A method to obtain generating functions for polynomials using Lucas-Analogues

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UID/04561/2025

March 24, 2025

Overview

1. **Combinatorial and Algebraic approaches**

2. **Lucas Analogues of recurrence relations**

The Lucas sequence as a generalized Fibonacci sequence

The Lucas sequence, $\{n\}$, is motivated by the intent to generalize the Fibonacci sequence to a sequence of rational functions. For this reason it is sometimes called the Generalized Fibonacci Sequence.

Fibonacci Sequence

The Fibonacci sequence, F_n, $n \in \mathbb{N}$, can be defined recursively as follows:

$$F_n = \begin{cases} 0 & n = 0 \\ 1 & n = 1 \\ F_n = F_{n-1} + F_{n-2} & n \geq 2 \end{cases}$$

The Lucas sequence as a generalized Fibonacci sequence

Lucas Sequence

The Lucas sequence, $\{n\}$, $n \in \mathbb{N}$, can be defined recursively as follows, for s, t two indeterminants:

$$\{n\} = \begin{cases} 0 & n = 0 \\ 1 & n = 1 \\ \{n\} = s\{n-1\} + t\{n-2\} & n \geq 2 \end{cases}$$

Examples

The first 5 Lucas numbers are: $\{0\} = 0$; $\{1\} = 1$; $\{2\} = s$; $\{3\} = s^2 + t$; $\{4\} = s^3 + 2st$

Combinatorial Interpretation of the Lucas Numbers

Consider a row of n squares, and let $\mathcal{T}(n)$ denote the set of tillings T of this stip by monominoes, which cover one square, and dominoes, which cover two adjacent squares. Given any configuration, we define its *weight* to be:

$$wt T = s^{nr\ of\ monominoes\ in\ T} t^{nr\ of\ dominoes\ in\ T}$$

Given any set of tilings $\mathcal{T}$, we define its weight to be:

$$wt \mathcal{T} = \sum_{T \in \mathcal{T}} wt T$$

The set $\mathcal{T}(3)$

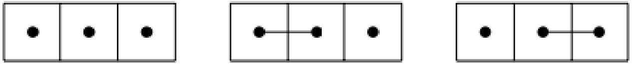


Figure: The set $\mathcal{T}(3)$.

Tiling words in $\mathcal{T}(3)$

The tiling words are: s^3, ts and st.

Polynomialty of the Lucas sequence

Proposition

For all $n \geq 1$: $\{n\} = wt(\mathcal{T}(n-1))$

Proof

We can prove this by the induction method. Consider $wt(\mathcal{T}(0))$, which only has one possible configuration, the empty tiling, so for $n = 1$ we verify the result. We want to show that $\{n+1\} = wt(\mathcal{T}(n))$, but getting back to the definition of the Lucas sequence we have: $\{n+1\} = s\{n\} + t\{n-1\}$ by the induction hypothesis this can be interpreted in the following way:

wt(T(n))=s $\times wt(\mathcal{T}(n-1)) + t \times wt(\mathcal{T}(n-2))$

We consider two cases for the possible configurations of n tiles, those that end with a monominoe, in which case the weight of all the possible $n-1$ tile configurations is given by $wt(\mathcal{T}(n-1))$; and those that end with a dominoe, in which case the weight of all the possible $n-2$ tile configuration is given by $wt(\mathcal{T}(n-2))$.

The Lucastorial and the Lucasnomial

Lucastorial

For $n \geq 0$, $n \in \mathbb{N}$, the Lucas analogue of the factorial is called the *Lucastorial*:

$$\{n\}! = \{1\}\{2\}...\{n\}$$

.

Now we are able to obtain a Lucas Analogue for the Binomial coefficient.

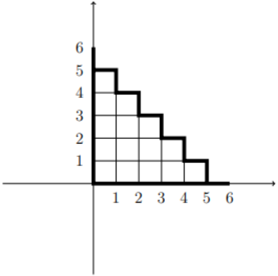
Lucasnomial

Given $0 \leq k \leq n$ natural numbers, we define the *Lucasnomial* to be:

$$\left\{ {n \atop k} \right\} = \frac{\{n\}!}{\{k\}!\{n-k\}!}$$

Combinatorial Interpretation of the Lucastorial

Consider the following set $\delta_n = (n-1, n-2, ..., 1)$. A tiling of λ is a tiling T of rows of the diagram with monominoes and dominoes. Let $\mathcal{T}(\lambda)$ denote the set of all such tilings. We write $wt\lambda$ instead of $wt\mathcal{T}(\lambda)$.



