Commutation classes of reduced words for elements in Coxeter groups of classical type

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Coxeter groups

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A Coxeter group is a group W generated by a finite set $S = \{s_1, s_2, \ldots, s_n\}$ such that for all $s, t \in S$,

$$(st)^{m(s,t)} = id \tag{1}$$

where $m : S \times S \to \mathbb{N} \cup \{\infty\}$ is a function satisfying $m(s,s) = 1$, and $m(s,t) = m(t,s)$ and $m(s,t) \geq 2$.

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$$\underbrace{stst\cdots}_{m(s,t)} = \underbrace{tsts\cdots}_{m(s,t)}.$$

Relation (1) is called

- commutation, if $m(s,t) = 2$
- braid relation, if $m(s,t) > 2$

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Two reduced words for the same element differ by a finite sequence of commutations and/or braid relations.

Problem: How ”far” are two reduced words??

Graphs structures on reduced words

Graph of reduced words

$G(w)$ is the graph with vertex set $R(w)$ and an edge between two reduced words if they differ by a single commutation or braid relation.

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Most of the work was done for elements in Coxeter groups of classical type.

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Then, $G_c(w)$ is the graph with vertex set $C(w)$ and an edge between $[a]$ and $[b]$ if there are $a' \in [a]$ and $b' \in [b]$ such that a' and b' differ by a single braid relation

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$d_w([a], [b])$ is the distance between $[a]$ and $[b]$ in $G_c(w)$

Coxeter groups of classical type

Type A_n

- $\Phi(A_n) = \{e_j - e_i : 1 \leq i \neq j \leq n+1\}$
- $\Phi^+(A_n) = \{e_j - e_i : 1 \leq i < j \leq n+1\}$
- $\Delta(A_n) = \{\alpha_i : 1 \leq i \leq n\}$, where $\alpha_i = e_{i+1} - e_i$.
- $W(\Phi(A_n)) \cong \mathfrak{S}_{n+1}$

Generators: $\{s_1, \dots, s_n\}$ with $s_i = (i\ i+1)$
Relations:

- $s_i s_j = s_j s_i$, if $|i - j| > 1$
- $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$

Type B_n

- $\Phi(B_n) = \{\pm e_j \pm e_i : 1 \leq i \neq j \leq n+1\} \cup \{\pm e_i : 1 \leq i \leq n+1\}$
- $\Phi^+(B_n) = \{e_j \pm e_i : 1 \leq i < j \leq n+1\} \cup \{e_i : 1 \leq i \leq n+1\}$
- $\Delta(B_n) = \{\alpha_i : 0 \leq i \leq n\}$, where $\alpha_0 = e_1$ and $\alpha_i = e_{i+1} - e_i$, for $i \geq 1$.
- $W(\Phi(B_n)) \cong \mathfrak{S}^B_{n+1}$, the group of signed permutations.

Generators: $\{s^B_0, s^B_1, \ldots, s^B_n\}$, with $s^B_0 = (\bar{1}\ 1)$ and $s^B_i = (i\ i+1)(\overline{i+1}\ \bar{i})$

Relations

- $s^B_i s^B_j = s^B_j s^B_i$, if $|i - j| > 1$
- $s^B_i s^B_{i+1} s^B_i = s^B_{i+1} s^B_i s^B_{i+1}$, if $i \neq 0$
- $s^B_0 s^B_1 s^B_0 s^B_1 = s^B_1 s^B_0 s^B_1 s^B_0$

Type D_n

$\Phi(D_n) = \{\pm e_j \pm e_i : 1 \leq i \neq j \leq n+1\}$
$\Phi^+(D_n) = \{e_j \pm e_i : 1 \leq i < j \leq n+1\}$
$\Delta(D_n) = \{\alpha_i : 0 \leq i \leq n\}$, where $\alpha_0 = e_2 + e_1$ and $\alpha_i = e_{i+1} - e_i$, for $i \geq 1$.
$W(\Phi(D_n)) \cong \mathfrak{S}^D_{n+1} \subseteq \mathfrak{S}^B_{n+1}$ denotes group of even signed permutations

Generators: $\{s^D_0, s^D_1, \ldots, s^D_n\}$, with $s^D_0 = s^B_0 s^B_1 s^B_0$ and $s^D_i = s^B_i$ if $i \neq 0$
Relations:

- $s^D_i s^D_j = s^D_j s^D_i$, if $|i-j| > 1$ and $\{i,j\} \neq \{0,2\}$
- $s^D_0 s^D_1 = s^D_1 s^D_0$
- $s^D_i s^D_{i+1} s^D_i = s^D_{i+1} s^D_i s^D_{i+1}$, if $i \neq 0$
- $s^D_0 s^D_2 s^D_0 = s^D_2 s^D_0 s^D_2$

Reduced words and orderings of $I(w)$

$I(w) = \{\alpha \in \Phi^+ : w\alpha \in -\Phi^+\}$ is the inversion set of w

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Question: How commutations and braid relations affect the orderings of $I(w)$?

