & & 0 & & 0 & & 1 & & 3 & & 3 & & 4 & & 5
\end{array}
$$

$\leqslant$ $\leqslant$ $\leqslant$

$$x_1^3 x_2 x_3^2 x_4^2 x_5^2 x_6^3 x_7^2 x_8$$

`weight:` $\prod_{i=1}^{n} x_i^{\# i}$ $\qquad$ $\prod_{i=1}^{n} x_i^{\sum \text{entries in row } i - \sum \text{entries in row } (i-1)}$

Definition

A down-arrowed monotone triangle (DAMT) is a Gelfand–Tsetlin pattern with strict increase along rows where each entry, except for those in the bottom row, is decorated with either $\swarrow$, $\downarrow$ or $\searrow$ subject to the following rule:

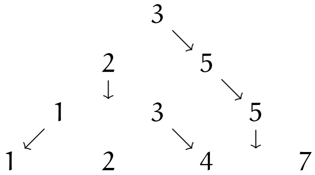
If an entry is equal to one of the entries in the row below, then those entries have to be connected by a slanted arrow ($\swarrow$ or $\searrow$).

$$
\begin{array}{ccccccc}
 & & & 3 & & & \\
 & & & & \searrow & & \\
 & & 2 & & 5 & & \\
 & & \downarrow & & & \searrow & \\
 & 1 & & 3 & & 5 & \\
 \swarrow & & & & \searrow & \downarrow & \\
1 & & 2 & & 4 & & 7
\end{array}
$$

Modified Robbins Polynomials

Definition
The (modified) Robbins polynomial $R^*_k(x_1, \ldots, x_n; u, v, w)$ is the generating function of DAMTs with bottom row k with weight

$$u^{\#\searrow} v^{\#\swarrow} w^{\#\downarrow} \times \prod_{i=1}^{n} x_i^{\sum \text{entries in row } i - \sum \text{entries in row } (i-1) + \#\searrow \text{ in row } (i-1) - \#\swarrow \text{ in row } (i-1)}.$$



$u^3 v w^2 x_1^3 x_2^5 x_3^3 x_4^5$

Main result I

We establish a Littlewood identity for Robbins polynomials:

Theorem (Fischer, H. 2025)
Let n be a positive integer. Then

$$\sum_{0 \leqslant k_1 < \cdots < k_n} R^*_{(k_1,\ldots,k_n)}(x_1,\ldots,x_n;1,1,w)$$
$$= \prod_{i=1}^{n} \frac{1}{1-x_i} \prod_{1 \leqslant i < j \leqslant n} \frac{x_i + x_j + w x_i x_j}{x_j - x_i}$$
$$\times \operatorname*{Pf}_{\chi_{\text{even}}(n) \leqslant i < j \leqslant n} \left(\begin{cases} 1, & i = 0, \\ \frac{(x_j - x_i)(1+(1+w)x_i x_j)}{(x_i + x_j + w x_i x_j)(1 - x_i x_j)}, & i \geqslant 1, \end{cases} \right)$$

where Pf denotes the Pfaffian of an upper triangular array and $\chi_{\text{even}}(n)$ equals 1 if n is even and 0 otherwise.

Pfaffians

- Consider all $(2n-1)!!$ partitions of $\{1, 2, \ldots, 2n\}$ into pairs.
- They can be written as $\{(i_1, j_1), \ldots, (i_n, j_n)\}$ with $i_1 < \cdots < i_n$ and $i_k < j_k$ for all $1 \leqslant k \leqslant n$.
- For a triangular array $A = (a_{i,j})_{1 \leqslant i < j \leqslant 2n}$, the Pfaffian $\mathrm{Pf}(A)$ is defined as

$$\sum_{\{(i_1,j_1),\ldots,(i_n,j_n)\}} \mathrm{sgn}(i_1 j_1 \ldots i_n j_n) a_{i_1,j_1} \cdots a_{i_n,j_n},$$

  where we sum over all pairings in consideration.
- If we complete A to the uniquely determined skew-symmetric matrix M with A being its upper triangular part, then it is well known that

$$\mathrm{Pf}(A)^2 = \det(M).$$

Alternating sign matrices

$$\begin{array}{ccccccc} & & & 3 & & & \\ & & 2 & & 3 & & \\ & 1 & & 2 & & 4 & \\ 1 & & 2 & & 3 & & 4 \end{array} \longleftrightarrow \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \longleftrightarrow \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Definition