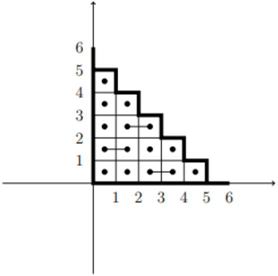


Figure: On the left we have δ_6 embedded in $\mathbb{R}^2$ and on the right we have the example of a Tiling of δ_6.

Combinatorial Interpretation of the Lucastorial

Theorem regarding the Lucastorial

For $n \geq 2$, $wt(\delta_n) = \{n\}!$

Combinatorial Interpretation of the Lucasnomial

It is our intent to partition $\mathcal{T}(\delta_n)$ into subsets, which we call *blocks* such that $wt\beta$ is evenly divisible by $p(s,t)$ for all blocks β. We use lattice paths inside δ_n to create the partitions where the choice of path will vary depending on which Lucas analogue we are considering. The path will start at $(k,0)$ and end at $(0,n)$, such that it can only take steps north, N, and west, W. There are some rules that we need to remember when constructing this lattice path. If p is at a lattice point (x,y), then it will move north to $(x,y+1)$ if it does not cross a domino when doing that, neither does it take p outside the diagram of δ_n. Otherwise, p moves West to $(x-1,y)$. The block containing the original tiling T consists of all tilings in $\mathcal{T}(\delta_n)$ which agree with T to the right of each NL steps and to the left of each NI steps.

Theorem

Let $0 \leq k \leq n$. There is a partition of $\mathcal{T}(\delta_n)$ such that $\{k\}!\{n-k\}!$ divides $wt\beta$ for every block β.

Combinatorial Interpretation of the Lucasnomial

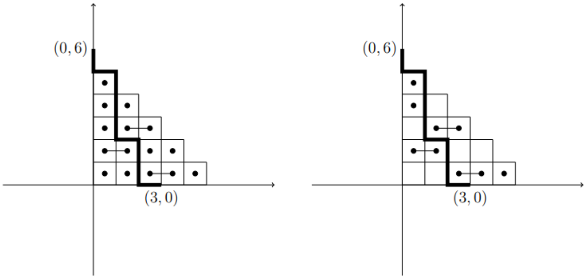


Figure: On the left we have a lattice path in δ_6 corresponding to the tiling in the previous figure, and on the right the corresponding Partial Tiling.

We call the tiling showing the fixed tiles for a given block β in the partition of the previous theorem as *binomial partial tiling B*. The previous theorem allows us to conclude the following: $wt\beta = \{k\}!\{n-k\}!wtB$.

Combinatorial Interpretation of the Lucasnomial

we can sum the weight all of this blocks associated with lattice paths from $(k, 0)$ to $(0, n)$ and we get the weight of $\mathcal{T}(\delta_n)$, therefore:

$$wt(\delta_n) = \sum_{\beta} wt\beta = \{n\}!$$

Theorem

Given $0 \leq k \leq n$, we have

$$\left\{ {n \atop k} \right\} = \sum_{B} wtB$$

where the sum is taken over all binomial partial tilings associated with lattice paths from $(k, 0)$ to $(0, n)$ in δ_n. Thus, we have proved that $\left\{ {n \atop k} \right\}$ is a polynomial in $\mathbb{N}[s, t]$.

Lucas-analogues of the Catalan numbers

One of the most well-known applications of Lucas analogues is the study of Catalan Combinatorics. We define the *Catalan numbers* by:

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

for $n \geq 0$. Naturally, its Lucas analogue is:

$$C_{\{n\}} = \frac{1}{\{n+1\}}\left\{ {2n \atop n} \right\}$$

Combinatorial interpretation of the Catalan Analogues

Theorem

Given $n \geq 0$ there is a partition of $\mathcal{T}(\delta_{2n})$ such that $\{n\}!\{n+1\}!$ divides $wt\beta$ for every block β.

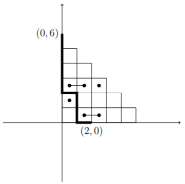


Figure: An example of a Catalan Partial Tiling.

In similarity to what was done regarding the Lucasnomials, we can associate with each block of the partition in the previous theorem a *Catalan partial tiling*.

Combinatorial Interpretation of the Catalan Analogues

Theorem

Given $n \geq 0$, we have

$$C_{\{n\}} = \sum_{C} wtC$$

where the sum is over all Catalan partial tilings C associated with lattice paths from $(n-1, 0)$ to $(0, 2n)$ in δ_{2n}. Thus, $C_{\{n\}} \in \mathbb{N}[s, t]$ and, therefore, sometimes $C_{\{n\}}$ is refered to as the *Lucas-Catalan polynomials*.

Algebraic Factorization of the Lucas Numbers

The primary objective of this approach is to factorize the n-th Lucas number into polynomials, which we term Atoms.

