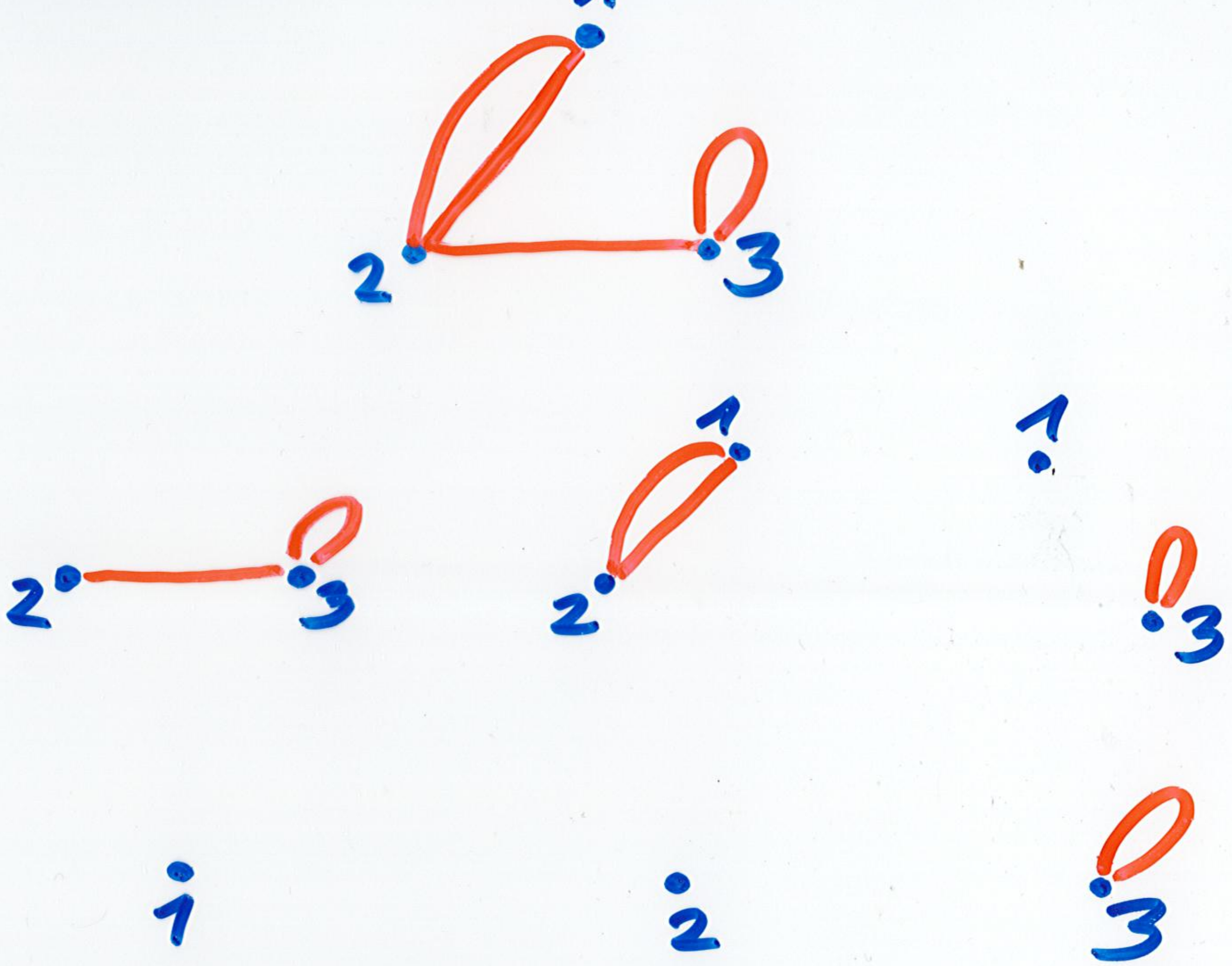
Séries génératrices de matroïdes

Bodo Lass

Lyon

Sous-structures:



$\emptyset$

Is the empty graph a pointless concept?

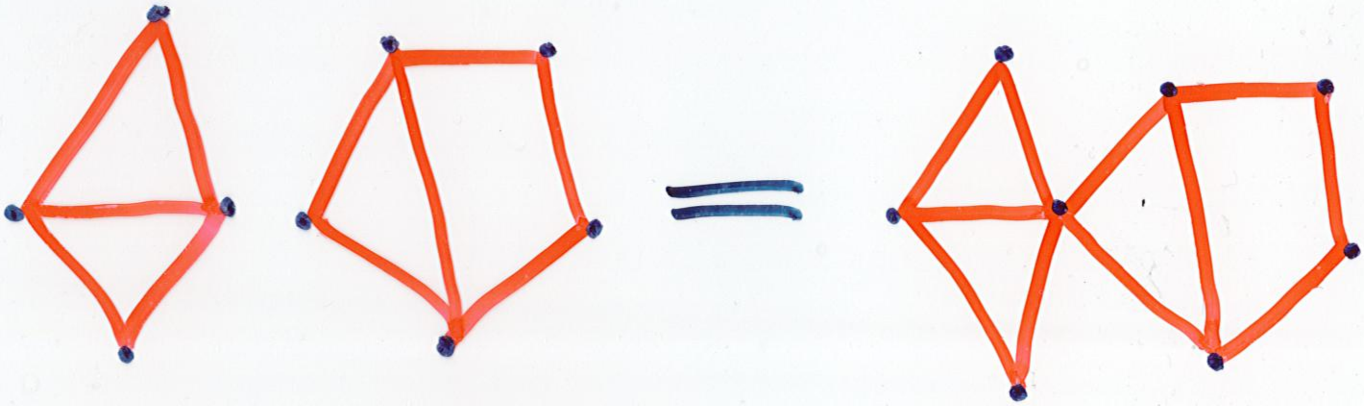
Matroïdes ?