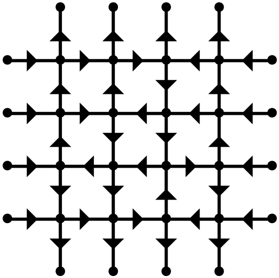
An alternating sign matrix is a square matrix with entries $\{-1, 0, +1\}$ such that

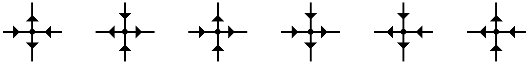
- entries in rows and columns sum to 1 and
- nonzero entries along rows and columns alternate.

Six-vertex model

Alternating sign matrices are in correspondence with six-vertex model configurations with domain wall boundary conditions:

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \longleftrightarrow$$



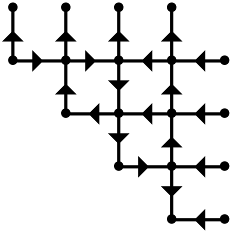


1	-1	0	0	0	0

Diagonally symmetric alternating sign matrices

Diagonally symmetric alternating sign matrices (DSASMs) correspond to six-vertex model configurations on a triangular grid:

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \longleftrightarrow$$



The generating function of all such six-vertex model configurations of size n, denoted by $6V_{\triangledown}(n)$, is called the partition function $Z_{\mathrm{DSASM}}(x_1, \ldots, x_n)$.

Main result II

We relate the Littlewood identity for Robbins polynomial to the partition function of diagonally symmetric alternating sign matrices:

Theorem (Fischer, H. 2025)
Let n be a positive integer. Then

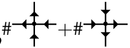
$$\sum_{0 \leqslant k_1 < \cdots < k_n} R^*_{(k_1,\ldots,k_n)}(x_1, \ldots, x_n; 1, 1, w)$$
$$= \prod_{i=1}^{n} \frac{1}{1 - x_i} \prod_{1 \leqslant i < j \leqslant n} \frac{1}{1 - x_i x_j} Z_{\text{DSASM}}(x_1, \ldots, x_n).$$

Main result III

We provide an explicit expression for the coefficient of the highest term in the polynomial expansion of $Z_{\mathrm{DSASM}}(x_1, \ldots, x_n)$:

Theorem (Fischer, H. 2025)
The coefficient of $x_1^{n-1} \cdots x_n^{n-1}$ in $Z_{\mathrm{DSASM}}(x_1, \ldots, x_n)$ is given by the generating function

$$\sum_{6V_{\bigtriangledown}(n)} w^{\# \quad + \#},$$



which equals

$$w^{\binom{n}{2}} \operatorname*{Pf}_{\chi_{\mathrm{odd}}(n) \leqslant i < j \leqslant n-1} \left(\langle u^i v^j \rangle \frac{(v-u)(1+uv+w)}{(1-uv)(w-u-v)} \right),$$

where $\langle u^i v^j \rangle f(u, v)$ denotes the coefficient of $u^i v^j$ in the expansion of $f(u, v)$.

Littlewood identity for Robbins polynomials

Main result I

Theorem (Fischer, H. 2025)
Let n be a positive integer. Then

$$\sum_{0 \leqslant k_1 < \cdots < k_n} R^*_{(k_1,\ldots,k_n)}(x_1,\ldots,x_n;1,1,w)$$
$$= \prod_{i=1}^{n} \frac{1}{1-x_i} \prod_{1 \leqslant i < j \leqslant n} \frac{x_i + x_j + w x_i x_j}{x_j - x_i}$$
$$\times \operatorname*{Pf}_{\chi_{\text{even}}(n) \leqslant i < j \leqslant n} \left(\begin{cases} 1, & i = 0, \\ \frac{(x_j - x_i)(1+(1+w)x_i x_j)}{(x_i + x_j + w x_i x_j)(1 - x_i x_j)}, & i \geqslant 1, \end{cases} \right)$$

where Pf denotes the Pfaffian of an upper triangular array and $\chi_{\text{even}}(n)$ equals 1 if n is even and 0 otherwise.