Reduced words and orderings of $I(w)$

Lemma (Humphreys,1990)
Let $w \in W$ and $a, b \in R(w)$

1. *If b differ by a single commutation aplied to a, then the effect on the total order $<_a$ is to replace two consecutive positive roots α, β by β, α, where $\alpha + \beta \notin I(w)$.*
2. *If b differ by a single 3-braid move aplied to a, then the effect on the total order $<_a$ is to replace three consecutive roots of the form $\alpha, \alpha+\beta, \beta$ by $\beta, \alpha+\beta, \alpha$.*
3. *If b differ by a single 4-braid move aplied to a, then the effect on the total order $<_a$ is to replace four consecutive roots of the form $\alpha, \alpha+\beta, \alpha+2\beta, \beta$ by $\beta, \alpha+2\beta, \alpha+\beta, \alpha$, or vice-versa*

Distances in $G_c(w)$

Lower bound

$r_w(a,b) = |\{(\alpha,\beta) \in I(w)^2 : \alpha <_a \beta, \beta <_b \alpha,\ \alpha+\beta \in I(w),\ 2\alpha+\beta, \alpha+2\beta \notin I(w)\}|$
is the rank of b with respect to a

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Properties of r_w:

- If $a' \in [a]$ and $b' \in [b]$, then $r_w(a',b') = r_w(a,b)$.
- If b and c differ by a single braid relation, then $|r_w(a,b) - r_w(a,c)| = 1$

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Question: Is the other inequality true?? No...

Smallest examples:

- Type A: $w = (6,5,4,3,2,1) \in \mathfrak{S}_6$
- Type B: $w = (\overline{1},\overline{2},\overline{3},\overline{4}) \in \mathfrak{S}_4^B$
- Type D: $w = (\overline{1},\overline{2},\overline{3},\overline{4}) \in \mathfrak{S}_4^D$

Necessary conditions

Proposition (Mamede, Santos and Soares)
Suppose that $d_w([a],[b]) = r_w([a],[b])$ *for all* $[a],[b] \in C(w)$.

1. *If* $w \in \mathfrak{S}_n$, *then* w *is* $654321-$*avoiding.*
2. *If* $w \in \mathfrak{S}_n^B$ *or* $w \in \mathfrak{S}_n^D$, *then* w *is* $\overline{1}\ \overline{2}\ \overline{3}\ \overline{4}$ *and* $654321-$*avoiding.*

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The proof of the previous proposition relys on the following lemma.

Lemma (Mamede, Santos and Soares)
Let $u, v \in W$ *such that* $u \leq_{WB} v$. *Then* $G_c(u)$ *can be isometrically embedded in* $G_c(v)$.

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The converse of the previous proposition is still unknown...

Longest elements

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$w_{0,n} := (n, n-1, \ldots, 1)$ is the longest elements of $\mathfrak{S}_n$

$w_{0,n}^B := (\overline{1}, \overline{2}, \ldots, \overline{n})$ is the longest elements of $\mathfrak{S}_n^B$

$w_{0,n}^D := \begin{cases} (\overline{1}, \overline{2}, \ldots, \overline{n}) & \text{if n is even} \\ (1, \overline{2}, \ldots, \overline{n}) & \text{if n is odd} \end{cases}$ is the longest elements of $\mathfrak{S}_n^D$

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Proposition (Mamede, Santos and Soares)

1. $d_{w_{0,n}} = r_{w_{0,n}}$ *if and only if* $n < 6$.
2. $d_{w_{0,n}^B} = r_{w_{0,n}^B}$ *if and only if* $n < 4$.
3. $d_{w_{0,n}^D} = r_{w_{0,n}^D}$ *if and only if* $n < 4$.

