A combinatorial interpretation of a particular ${}_3\phi_2$ transformation

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25 March 2025

*Supported by the FCT project UIDP/00208/2020; DOI: 10.54499/UIDP/00208/2020.

q-series notation

For $0 < |q| < 1$, we define

$$(x)_\infty = \prod_{n=0}^{\infty}(1-xq^n), \qquad (x)_n = \frac{(x)_\infty}{(xq^n)_\infty}, \qquad (x_1, x_2, \ldots, x_k)_n = (x_1)_n(x_2)_n \ldots (x_k)_n.$$

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Thus $(x)_0 = 1$ and, for $n \geq 1$,

$$(x)_n = (1-x)(1-qx)(1-q^2x)\dots(1-q^{n-1}x),$$

while for $n \leq -1$,

$$1/(x)_n = (1-xq^{-1})(1-xq^{-2})\dots(1-xq^n).$$

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The standard notation for q-series is as follows:

$${}_r\phi_s\left(\begin{matrix}\alpha_1, & \alpha_2, \ldots, & \alpha_r \\ \beta_1, & \beta_2, \ldots, & \beta_s\end{matrix}; x\right) = \sum_{n=0}^{\infty} \frac{(\alpha_1, \alpha_2, \ldots, \alpha_r)_n}{(q, \beta_1, \beta_2, \ldots, \beta_s)_n}\left((-1)^n q^{\frac{n(n-1)}{2}}\right)^{s-r+1} x^n,$$

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Integer partitions

Let n be any positive integer. A *partition* of n is an ordered tuple $\pi = (\pi_1, \pi_2, \ldots, \pi_k)$ such that

$$\pi_1 \geq \pi_2 \geq \ldots \geq \pi_k \qquad \text{and} \qquad \pi_1 + \pi_2 + \ldots + \pi_k = n.$$

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Example. There are seven partitions of 5:

$$\begin{array}{llll} (5) & (2,2,1) & (4,1) & (2,1,1,1) \\ (3,2) & (1,1,1,1,1) & (3,1,1) & \end{array}$$

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For a partition $\pi \in \mathcal{P}$, we define the following functions:

$$\begin{aligned} \sigma(\pi) &= \text{ the sum of the parts of } \pi, \\ \nu(\pi) &= \text{ the number of parts of } \pi, \\ \lambda(\pi) &= \text{ the largest part of } \pi. \end{aligned}$$

Ferrers diagrams

Definition

Let $\pi = (\pi_1, \pi_2, \ldots, \pi_k)$ be a partition. The *Ferrers diagram* of π is

$$\{(a, b) \in \mathbb{Z}^2 \mid 1 \leq b \leq k \text{ and } 1 \leq a \leq \pi_b\}.$$

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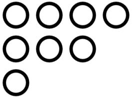
Example. The Ferrers diagram of $\pi = (4, 3, 1)$ is



For this partition,

$$\sigma(\pi) = 8, \qquad \nu(\pi) = 3, \qquad \lambda(\pi) = 4.$$

Generating functions for partitions

Euler (1748) observed that

$$\sum_{\pi\in\mathcal{P}} q^{\sigma(\pi)}x^{\nu(\pi)} = \prod_{n=1}^{\infty}(1 + xq^n + x^2q^{2n} + \ldots) = \frac{1}{(qx)_\infty} = \sum_{n=0}^{\infty}\frac{(qx)^n}{(q)_n}.$$

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If we also keep track of $\lambda(\pi)$ then we have only a series form available:

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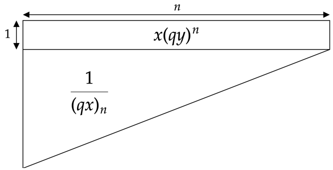
The series can be obtained by considering the Ferrers diagram as follows:



Conjugation of partitions

The *conjugate* of the Ferrers diagram for a given partition $\pi = (\pi_1, \pi_2, \ldots, \pi_k)$ is

$$\{(a, b) \in \mathbb{Z}^2 \mid 1 \leq a \leq k \text{ and } 1 \leq b \leq \pi_a\}.$$

The partition represented by this diagram is called the conjugate of π and is denoted π^*.