Antisymmetriser formula for Robbins polynomials

Theorem (Fischer, Schreier-Aigner 2021)

The Robbins polynomial $R^*_{\mathbf{k}}(x_1, ..., x_n; u, v, w)$ are given by

$$\frac{\mathbf{ASym}_{\mathbf{x}}\left[\prod_{1\leqslant i<j\leqslant n}(ux_ix_j + v + wx_i)\prod_{i=1}^{n} x_i^{k_i}\right]}{\prod_{1\leqslant i<j\leqslant n}(x_j - x_i)},$$

where

$$\mathbf{ASym}_{\mathbf{x}} F(x_1, ..., x_n) \coloneqq \sum_{\sigma\in\mathfrak{S}_n} \operatorname{sgn}(\sigma) F(x_{\sigma(1)}, ..., x_{\sigma(n)}).$$

Reformulation of the Littlewood identity

Using the antisymmetriser, we obtain

$$s_\lambda(x_1,\ldots,x_n) = \frac{\det\limits_{1\leqslant i,j\leqslant n}\left(x_i^{\lambda_j+n-j}\right)}{\det\limits_{1\leqslant i,j\leqslant n}\left(x_i^{n-j}\right)} = \frac{\mathbf{ASym}_{\mathbf{x}}\left(\prod_{i=1}^{n} x_i^{\lambda_i+n-i}\right)}{\prod\limits_{1\leqslant i<j\leqslant n}(x_i-x_j)}.$$

Thus the Littlewood identity reads as

$$\frac{\mathbf{ASym}_{\mathbf{x}}\left(\sum\limits_{0\leqslant k_1<\cdots<k_n}\prod_{i=1}^{n} x_i^{k_i}\right)}{\prod\limits_{1\leqslant i<j\leqslant n}(x_j-x_i)} = \prod_{i=1}^{n}\frac{1}{1-x_i}\prod_{1\leqslant i<j\leqslant n}\frac{1}{1-x_ix_j}.$$

Symmetric functions

Robbins polynomials are connected to other symmetric functions:

- Schur polynomials
- symmetric Grothendieck polynomials
- Hall–Littlewood polynomials
- fully inhomogeneous spin Hall–Littlewood symmetric rational functions

We can recover Schur polynomials from Robbins polynomials:

Proposition

$$R^*_{(k_1,\ldots,k_n)}(x_1,\ldots,x_n;0,0,1) = s_{(k_n-2(n-1),k_{n-1}-2(n-2),\ldots,k_2-4,k_1)}(x_1,\ldots,x_n)\prod_{i=1}^{n} x_i^{n-1}$$

Fully inhomogeneous spin Hall–Littlewood symmetric rational functions

Borodin and Petrov (2018) introduced fully inhomogeneous spin Hall–Littlewood symmetric rational functions $F_\lambda(u_1, \ldots, u_n)$ in the context of higher spin six vertex model:

$$\frac{\mathbf{ASym_u}\left[\prod_{1\leqslant i<j\leqslant n}(u_i - tu_j)\prod_{i=1}^{n}\left(\frac{1-t}{1-s_{\lambda_i}\xi_{\lambda_i}u_i}\prod_{j=0}^{\lambda_i-1}\frac{\xi_j u_i - s_j}{1-\xi_j s_j u_i}\right)\right]}{\prod_{1\leqslant i<j\leqslant n}(u_i-u_j)},$$

depending on a parameters q and inhomogeneities ξ_x and s_x.