Longest elements

Theorem (Gutierres, Mamede and Santos; Mamede, Santos and Soares)

1. *The graph $G_c(w_{0,n})$ is plannar if and only if $n < 6$.*
2. *The graph $G_c(w^B_{0,n})$ is plannar if and only if $n < 4$.*

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Corollary (Mamede, Santos and Soares)
Let w_0 denote the longest elements of type A and B. Then, $d_{w_0} = r_{w_0}$ if and only if $G_c(w_0)$ is plannar.

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We conjecture that previous result is also true for type D (It remains to check the plannarity of $G_c(w^D_{0,4})$).

Longest elements

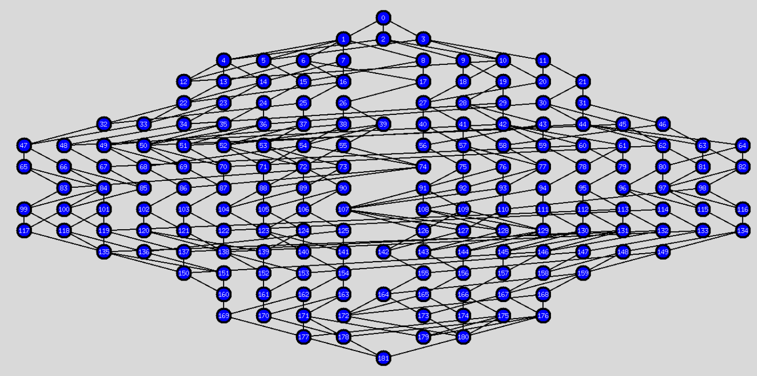

Figure 1: The graph $G_c(w_{0,4}^D)$

Longest elements

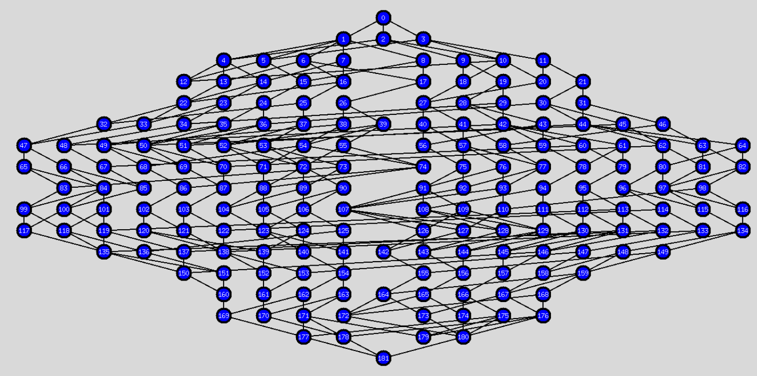

Figure 1: The graph $G_c(w_{0,4}^D)$

The previous result is not true in general! For instance, if $w = [\overline{3}, \overline{1}, \overline{2}, \overline{4}] \in \mathfrak{S}_4^B$ we have that $d_w = r_w$ but $G_c(w)$ is not plannar.

Type B and new graph structures

Type B

Recall that $\mathfrak{S}_{n+1}^B$ has the following presentation:

Generators: $\{s_0^B, s_1^B, \ldots, s_n^B\}$, with $s_0^B = (\bar{1}\ 1)$ and $s_i^B = (i\ i+1)(\overline{i+1}\ \bar{i})$

Relations

- $s_i^B s_j^B = s_j^B s_i^B,$ if $|i - j| > 1$
- $s_i^B s_{i+1}^B s_i^B = s_{i+1}^B s_i^B s_{i+1}^B,$ if $i \neq 0$
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Recall that $\mathfrak{S}^B_{n+1}$ has the following presentation:

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Relations

- $s^B_i s^B_j = s^B_j s^B_i$, if $|i - j| > 1$
- $s^B_i s^B_{i+1} s^B_i = s^B_{i+1} s^B_i s^B_{i+1}$, if $i \neq 0$
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- $s^B_i s^B_{i+1} s^B_i = s^B_{i+1} s^B_i s^B_{i+1}$, if $i \neq 0$
- $s^B_0 s^B_1 s^B_0 s^B_1 = s^B_1 s^B_0 s^B_1 s^B_0$

3-braid relation

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- $s^B_i s^B_j = s^B_j s^B_i$, if $|i - j| > 1$
- $s^B_i s^B_{i+1} s^B_i = s^B_{i+1} s^B_i s^B_{i+1}$, if $i \neq 0$
- $s^B_0 s^B_1 s^B_0 s^B_1 = s^B_1 s^B_0 s^B_1 s^B_0$

4-braid relation

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$C_3(w)$ is the set of equivalence classes for the relation $\sim_3$

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For $w \in \mathfrak{S}_n^B$, we have that $G_3(w)$ is the Hasse diagram of an interval $[id, \sigma]$ in the weak bruhat order of $\mathfrak{S}_n$, where $\sigma = neg(w) \cdot w_{0,k}$ and $k = |\{i \in \{1, 2, \ldots, n\} : w(i) < 0\}$.