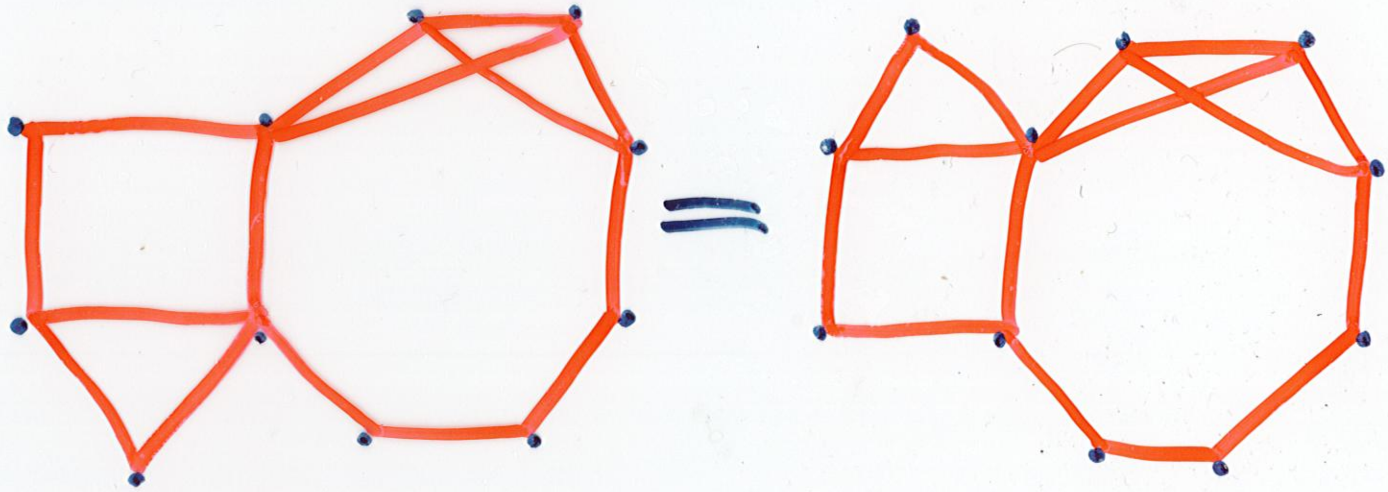
It is a pointless concept !

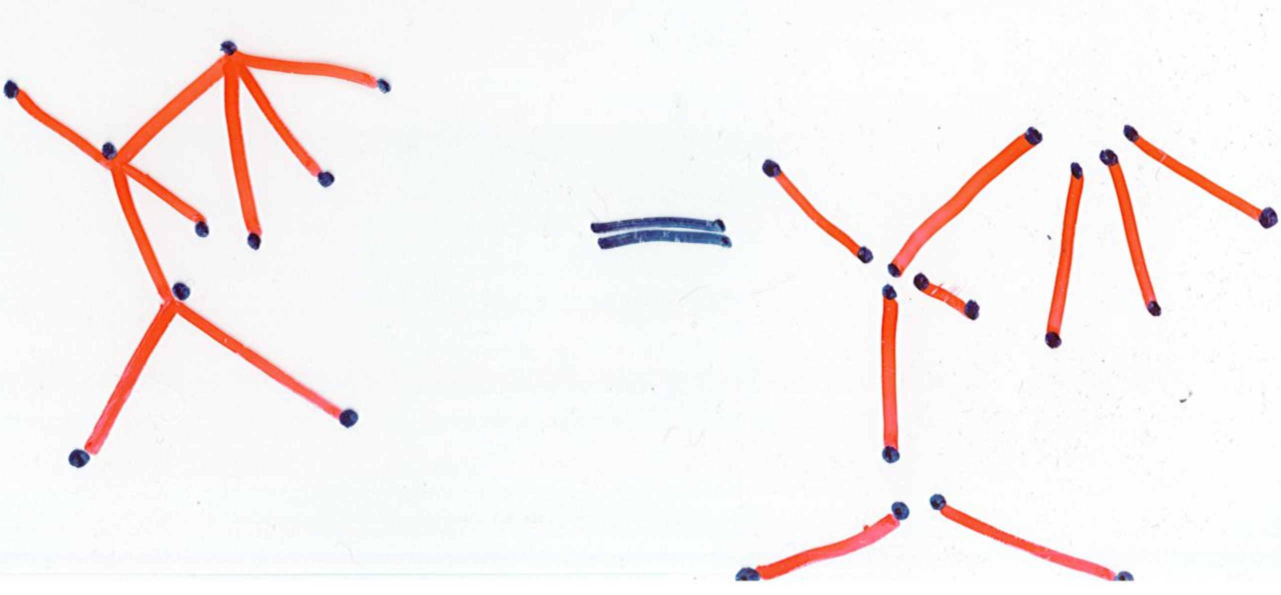
(Harary)

Pourquoi ?