Lucas Atoms

The goal is to define a sequence of polynomials, $P_n(s, t)$ such that they satisfy:

$$\{n\} = \prod_{d|n} P_d(s, t)$$

Palindromic polynomials

Let $p(q) = \sum_{i \geq 0} a_i q^i$ be a nonzero polynomial in q with coefficients in $\mathbb{C}$, where $\mathbb{C}$ stands, as usual, for the complex numbers.

Properties of palindromic polynomials

(i) $p(q)$ is palindromic if, and only if, $q^d p(1/q) = p(q)$;
(ii) The product of palindromic polynomials is palindromic.

Palindromic polynomials

Consider

$$\mathbb{P}_d(q) = \{p(q) \in \mathbb{C}[q] | p(q) \text{is palindromic with totdegp(q)=d}\} \cup \{0\}$$

Notice that the polynomials $(1+q)^d$, $q(1+q)^{d-2}$, $q^2(1+q)^d - 4$, etc, form a basis for $\mathbb{P}_d$. Therefore, if $p(q) \in \mathbb{P}_d$ then it has *gamma expansion*:

$$p(q) = \sum_{j \geq 0} \gamma_j q^j (1+q)^{d-2j},$$

where the scalars γ_j are called the *gamma coefficients* of $p(q)$.

The n-th Lucas Number as a palindromic polynomial

Expressing the n-th Lucas Numbers as a palindromic polynomial

$\{n\} = \sum_{j \geq 0} a_j s^{n-2j-1} t^j$, for certain $a_j \in \mathbb{N}$.

Consider the following:

$$\mathbb{P}(q) = \bigcup_{d \geq 0} \mathbb{P}_d(q)$$

We can define the following map: $\Gamma : \mathbb{P}(q) \longrightarrow \mathbb{C}[s, t]$, which we will call the *Gamma map*, as such:

$$\Gamma(p(q)) = \sum_{j \geq 0} \gamma_j s^{d-2j} (-t)^j,$$

where $p(q)$ is as defined previously.

Properties of the Gamma map

Properties

(a) If $p(q)$, $r(q) \in \mathbb{P}(q)$, then: $\Gamma(p(q)r(q)) = \Gamma(p(q))\Gamma(r(q))$;
(b) For any d, the restriction of Γ to $\mathbb{P}_d(q)$ is linear;
(c) The map Γ is injective;
(d) If $\Gamma(p(q)) = f(s,t)$, then $f(1+q,-q) = p(q)$;
(e) If $p(q) \in \mathbb{Z}[q]$, then $\Gamma(p(q)) \in \mathbb{Z}[s,t]$.

Lucas Atoms as images through the Gamma map of the n-th cyclotomic polynomial

We can define the *n-th cyclotomic polynomial* as:

$$\phi_n(q) = \prod_{\zeta}(q - \zeta),$$

where the product is over all primite n-th roots of unity. We know that $\phi_n(q)$ is palindromic for the range $n \geq 2$, because all the constant coefficients of $\phi_n(q)$ are equal to 1 (for $n \geq 2$) and using the fact that ζ is a primitive n-th root if, and only if, $\frac{1}{\zeta}$ also is. Now can define the *n-th Lucas atom* as $P_1(s,t) = 1$ and $P_n(s,t) = \Gamma(\phi_n(q))$, for $n \geq 2$.

The Lucas Numbers decomposition

For all $n \geq 1$, the following is true:
(a) $\{n\} = \prod_{d|n} P_d(s,t)$;
(b) $P_n(s,t) \in \mathbb{N}[s,t]$

Lucas Analogues of recurrence relations

Consider the following recurrence relation:

$$B_{n,k} = (k+1)B_{n-1,k} - (n-k-1)B_{n-1,k-1}$$

Usually we consider its analogue to be:

$$B_{\{n,k\}} = \{k+1\}B_{\{n-1,k\}} + t\{n-k-1\}B_{\{n-1,k-1\}}$$

Why not $A_{\{n,k\}} = s\{k+1\}B_{\{n-1,k\}} + t\{n-k-1\}B_{\{n-1,k-1\}}$?

Transformation T_1 and Its Properties

The transformation T_1 acts as follows:

- Maps constants: $1 \mapsto s$, $-1 \mapsto t$.
- Transforms functions $a(n,k)$ (where $n \geq k \geq 0$) to their analogue:

$$a(n,k) \mapsto \{a(n,k)\}.$$

- Factorial and Binomial terms are replaced by their *Lucas analogues*, which are known to be polynomial.
- Preserves the power. For example:

$$(n-1)^k \mapsto \{n-1\}^k.$$

- Ensures consistency with expressions like $(n+1)^k$.
- If an array $A_{n,k}$ satisfies a recurrence relation with an initial condition $A_{n,0}$, then T_1 preserves these conditions:

$$A_{n,0} \mapsto A_{\{n,0\}}.$$

Lucas Analogues of recurrence relations

Theorem

Let $n \in \mathbb{N}_0$. Let a_n represent an array of numbers that satisfies a recurrence relation in $\mathbb{Z}$ that can be written as:

$$a_n = \sum_{i \in I} (-1)^{\alpha_i} b_i^{\beta_i}(n) a_{n-p(i)},$$

with a set of initial conditions, where I is an index set and for some map $b_i(n)$ that is not expressed as a quocient and may change its form depending on i, $\alpha_i \in \{0, 1\}$ and $\beta_i \in \mathbb{N}_0$ some power that may change with i, and $p(i)$ a map that maps i into $\mathbb{N}$. Then its analogue through T_1 is polynomial.

Lucas Analogues of recurrence relations

Example

Consider the following recursive relation with initial conditions $a(0) = 0$ and $a(1) = 1$ and for $n \geq 2$, the general term is given by:

$$a_n = a_{n-1} - a_{n-2}.$$

Sometimes this is mentioned as *The Alternating Fibonacci Sequence*. We can write it in the form of the previous theorem with $I = \{1, 2\}$; $\alpha_1 = 0$ and $\alpha_2 = 1$; $b_i(n) = i$, $\forall_{i \in I}$, $p(1) = 1$ and $p(2) = 2$. Therefore, the previous theorem guarantees that its analogue via T_1 is polynomial. Furthermore, notice that it is the same analogue as the original Fibonacci Sequence, which is the Lucas Sequence.

Lucas Analogues of recurrence relations

Theorem

Let $n, k \in \mathbb{N}_0$, such that $n \geq k \geq 0$. Let $A_{n,k}$ represent an array of numbers that satisfies a recurrence relation in $\mathbb{Z}$ that can be written as:

$$A_{n,k} = \sum_{i \in I} (-1)^{\alpha_i} a_i^{\beta_i}(n,k) B_{n-p(i),k-q(i)},$$

with initial conditions $B_{n,0} = B_{n,n} = 1$ for $n \geq 0$ and $B_{n,k} = 0$ whenever $k<0$ or $n<k$, where I is an index set and for some map $a_i(n,k)$ that is not expressed as a quocient and may change its form depending on i, $\alpha_i \in \{0,1\}$ and $\beta_i \in \mathbb{N}_0$ some power that may change with i, and $p(i), q(i)$ map i into $\mathbb{N}_0$. Then its analogue through T_1 is polynomial.

Lucas Analogues of recurrence relations

Example

Consider $n \geq k \geq 0$, where $n, k \in \mathbb{N}_0$. Recall that $B_{n,0} = B_{n,n} = 1$ for $n \geq 0$ and $B_{n,k} = 0$ whenever $k{<}0$ or $n{<}k$. Let $A_{n,k}$ be the following recurrence relation:

$$A_{n,k} = n! B_{n-1,k} - (n-k+2)^3 B_{n-1,k-1}$$

We obtain integers through this recurrence relation. Taking its Analogue using the T_1 transformation, we obtain:

$$A_{\{n,k\}} = s\{n\}! B_{\{n-1,k\}} + t\{n-k+2\}^3 B_{\{n-1,k-1\}}$$

Notice that $A_{n,k}$ can be written in the form of the theorem with $I = \{1, 2\}$, $\alpha_1 = 0$, $\alpha_2 = 1$, $\beta_1 = 1$, $\beta_2 = 3$, $a_1(n,k) = n!$, $a_2(n,k) = (n-k+1)$, $p(1) = 1$, $q(1) = 0$, $p(2) = 1$ and $q(2) = 1$. Therefore, the previous theorem guarantees that its analogue via T_1 is polynomial.

Lucas Analogues of recurrence relations

Theorem

Let $n, k \in \mathbb{N}_0$, such that $n \geq k \geq 0$. Let $A_{n,k}$ represent an array of numbers that satisfies a recurrence relation that can be written as:

$$A_{n,k} = \sum_{i \in I} (-1)^{\alpha_i} a_i^{\beta_i}(n,k) B_{w_i(n), v_i(k)} A_{n-p(i), k-q(i)},$$

where I is an index set and for some map $a_i(n,k)$ that is not expressed as a quocient and may change its form depending on i, $\alpha_i \in \{0,1\}$ and $\beta_i \in \mathbb{N}_0$ some power that may change with i, and $p(i), q(i)$ map i into $\mathbb{N}_0$. Let w_i and v_i be maps into $\mathbb{N}_0$ Furthermore, let $A_{n,k}$ have a set of initial conditions that in $\mathbb{Z}$. Then, $A_{n,k}$ has a polynomial Lucas Analogue using the T_1 transformation.

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