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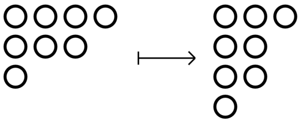

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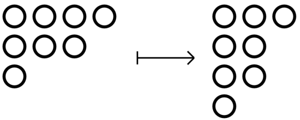

For any partition π, we have

$$\sigma(\pi^*) = \sigma(\pi), \qquad \nu(\pi^*) = \lambda(\pi), \qquad \lambda(\pi^*) = \nu(\pi).$$

A symmetry of the generating function for partitions

Since $\pi \mapsto \pi^*$ is a bijection, we may transform the generating function as follows:

$$\sum_{\pi \in \mathcal{P}} x^{\nu(\pi)} y^{\lambda(\pi)} q^{\sigma(\pi)} = \sum_{\pi \in \mathcal{P}} x^{\nu(\pi^*)} y^{\lambda(\pi^*)} q^{\sigma(\pi^*)} = \sum_{\pi \in \mathcal{P}} x^{\lambda(\pi)} y^{\nu(\pi)} q^{\sigma(\pi)}.$$

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The generating function is therefore symmetrical in x and y. Consequently

$$x \sum_{n=1}^{\infty} \frac{(qy)^n}{(qx)_n} = y \sum_{n=1}^{\infty} \frac{(qx)^n}{(qy)_n}.$$

This identity is a special case of a transformation formula which appears in the monograph of Fine (1988).

Overpartitions

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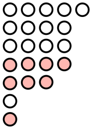
Example. Let $\pi = \big((5,4,4,1),(4,3,1)\big)$. We represent this with a Ferrers diagram:



It is conventional to list the marked and unmarked parts together—so here we may write

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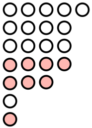

It is conventional to list the marked and unmarked parts together—so here we may write

$$\pi = (5,4,4,{\color{red}4},{\color{red}3},1,{\color{red}1}).$$

By identifying a partition $\pi \in \mathcal{P}$ with the overpartition $(\pi, \varnothing) \in \mathcal{P}^{(1)}$, we may regard $\mathcal{P}$ as a subset of $\mathcal{P}^{(1)}$.

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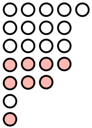

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By identifying a partition $\pi \in \mathcal{P}$ with the overpartition $(\pi, \varnothing) \in \mathcal{P}^{(1)}$, we may regard $\mathcal{P}$ as a subset of $\mathcal{P}^{(1)}$. It matters which parts are marked. For example,

$$(5,4,4,4,3,1,1)$$

is different from the overpartition π shown above.

Conjugation of overpartitions

Following Corteel and Lovejoy (2004), we form the conjugate of an overpartition π as follows: We first transpose the rows and columns of its Ferrers diagram. The marked parts of the conjugate π^* correspond to the rows which contain a corner node that was marked in the original diagram.

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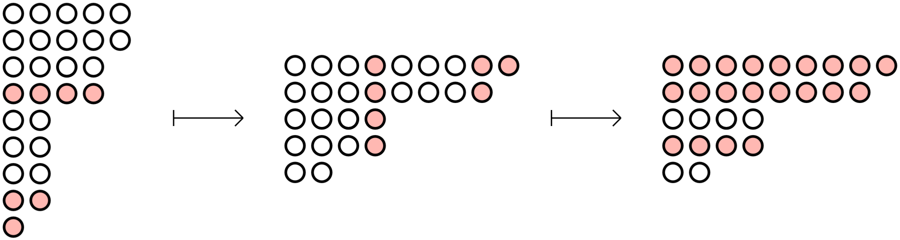
Example. Let $\pi = (5, 5, 4, \color{red}{4}, 2, 2, 2, \color{red}{2}, \color{red}{1})$. Its conjugate $\pi^* \in \mathcal{P}^{(1)}$ is given by

$$\pi^* = (\color{red}{9}, \color{red}{8}, 4, \color{red}{4}, 2).$$



The generating function for overpartitions and its symmetry

The generating function for overpartitions is

$$\sum_{\pi \in \mathcal{P}^{(1)}} a^{r(\pi)} q^{\sigma(\pi)} x^{\nu(\pi)} y^{\lambda(\pi)} = 1 + x(1+a) \sum_{n=1}^{\infty} \frac{(-aqx)_{n-1}}{(qx)_n} (qy)^n,$$

where $r(\pi)$ denotes the number of marked parts of π.