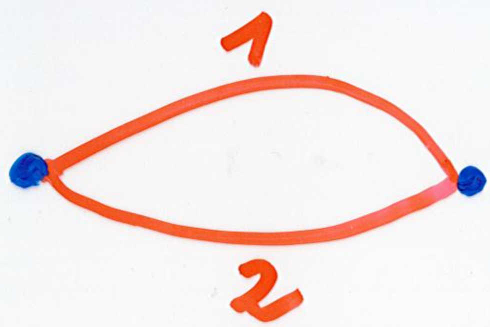
$G = (V, E)$

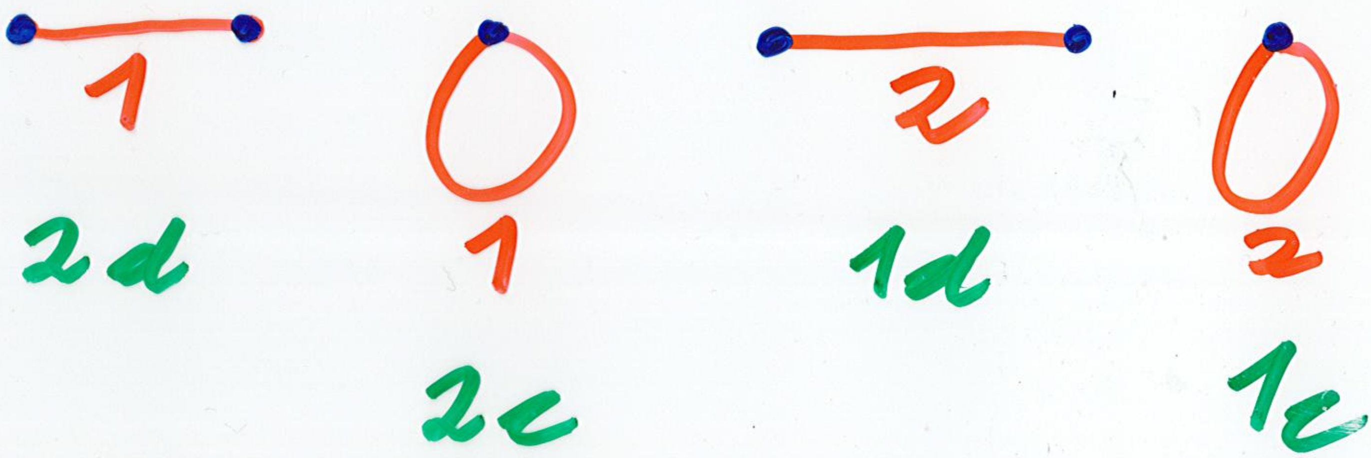






Mineurs !





$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
1 d 2 d	1 d 2 c	1 c 2 d	1 c 2 c

chaque $e \in E$:

supprimer — contracter — garder

d — c — e

(delete) — (contract)

$3^{|E|}$ mineurs

Valeurs :

cardinalité : $|E|$

rang : $r(G) = |V| - c(G)$

G : $c(G)$ composantes connexes

$r(G)$: cardinalité de chaque base = forêt max.

Matroïde

$$M = (E, r),$$

$$|M| := |E|.$$

$$\underbrace{\overline{}_{D'} \quad \overline{}_{E'} \quad \overline{}_{C'}}_{E}$$

$$E = E' \,\dot\cup\, D' \,\dot\cup\, C'$$

mineur: $M \setminus D' / C'$

$d \in D'$: supprimer

$c \in C'$: contracter

$$|M \setminus D' / C'| = |E'|$$

ensemble des mineurs : 3^M,

$|3^M| = 3^{|M|}$.

$\nu : 3^M \to \mathbb{R}$

axiomes :

$$\underbrace{\overline{}}_{D'}\quad \underbrace{\underbrace{\overline{}}_{E''}\quad \underbrace{\overline{}}_{E'''}}_{E'}\quad \underbrace{\overline{}}_{C'}$$

$M' = M \setminus D' / C'$ sur E'

$(E = E' \cup D' \cup C'), (E' = E' \cup E''')$

$M' \setminus E'' = M \setminus (D' \cup E'') / C'$

$M' / E''' = M \setminus D' / (C' \cup E''')$

a) $|M'| = |M'/E'''| + |M' \setminus E''|$

$r(M') = r(M'/E''') + r(M' \setminus E'')$

b) $\forall M' \in 3^M, \ |M'| = 1$

$\Rightarrow r(M') \in \{0, 1\}$

c) $\forall M' \in 3^M, \ |M'| = 2$

$\Rightarrow r(M') \leq r(M' \setminus e) + r(M' \setminus f$

$\Leftrightarrow r(M'/e) \leq r(M' \setminus e)$

(a) $|M'| = 0 \Rightarrow r(M') = 0$

$\Rightarrow$ un seul matroïde sur $\emptyset$

(b) $|M'| = 1 \Rightarrow$

$M' \cong E_0$ (rang 0) 

ou $M' \cong E_1$ (rang 1) 

$|M'| = 2 \Rightarrow$ 

$M' \cong U_{0,2}$, $M' \cong U_{2,2}$

$M' \cong U_{1,2}$, $\underbrace{M' \cong \widetilde{U_{1,2}}}$

axiome 3 : $\widetilde{U_{1,2}}$ est interdit !