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Corollary
$G_3(w_{0,n}^B)$ is the Hasse diagram of $\mathfrak{S}_n$ with respect to weak Bruhat order. Moreover, $|C_3(w_{0,n}^B)| = n!$.

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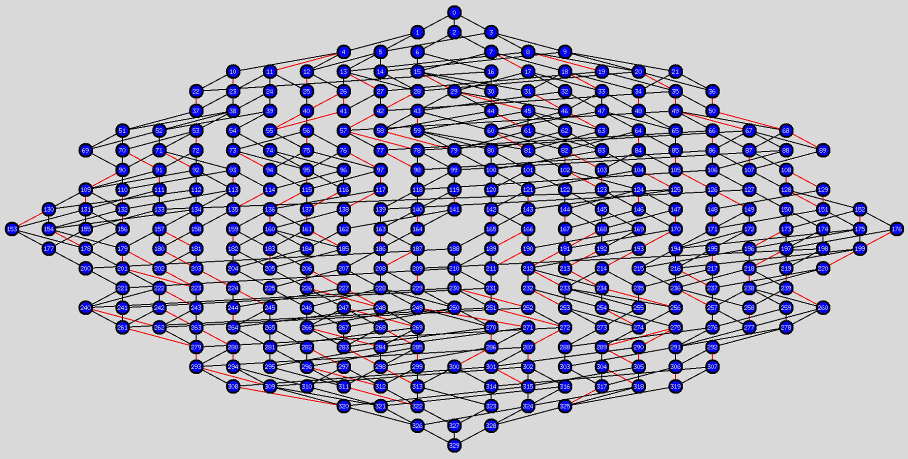

Figure 2: The graph $G_c(w^B_{0,4})$

The graph $G_3(w)$

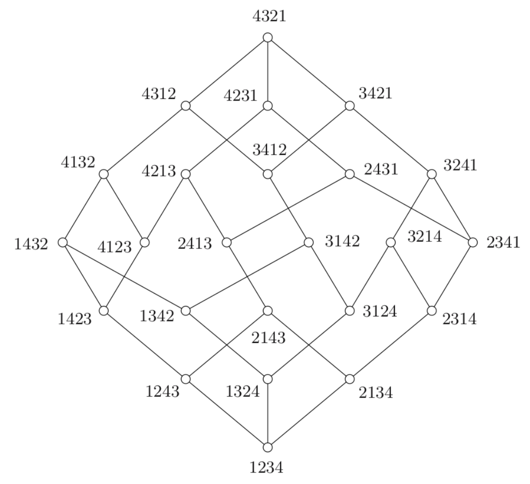

Figure 3: The graph $G_3(w_{0,4}^D)$

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The commutation relation $s_0^D s_1^D = s_1^D s_0^D$ in $\mathfrak{S}_n^D$ is equivalent to the 4-braid relation $s_0^B s_1^B s_0^B s_1^B = s_1^B s_0^B s_1^B s_0^B$ in $\mathfrak{S}_n^B$.

Connections with type D

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In the proof of the previous theorem, we rely on the following well know result.

Theorem (Björner)
An ordering $<$ of $I(w)$ coincides with an ordering of a reduced word $a \in R(w)$ if and only if for all the triples $\alpha, \beta, \alpha+\beta \in \Phi^+$ such that $\alpha, \alpha+\beta \in I(w)$,

1. *$\alpha < \alpha+\beta$, if $\beta \notin I(w)$,*
2. *$\alpha < \alpha+\beta < \beta$, or $\beta < \alpha+\beta < \alpha$ if $\beta \in I(w)$.*

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$$\psi : C_4(w_B) \to C(w_D)$$
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Question: In what cases do we have a bijection??

Conclusion

$w \in \mathfrak{S}_n^B$

- $G_3(w)$ is a closed interval in the weak Bruhat order of $\mathfrak{S}_n$.
- if $w \in \mathfrak{S}_n^D$, then $G_4(w_B)$ is a subgraph of $G_c(w)$.

Thank you for your attention!

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