After setting $\xi_x = 1$ and $s_x = t^{-1/2}$ and some suitable variable transformations, we obtain

$$F_\lambda(u_1, \ldots, u_n) \rightsquigarrow t^{\binom{n+1}{2}}\prod_{i=1}^{n}(t+x_i)R^*_{\mathbf{k}}(x_1, ..., x_n; 1, 1, t+t^{-1}),$$

where $\mathbf{k}$ is the reverse sequence of λ.

Related Littlewood-type identities

- Littlewood-identities for Hall–Littlewood polynomials by Macdonald
- Refinement by Betea, Wheeler, Zinn-Justin (2015):

$$\sum_{\lambda' \text{ even}} \prod_{i=0}^{\infty} \prod_{j=2,4,\dots}^{m_i(\lambda)} P_\lambda(x_1,\dots,x_{2n};t) = \prod_{1\leqslant i<j\leqslant 2n} \frac{1-tx_ix_j}{x_i-x_j} \operatorname*{Pf}_{1\leqslant i<j\leqslant 2n} \left(\frac{(x_i-x_j)(1-t)}{(1-tx_ix_j)(1-x_ix_j)} \right).$$

- Gavrilova (2023):

$$\sum_{\lambda' \text{ even}} \frac{1}{(t;t)_{m_0(\lambda)}} \prod_{j=1}^{m_0(\lambda)/2} (1-\frac{s_0^2}{\gamma}t^{2j-1})(1-\gamma t^{2j-1}) \prod_{j=1}^{2n}(1-s_0u_j)$$
$$\times \prod_{i=1}^{\infty} \prod_{j=1}^{m_i(\lambda)/2} \frac{1-s_i^2t^{2j-2}}{1-t^{2j}} F_\lambda(u_1,\dots,u_{2n}) = \prod_{1\leqslant i<j\leqslant 2n} \frac{1-tu_iu_j}{u_i-u_j}$$
$$\operatorname*{Pf}_{1\leqslant i<j\leqslant 2n} \left(\frac{(u_i-u_j)((1-t)(1-s_0u_i)(1-s_0u_j)+(1-\gamma)(t-\frac{s_0^2}{\gamma})(1-u_iu_j))}{(1-tu_iu_j)(1-u_iu_j)} \right).$$

$\rightarrow$ These identities are of different type than ours!

Combinatorial interpretation of the right-hand side of the Littlewood identity

Main result II

Theorem (Fischer, H. 2025)
Let n be a positive integer. Then

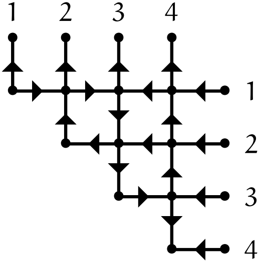
$$\sum_{0 \leqslant k_1 < \cdots < k_n} R^*_{(k_1,\ldots,k_n)}(x_1,\ldots,x_n;1,1,w) = \prod_{i=1}^{n} \frac{1}{1-x_i} \prod_{1 \leqslant i < j \leqslant n} \frac{1}{1-x_i x_j} Z_{\mathrm{DSASM}}(x_1,\ldots,x_n).$$

Diagonally symmetric alternating sign matrices

- Alternating sign matrices were introduced in the early 1980s by Robbins and Rumsey.
- The ASM enumeration formula was first established by Zeilberger in 1996.
- There are eight different symmetry classes of ASMs that are induced by the symmetry group of the square. The enumeration of these symmetry classes was initiated by Stanley.
- In $5\frac{1}{2}$ cases, a product formula has been established (Behrend, Fischer, Konvalinka, Kuperberg, Razumov, Stroganov, Okada, Zeilberger).
- DSASMs are the first and only of the remaining symmetry classes for which an enumeration formula is known (Behrend, Fischer, Koutschan 2023):

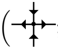
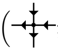
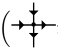
$$\mathrm{DSASM}(n) = \operatorname*{Pf}_{\chi_{\mathrm{odd}}(n) \leqslant i < j \leqslant n-1} \left(\langle u^i v^j \rangle \frac{(v-u)(2+uv)}{(1-uv)(1-u-v)} \right).$$

Six-vertex model configurations I



We assign weights to all vertices:

- Top vertices and right boundary vertices have weight 1.
- Bulk vertices at position (i, j) with local configuration c have weight $W(c; x_i, x_j)$.
- Left boundary vertices at (i, i) with local configuration c have weight $W(c; x_i)$.

bulk weights	left boundary weights
$W\big($$; x, y\big) = W\big($$; x, y\big) = \tau(x)\tau(y)$	$W\big($$; x\big) = 1$
$W\big($$; x, y\big) = W\big($$; x, y\big) = x + y + wxy$	$W\big($$; x\big) = -1$
$W\big($$; x, y\big) = W\big($$; x, y\big) = 1 - xy$	$W\big($$; x\big) = W\big($$; x\big) = \sqrt{w + 2\frac{x}{\tau(x)}}$

We set $\tau(x) \coloneqq \sqrt{-1 - wx - x^2}$.

Six-vertex model configurations II

- The weight of a six-vertex model configuration is the product of the weights of all vertices.
- The sum of these weights over all six-vertex model configurations in $6V_{\triangledown}(n)$ is the partition function of DSASMs of order n, denoted by $Z_{\mathrm{DSASM}}(x_1, \ldots, x_n)$.

Theorem (Fischer, H. 2025)

The partition function $Z_{\mathrm{DSASM}}(x_1, \ldots, x_n)$ of DSASMs of order n is

$$\prod_{1 \leqslant i < j \leqslant n} \frac{(1 - x_i x_j)(x_i + x_j + w x_i x_j)}{x_j - x_i}$$
$$\times \operatorname*{Pf}_{\chi_{\mathrm{even}}(n) \leqslant i < j \leqslant n} \left(\begin{cases} 1, & i = 0, \\ \frac{(x_j - x_i)(1 + (1+w) x_i x_j)}{(x_i + x_j + w x_i x_j)(1 - x_i x_j)}, & i \geqslant 1. \end{cases} \right).$$

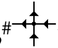
$\longrightarrow$ follows from Behrend, Fischer, Koutschan (2023)

Coefficient of the highest term in the polynomial expansion of $Z_{\mathrm{DSASM}}(x_1, \ldots, x_n)$

Coefficient of the highest term

The partition function $Z_{\mathrm{DSASM}}(x_1, \ldots, x_n)$ is a symmetric polynomial in $x_1, \ldots, x_n$. What can we say about its Schur expansion?
$\longrightarrow$ Work in progress

Theorem (Fischer, H. 2025)
The coefficient of $x_1^{n-1} \cdots x_n^{n-1}$ in $Z_{\mathrm{DSASM}}(x_1, \ldots, x_n)$ is given by the generating function

$$\sum_{6V_{\bigtriangledown}(n)} w^{\#}$$  $^{+\#}$  ,

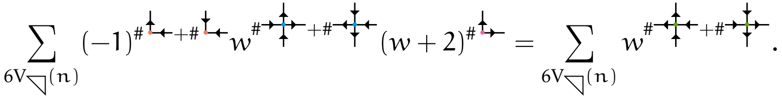
which equals

$$w^{\binom{n}{2}} \operatorname*{Pf}_{\chi_{\mathrm{odd}}(n) \leqslant i < j \leqslant n-1} \left(\langle u^i v^j \rangle \frac{(v-u)(1+uv+w)}{(1-uv)(w-u-v)} \right).$$

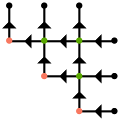
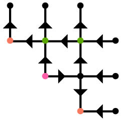
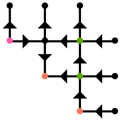
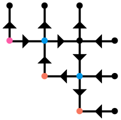
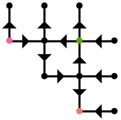
Open problem

Problem

Find a bijective proof of the following identity:



Here is an illustration of the case $n = 3$, for which both sides sum to $1 + w + 2w^2 + w^3$.

DSASM	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$
$6V_{\triangledown}$					
LHS	-1	$w+2$	$w+2$	$w^2(w+2)$	$-(w+2)$
RHS	w^3	w^2	w^2	1	w