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where $r(\pi)$ denotes the number of marked parts of π. Under conjugation, $\pi \mapsto \pi^*$, we have

$$r(\pi^*) = r(\pi), \qquad \sigma(\pi^*) = \sigma(\pi), \qquad \nu(\pi^*) = \lambda(\pi), \qquad \lambda(\pi^*) = \nu(\pi).$$

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Consequently, the generating function is symmetrical in x and y. After making the substitution $(a, x, y) \mapsto (-a, x/q, y/q)$, this transformation formula may be written in the form

$$\sum_{n=0}^{\infty} \frac{(ay)_n}{(y)_{n+1}} x^n = \sum_{n=0}^{\infty} \frac{(ax)_n}{(x)_{n+1}} y^n.$$

The connection between overpartitions and this identity is mentioned by Pak (2006).

q-series which share the same symmetry

Each of the following series is a symmetrical function of the variables x and y:

$$\sum_{n=0}^{\infty} \frac{x^n}{(y)_{n+1}},$$

$$\sum_{n=0}^{\infty} \frac{(ay)_n}{(y)_{n+1}} x^n,$$

$$\sum_{n=0}^{\infty} \frac{(ay, by)_n}{(y)_{n+1}(abxy)_n} x^n,$$

$$\sum_{n=0}^{\infty} \frac{(ay, by, cy, \frac{abcxy}{q})_n}{(y)_{n+1}(abxy, acxy, bcxy)_n} (1 - abcxy^2 q^{2n-1}) x^n,$$

$$\sum_{n=0}^{\infty} \frac{\left(\overset{\times 4}{ay}, \overset{\times 4}{\frac{abcxy}{q}}\right)_n \left(\frac{abcdx^2y^2}{q^2}\right)_{2n}}{(y)_{n+1}\left(\underset{\times 6}{abxy}, \frac{abcdx^2y}{q}\right)_n (abcdxy^2)_{2n}} p(x, yq^n) x^n.$$

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do these have combinatorial interpretations as generating functions for partitions of some sort?

Let $\mathcal{P}^{(2)}$ be the set of partitions π such that each part of π is either marked with one of three colours (red, blue, purple) or is left unmarked, and such that:

- The red parts are distinct.
- The blue parts are distinct.
- Purple and unmarked parts may be repeated.

‘Second-order’ overpartitions

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\lambda(\pi) &= \text{ the largest part of } \pi \quad + \quad \text{the number of purple parts of } \pi \quad - \quad \varepsilon(\pi),
\end{aligned}$$

where

$$\varepsilon(\pi) = \begin{cases} 1 & \text{if every occurrence of the largest part is purple,} \\ 0 & \text{otherwise.} \end{cases}$$

The generating function for second-order overpartitions

The generating function for these is

$$\sum_{\pi \in \mathcal{P}^{(2)}} a^{r(\pi)} b^{b(\pi)} q^{\sigma(\pi)} x^{\nu(\pi)} y^{\lambda(\pi)} = 1 + x(1+a)(1+b) \sum_{n=1}^{\infty} \frac{(-aqx, -bqx)_{n-1}}{(qx, abqxy)_n} (qy)^n.$$

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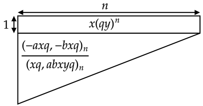
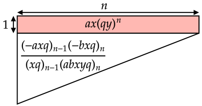
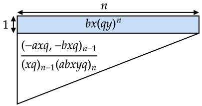


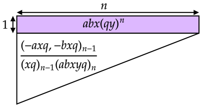
At least one part of size n is unmarked.



No part of size n is unmarked. At least one part of size n is marked red.



No red or unmarked parts of size n occur. At least one part of size n is marked blue.



Every part of size n is marked purple.

Conjugation of second-order overpartitions?

Theorem (B.-T. 2023)

Let h, j, ℓ, m, n be any non-negative integers. Among all second-order overpartitions, π, there are as many satisfying the conditions

$$r(\pi) = h, \quad b(\pi) = j, \quad \sigma(\pi) = \ell, \quad \nu(\pi) = m, \quad \lambda(\pi) = n$$

as there are satisfying the conditions

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Question: Is it possible to produce a bijective proof of this? I.e. to produce an involution $\pi \mapsto \pi^*$ on $\mathcal{P}^{(2)}$, defined solely in terms of manipulations of the Ferrers diagram of π, with the property that

(i) for all $\pi \in \mathcal{P}^{(2)}$,

$$r(\pi^*) = r(\pi), \quad b(\pi^*) = b(\pi), \quad \sigma(\pi^*) = \sigma(\pi), \quad \nu(\pi^*) = \lambda(\pi), \quad \lambda(\pi^*) = \nu(\pi)$$

(ii) restriction to $\mathcal{P}^{(1)}$ reduces to conjugation of overpartitions.

Examples

Among second-order overpartitions which satisfy

$$r(\pi) = 2, \qquad b(\pi) = 1, \qquad \sigma(\pi) = 10,$$

there are three which satisfy $\nu(\pi) = 3$ and $\lambda(\pi) = 4$ and there are three which satisfy $\nu(\pi) = 4$ and $\lambda(\pi) = 3$.