L'Algorithme glouton

$$\begin{cases} c : E \to \{w_1, w_2, w_3, \dots, w_k\} \\ w_1 > w_2 > w_3 > \dots > w_k \end{cases}$$

$$c(B) = l_1 w_1 + l_2 w_2 + l_3 w_3 + \dots + l_k w_k$$

$$= l_1 (w_1 - w_2) +$$
$$(l_1 + l_2)(w_2 - w_3) +$$
$$(l_1 + l_2 + l_3)(w_3 - w_4) + \dots +$$
$$(l_1 + l_2 + l_3 + \dots + l_k) w_k$$

$$\leq r(M \setminus c^{-1}\{w_2, w_3, \dots, w_k\})(w_1 - w_2)$$
$$r(M \setminus c^{-1}\{w_3, \dots, w_k\})(w_2 - w_3) +$$
$$r(M \setminus c^{-1}\{w_4, \dots, w_k\})(w_3 - w_4) + \dots +$$
$$r(M) w_k$$

Optimalité $\Longleftrightarrow$

Égalité partout $\Longleftrightarrow$

$B|_{M \setminus c^{-1}\{w_2, w_3, \dots, w_k\}}$ base,

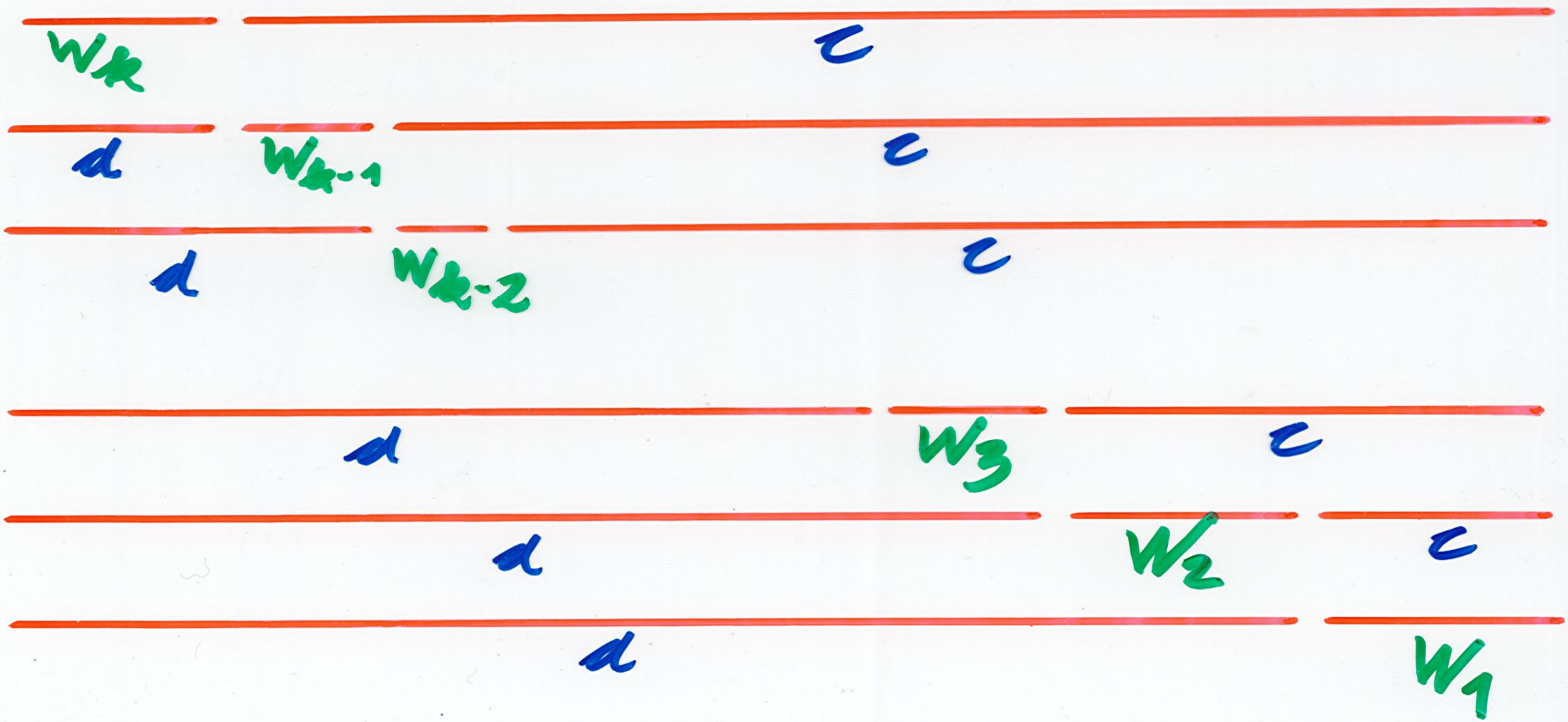
$B|_{M \setminus c^{-1}\{w_3, \dots, w_k\}}$ base,

$B|_{M \setminus c^{-1}\{w_4, \dots, w_k\}}$ base, …,

$B|_{M}$ base $\Longleftrightarrow$

B base de $\bigoplus_{i=1}^{k} M_i$)

$$M_i = M \setminus c^{-1}\{w_{i+1}, w_{i+2}, \dots, w_k\}$$
$$/ \, c^{-1}\{w_{i-1}, w_{i-2}, \dots, w_1\}$$