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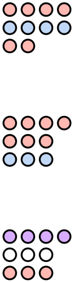
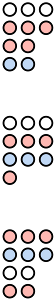


$\nu(\pi) = 4,\ \lambda(\pi) = 3$



Examples

Consider now the second-order overpartitions which satisfy

$$r(\pi) = 3, \qquad b(\pi) = 4, \qquad \sigma(\pi) = 9.$$

There are six of these satisfying $\nu(\pi) = 7$ and $\lambda(\pi) = 5$ and six satisfying $\nu(\pi) = 5$ and $\lambda(\pi) = 7$.

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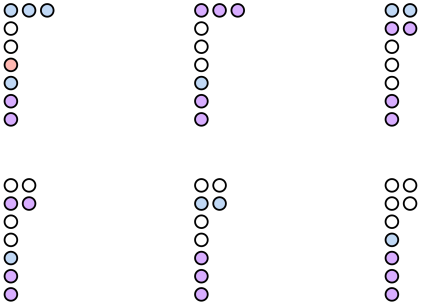
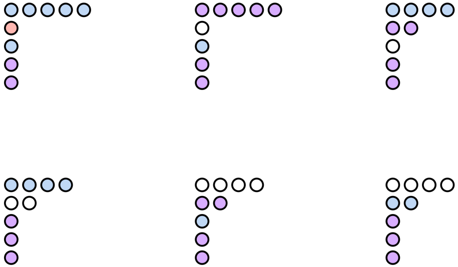
$\nu(\pi) = 7,\ \lambda(\pi) = 5$



$\nu(\pi) = 5,\ \lambda(\pi) = 7$



A generalization of the Rogers–Fine identity

The Rogers–Fine identity may be written in the form

$$\sum_{n=0}^{\infty} \frac{(ay)_n}{(y)_{n+1}} x^n = \sum_{n=0}^{\infty} \frac{(ax, ay)_n}{(x, y)_{n+1}} \left(1 - axyq^{2n}\right) q^{n^2} (xy)^n.$$

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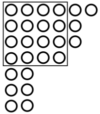
It can be proved combinatorially by considering the Durfee square of an overpartition. (The Durfee square is the largest square of nodes contained in the Ferrers diagram; e.g. for $(6, 5, 5, 4, 2, 2, 2)$ the Durfee square has size 4.) Alladi (2009) gives an argument equivalent to this.



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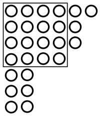

Theorem (B.-T. 2023)

The following generalization of the Rogers–Fine identity holds:

$$\sum_{n=0}^{\infty} \frac{(ay, by)_n}{(y)_{n+1}(abxy)_n} x^n = \sum_{n=0}^{\infty} \frac{(ax, bx, ay, by)_n}{(x, y)_{n+1}(abxy)_{2n+1}} \Big(1 - (a + b + ab)xyq^{2n} + abxy(x + y)q^{3n}\Big) q^{n^2} (xy)^n.$$

This can be proved by considering the Durfee square of a second-order overpartition.

A further generalization of the Rogers–Fine identity

Theorem (B.-T. 2023)

For $|x| < 1$,

$$\sum_{n=0}^{\infty} \frac{\overset{\times 4}{(ay)_n}\,\overset{\times 4}{(abcxy)_{n-1}}\,(abcdx^2y^2)_{2n-2}}{(y)_{n+1}\,\underset{\times 6}{(abxy)_n}\,(abcdx^2y)_{n-1}\,(abcdxy^2)_{2n}}\; p(x, yq^n)\; x^n$$

$$= \sum_{n=0}^{\infty} \frac{(\overset{\times 4}{ax}, \overset{\times 4}{ay})_n\,\overset{\times 4}{(abcxy)_{2n-1}}\,(abcdx^2y^2)_{4n-2}}{(x, y)_{n+1}\,\underset{\times 6}{(abxy)_{2n+1}}\,(abcdx^2y, abcdxy^2)_{3n+1}}\; P(xq^n, yq^n)\; q^{n^2} x^n y^n.$$

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$$= \sum_{n=0}^{\infty} \frac{(\overset{\times 4}{ax}, \overset{\times 4}{ay})_n (\overset{\times 4}{abcxy})_{2n-1} (abcdx^2y^2)_{4n-2}}{(x, y)_{n+1} (\underset{\times 6}{abxy})_{2n+1} (abcdx^2y, abcdxy^2)_{3n+1}} \underbrace{P(xq^n, yq^n)}_{\text{20,990 monomial terms}} q^{n^2} x^n y^n.$$

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The polynomial P is symmetrical in its last two arguments. This identity therefore makes explicit the symmetry in x and y of the series on its left-hand side.

‘Third-order’ overpartitions

Let $\mathcal{P}^{(3)}$ be the set of partitions π such that each part of π is either left unmarked or else marked with one of seven colours—three primary (red, blue, yellow), three secondary (purple, orange, green), and one tertiary (brown)—and such that no two primary or tertiary parts of the same colour may have the same size.

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\nu(\pi) &= \text{ the number of parts of } \pi,\\
\lambda(\pi) &= \text{ the largest part of } \pi\\
&\quad + \text{ the number of secondary parts of } \pi\\
&\quad + \text{ the number of tertiary parts of } \pi\\
&\quad - \varepsilon(\pi).
\end{aligned}$$

'Third-order' overpartitions

In this case, ε is defined by

$$\varepsilon(\pi) = \begin{cases} 2 & \text{if every occurrence of the largest part is secondary and each of the three secondary colours occurs at least once among them,} \\ 1 & \text{if every occurrence of the largest part is secondary or tertiary but the preceding case does not apply,} \\ 1 & \text{if among all occurrences of the largest part exactly one is unmarked, none are primary, and one is tertiary,} \\ 0 & \text{otherwise.} \end{cases}$$

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The generating function is

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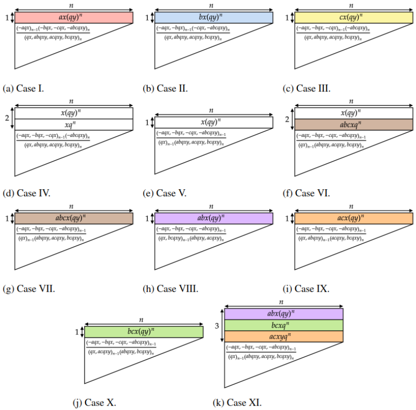
which is a symmetrical function of x and y.



$$1+x\sum_{n=1}^{\infty}\frac{(-aqx,-bqx,-cqx,-abcqxy)_{n-1}}{(qx,abqxy,acqxy,bcqxy)_n}$$
$$\times\Big(a(1+bxq^n)(1+cxq^n)(1+abcxyq^n)+b(1+cxq^n)(1+abcxyq^n)$$
$$+c(1+abcxyq^n)+xq^n(1+abcxyq^n)+(1-xq^n)+abcxq^n(1-xq^n)$$
$$+abc(1-xq^n)+ab(1-xq^n)(1-bcxyq^n)+ac(1-xq^n)(1-abxyq^n)$$
$$+bc(1-xq^n)(1-acxyq^n)+(abcx)^2yq^{2n}(1-xq^n)\Big)(qy)^n.$$

An example

Among third-order overpartitions which satisfy

$$r(\pi) = 3, \qquad b(\pi) = 3, \qquad y(\pi) = 2, \qquad \sigma(\pi) = 8,$$

there are three which satisfy $\nu(\pi) = 7$ and $\lambda(\pi) = 3$ and there are three which satisfy $\nu(\pi) = 3$ and $\lambda(\pi) = 7$.

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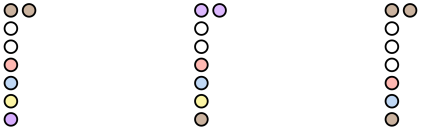

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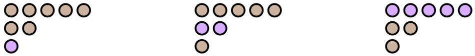

Open questions

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$$1 + x\overbrace{(1+a)}^{\times 4}\sum_{n=1}^{\infty} \frac{(-\overset{\times 4}{a}qx, -ab\overset{\times 4}{c}qxy)_{n-1}(abcdqx^2y^2)_{2n-1}}{(qx)_n(a\underset{\times 6}{b}qxy)_n(abcdq^3x^2y)_{2n-2}(abcdq^2xy^2)_{n-1}(1-abcdx^3yq^{3n})}$$

$$\times\Bigg(\overbrace{(1+abcx^2yq^{2n})}^{\times 4}(1-abcdx^2yq^{2n})(1-abcdxy^2q^n)$$

$$+ abcdxy^2q^n(1-xq^n)\underbrace{(1+axq^n)}_{\times 4}(1-(abcd)^2x^4y^3q^{4n})\Bigg)(qy)^n?$$

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- Can the list of symmetrical q-series be continued? If so, do later series in the list have interesting combinatorial interpretations?

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