Wk
c
d
Wk-1
c
d
Wk-2
c
d
W3
c
d
W2
c
d
W1

$$f : 3^M \to A$$

$$M_f(e, \tfrac{d}{c}) := \sum_{E = E' \cup D' \cup C'} f(M \setminus D'/C')\, e^{E'} d^{D'} c^{C'}$$

$$(e^{E'} d^{D'} c^{C'}) \cdot (e^{E''} d^{D''} c^{C''})$$

$$:= e^{E' \cup E''} \, d^{(D' \cup D'') \setminus (E' \cup E'')} \, c^{(C' \cup C'') \setminus (E' \cup E'')}$$

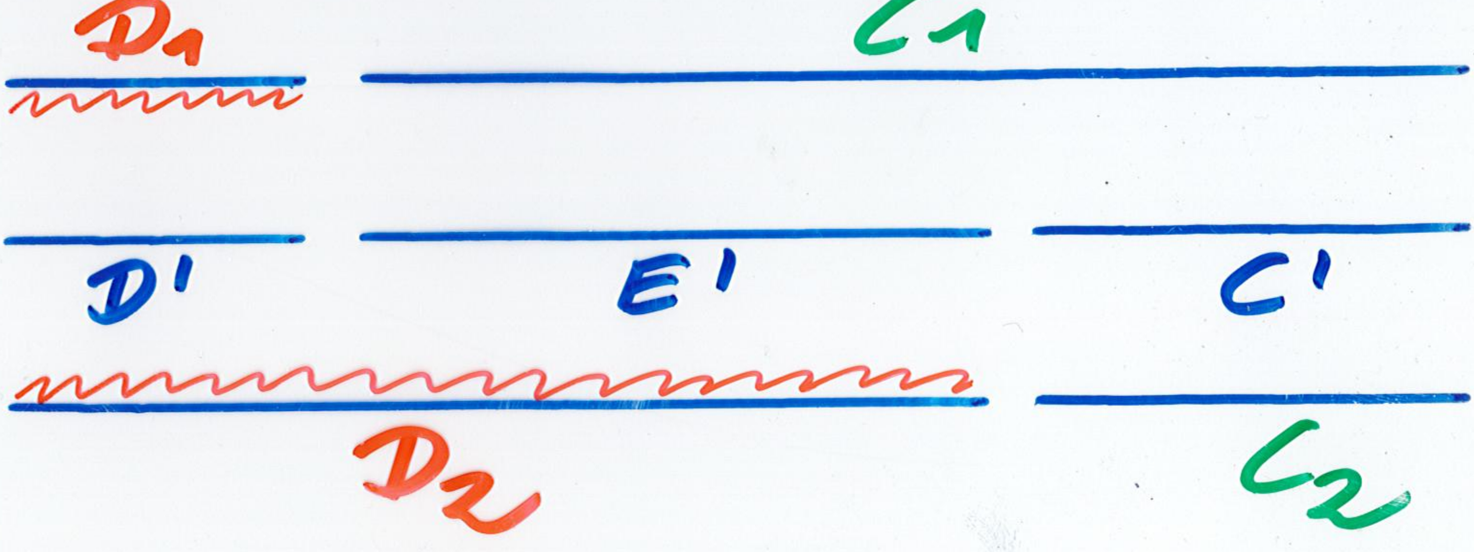
si $E' \cap E'' = E' \cap C'' = D' \cap E''$
$= D' \cap C'' = C' \cap D'' = \emptyset$,

sinon : zéro !

$$1 = \sum_{D \dot{\cup} C = E} d^{D}{}_{C}{}^{G}$$

$\Rightarrow$ algèbre $A[M]$

$\cong$ algèbre d'incidence (Rota) de l'ensemble partiellement ordonné des sous-ensembles de E



Exemple:

$$(fg)(M\backslash D/C) :=$$

$$\sum_{E'\dot\cup E'' = E\backslash(D\cup C)} f(M\backslash D/(\underline{C\cup E''}))\, g(M\backslash(\underline{D\cup E'})/C)$$

$$\Rightarrow M_{fg}\left(e, \frac{d}{c}\right) =$$

$$M_f\left(e, \frac{d}{c}\right) M_g\left(e, \frac{d}{c}\right)$$

$\forall e \in E$: x_e variable

$$\boxed{M_f(x_e e, z_c^d) := \sum_{E' \dot\cup D' \dot\cup C' = E} f(M \setminus D'/C') \cdot \left(\prod_{e \in E'} x_e\right) \cdot z^{r(M \setminus D'/C')} \cdot e^{E'} d^{D'} c^{C'}}$$

$M_f(0, {}_c^d) = 0 \Rightarrow$ nilpotent

$$E_M(e, {}_c^d) := \sum_{e \in E} \sum_{D \dot\cup C = E \setminus e} e^{\{e\}} d^D c^C$$

$$\Downarrow$$

$$\exp[E_M(x_e e, z_c^d)]$$

$$= \sum_{E' \dot\cup D' \dot\cup C' = E} \left(\prod_{e \in E'} x_e\right) \cdot z^{r(M \setminus D'/C')} \cdot e^{E'} d^{D'} c^{C'}$$

Flots et Tensions

$G=(V,E)$; ∂ : matrice d'incidence

$(v,e)\in V\times E \Rightarrow \partial(v,e)\in\{0,\pm1\}$

$$0 \to C_n(E) \xrightarrow{\partial} C_n(V) \to 0$$

$$0 \leftarrow C_n(E) \xleftarrow{\partial^T} C_n(V) \leftarrow 0$$

C_n : group abélien, card n

n flots : $\ker \partial$

n tensions : $\operatorname{im} \partial^T$

diff. potentiel

n-tensions(T_n) : $n^{r(G)}$

n-flots(F_n) : $n^{|E|-r(G)}$

$$E' = E'' \cup E'''$$

$$0 \longrightarrow T_n(M'/E''') \rightarrow T_n(M') \longrightarrow T_n(M' \setminus E'') \rightarrow 0$$

$$0 \leftarrow F_n(M'/E''') \leftarrow F_n(M') \leftarrow F_n(M' \setminus E'') \leftarrow 0$$

Théorème de dualité

$$\exp\left[E_M\left(e, n\tfrac{d}{e}\right)\right] =$$

$$\left[1+\theta_{M,\overline{C_n(e)}}\left(e,\tfrac{d}{e}\right)\right]\left[1+\theta_{M,C_n(e)}\left(-e,\tfrac{d}{e}\right)\right]^{-1}$$

$$\exp\left[E_M\left(n, \tfrac{1}{n}\tfrac{d}{e}\right)\right] =$$

$$\left[1+\phi_{M,C_n(e)}\left(-e,\tfrac{d}{e}\right)\right]^{-1}\left[1+\phi_{M,\overline{C_n(e)}}\left(e,\tfrac{d}{e}\right)\right]$$

$$1+\Theta_{M,\lambda}\left(e,\frac{d}{c}\right)=$$

$$\exp\left[E_M\left(e,\lambda\frac{d}{c}\right)\right]\exp\left[E_M\left(-e,\frac{d}{c}\right)\right]$$

$$1+\Phi_{M,\mu}\left(e,\frac{d}{c}\right)=$$

$$\exp\left[E_M\left(-e,\frac{d}{c}\right)\right]\exp\left[E_M\left(\mu e,\frac{1}{\mu}\frac{d}{c}\right)\right]$$

$$\int \chi_G(\lambda)=\lambda^{c(G)}\,\Theta_G(\lambda)$$

On a

$$E_M\left(x_e e,\frac{d}{c}\right)\cdot E_M\left(y_e e,\frac{d}{c}\right)=$$

$$E_M\left(y_e e,\frac{d}{c}\right)\cdot E_M\left(x_e e,\frac{d}{c}\right),$$

où $\frac{d}{c}$ peut être remplacé

par $z\frac{d}{c}$

Lemme fondamental

$E_M(x_e e, z\frac{d}{e})$ et

$E_M(y_e e, z\frac{d}{e})$ commutent,

$E_M(x_e e, z\frac{d}{e}) + E_M(y_e e, z\frac{d}{e})$

$= E_M((x_e + y_e)e, z\frac{d}{e})$

$\Rightarrow \exp\left[E_M((x_e+y_e)e, z\frac{d}{e})\right]$

$= \exp\left[E_M(x_e e, z\frac{d}{e})\right] \cdot \exp\left[E_M(y_e e, z\frac{d}{e})\right]$

Def.: $1 + R_{M,\lambda,\mu,x_e,y_e}(e, \frac{d}{e})$

$:= \exp\left[E_M(x_e e, \lambda\frac{d}{e})\right]$

$\exp\left[E_M(y_e \mu e, \frac{1}{\mu}\frac{d}{e})\right]$

Théorème

$$\exp\left[E_M\left(t_e' e, \lambda \tfrac{d}{e}\right)\right]\left[1+\Theta_{M,\lambda}\left(t_e e, \tfrac{d}{e}\right)\right]$$

$$\exp\left[E_M\left((t_e+f_e)e, \tfrac{d}{e}\right)\right]$$

$$\left[1+\Phi_{M,\mu}\left(f_e e, \tfrac{d}{e}\right)\right]\exp\left[E_M\left(f_e' \mu e, \tfrac{1}{\mu}\tfrac{d}{e}\right)\right]$$

$$= \exp\left[E_M\left((t_e+t_e')e, \lambda \tfrac{d}{e}\right)\right]$$

$$\exp\left[E_M\left((f_e+f_e')\mu e, \tfrac{1}{\mu}\tfrac{d}{e}\right)\right]$$

$$= 1 + R_{M,\lambda,\mu,t_e+t_e',f_e+f_e'}\left(e, \tfrac{d}{e}\right)$$

Corr.: $\left[1+\Theta_{M,-\lambda}\left(e,-\tfrac{d}{e}\right)\right]\left[1+\Phi_{M,-\mu}\left(-e,-\tfrac{d}{e}\right)\right]$

$$= 1 + R_{M,\lambda,\mu}\left(e, \tfrac{d}{e}\right)$$

Stanton, Kook, Reiner, Las Vergnas, ...

$$1+\Theta_{M,q,t}\left(e,\frac{d}{c}\right)$$

$$:=\left[1+\Theta_{M,q}\left(te,\frac{d}{c}\right)\right]\exp\left[E_M\left(e,\frac{d}{c}\right)\right]$$

$$=\exp\left[E_M\left(te,q\frac{d}{c}\right)\right]\exp\left[E_M\left((1-t)e,\frac{d}{c}\right)\right]$$

$$=1+R_{M,\underset{\nwarrow 4.}{\frac{q\cdot t}{1-t}},\underset{\nwarrow 2.}{\frac{1-t}{t}}}\left(\underset{\nwarrow 1.}{te},\underset{\nwarrow 3.}{\frac{1-t}{t}}\frac{d}{c}\right)$$

$\iint$C. Greene

Ю. В. Матиясевич:

$$\chi(G,\lambda) = \lambda^{c(G)}\,\theta(G,\lambda)$$

$$= \frac{(\lambda-1)^{e(G)}}{\lambda^{e(G)-|V|}} \sum_{H\leq G} \frac{\mathcal{C}(H,\lambda)}{(1-\lambda)^{e(H)}}$$

$$\Updownarrow$$

$$\sum_{H\leq G} \mathcal{C}(H,\lambda)\,(1-\lambda)^{e(G)-e(H)}$$

$$= \theta(G,\lambda)\cdot(-1)^{e(G)}\,\lambda^{e(G)-|V|+c(G)}$$

Значит: $\exp\left[E_M\left((1-\lambda)e\,\frac{d}{e}\right)\right]\left[1+\mathcal{C}_{M,\lambda}\left(e,\frac{d}{e}\right)\right]$

$$= \exp\left[E_M\left((1-\lambda)e,\frac{d}{e}\right)\right]\exp\left[E_M\left(-e,\frac{d}{e}\right)\right]$$

$$\exp\left[E_M\left(\lambda e,\frac{1}{\lambda}\frac{d}{e}\right)\right]$$

$$= \exp\left[E_M\left(-\lambda e,\frac{d}{e}\right)\right]\exp\left[E_M\left(\lambda e,\frac{1}{\lambda}\frac{d}{e}\right)\right]$$

$$= 1+\theta_{M,\lambda}\left(-\lambda e,\frac{1}{\lambda}\frac{d}{e}\right)$$

$f : 3^M \to A$ ne dépend pas des boucles

$\Downarrow$

$F_f(e, \tfrac{d}{c}) \cdot \exp[-E_0]$

compte f uniquement pour des mineurs sans boucles

$1 + \theta_{M,\lambda}(e, \tfrac{d}{c})$

$= \exp[E(e, \lambda \tfrac{d}{c})] \exp[E(-e, \tfrac{d}{c})]$

$= \exp[E(e, \lambda \tfrac{d}{c})] \exp[-E_0] \cdot$

$\underbrace{\exp[E_0] \exp[E(-e, \tfrac{d}{c})]}_{\text{fonction de Möbius}}$

$\underbrace{\exp[E(e, \tfrac{d}{c})] \exp[-E_0]}_{\text{fonction zeta}}$

L'algèbre d'incidence du treillis des sous-ensembles clos de M est isomorphe aux séries génératrices de matroïdes qui valent zéro en présence de boucles.

Dém : produit : pour chaque partition, une boucle se trouve dans l'un des blocs

inversion : v. Neum.

Lemme de Minty

$$\left[1+A_m\left(e,\frac{d}{e}\right)\right]\left[1+S_m\left(e,\frac{d}{e}\right)\right]$$

$$=\exp\left[E_M\left(2e,\frac{d}{e}\right)\right]$$

$$1 + A_m\left(e, \frac{d}{c}\right) = 1 + \Theta_{M,-1}\left(e, -\frac{d}{c}\right)$$

$$= \exp\left[E_M\left(e, \frac{d}{c}\right)\right] \exp\left[E_M\left(-e, -\frac{d}{c}\right)\right]$$

$$1 + S_M\left(e, \frac{d}{c}\right) = 1 + e_{M,-1}\left(-e, -\frac{d}{c}\right)$$

$$= \exp\left[E_M\left(e, -\frac{d}{c}\right)\right] \exp\left[E_M\left(e, \frac{d}{c}\right)\right]$$

$$\partial^e \left[1 + T_{M,\lambda,\mu}\left(e, \frac{d}{c}\right)\right]$$

$$= \exp\left[\lambda E_1 + E_M\left(e, -\frac{d}{c}\right)\right]$$

$$\left[\lambda \partial^e E_1 + \mu \partial^e E_0\right]$$

$$\exp\left[\mu E_0 - E_M\left(e, -\frac{d}{c}\right)\right]$$

$$\beta_1(M) = \frac{\partial}{\partial \lambda} T(M, \lambda, \mu)\Big|_{\lambda=\mu=0}$$

$$\beta_0(M) = \frac{\partial}{\partial \mu} T(M, \lambda, \mu \Big|\Big|_{\lambda=\mu=0}$$

$$\partial^e \beta_1 = \exp\left[E_n\left(e_1 - \frac{d}{c}\right)\right] \partial^e E_1 \exp\left[-E_n\left(e_1 - \frac{d}{c}\right)\right]$$

$$\partial^e \beta_0 = \exp\left[E_n\left(e_1 - \frac{d}{c}\right)\right] \partial^e E_0 \exp\left[-E_n\left(e_1 - \frac{d}{c}\right)\right]$$

$$= \exp\left[\mathrm{ad}\, E_n\left(e_1 - \frac{d}{c}\right)\right] \left\{ \begin{matrix} \partial^e E_1 \\ \partial^e E_0 \end{matrix} \right.$$

$$\partial^e \beta_0 - \partial^e \beta_1 = \partial^e E_n\left(e_1 - \frac{d}{c}\right)$$

$$\partial^v \chi_{G,-\lambda}(-v) =$$

$$[1+\chi_{G,-\lambda}(-v)]\cdot\lambda\cdot\partial^v A^*_G(v)$$

$$\partial^e \theta_{M,-\lambda}\left(e,-\tfrac{d}{e}\right) =$$

$$\left[1+\theta_{M,-\lambda}\left(e,-\tfrac{d}{e}\right)\right]\cdot(\lambda+1)\cdot\partial^e \beta^1_M\left(e,\tfrac{d}{e}\right)$$

$$\Downarrow \lambda = 1$$

Gioan,

Las Vergnas

$$\partial^e A_M\left(e,\tfrac{d}{e}\right) =$$

$$\left[1+A_M\left(e,\tfrac{d}{e}\right)\right]\cdot 2\cdot\partial^e \beta^1_M\left(e,\tfrac{d}{e}\right)$$

$$\partial^e \Theta_{M,-\lambda}\left(e,-\frac{d}{c}\right) =$$

$$\left[1+\Theta_{M,-\lambda}\left(e,-\frac{d}{c}\right)\right]\cdot(\lambda+1)\cdot\partial^e\beta^1_M\left(e,\frac{d}{c}\right)$$

$$\Big\Downarrow \ \lambda = 1$$

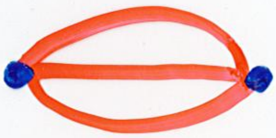
$$\partial^e A_M\left(e,\frac{d}{c}\right) =$$

$$\left[1+A_M\left(e,\frac{d}{c}\right)\right]\cdot 2\cdot\partial^e\beta^1_M\left(e,\frac{d}{c}\right)$$

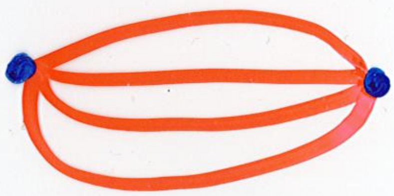
changer l'orientation de $\mathcal{G}$

(Gioan, Las Vergnas)

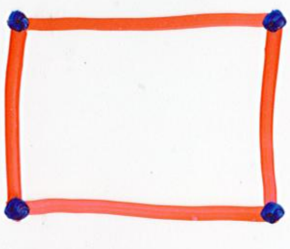
Matroïdes connexes :

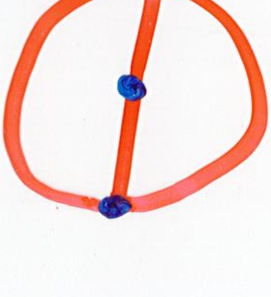
$\mathcal{U}_{1,3}$: 

$\mathcal{U}_{2,3}$: 

$\mathcal{U}_{1,4}$: 

$\mathcal{U}_{2,4}$: ?

$\mathcal{U}_{3,4}$: 

W_2 : 

Algèbre de Lie

$$[E_0, E_1] = 2\, u_{1,2}$$

$$[E_0, u_{1,2}] = 3\, u_{1,3}$$

$$[u_{1,2}, E_1] = 3\, u_{2,3}$$

$$[E_0, u_{1,3}] = 4\, u_{1,4}$$

$$[u_{2,3}, E_1] = 4\, u_{3,4}$$

$$[E_0, u_{2,3}] = [u_{1,3}, E_1]$$

$$= 4\, u_{2,4} + 2\, w_2$$

$$[\partial^{\{e\}} E_0, \partial^{\{f\}} E_1] = \partial^{\{e,f\}} \mathcal{U}_{1,2}$$

$$[\partial^{\{e\}} E_0, \partial^{\{f,g\}} \mathcal{U}_{1,2}] = \partial^{\{e,f,g\}} \mathcal{U}_{1,3}$$

$$[\partial^{\{f,g\}} \mathcal{U}_{1,2}, \partial^{\{e\}} E_1] = \partial^{\{e,f,g\}} \mathcal{U}_{2,3}$$

...

Matroïdes ?

It is a

Lie algebra

concept !

Axiomes

$$\partial^e E_0,\ \partial^e E_1 \geq 0$$

$$\partial^e E_0 + \partial^e E_1 = \partial^e E$$

$$[\partial^e E_0, \partial^f E_0] =$$

$$[\partial^e E_1, \partial^f E_1] = 0$$

prae-matroïdes

matroïdes:

$$[\partial^e E_0, \partial^f E_1] \geq 0$$

$$\begin{vmatrix} 1 & y & yz & x & xz & xyz \\ y & 1 & z & xy & xyz & xz \\ yz & z & 1 & xyz & xy & x \\ x & xy & xyz & 1 & z & yz \\ xz & xyz & xy & z & 1 & y \\ xyz & xz & x & yz & y & 1 \end{vmatrix}$$

$$= (1-x^2)^2(1-y^2)^2(1-z^2)^2$$
$$(1-x^2y^2z^2)^1$$

Varchenko