Quilts of alternating sign matrices

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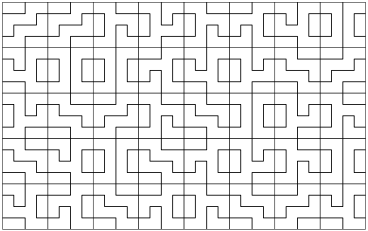
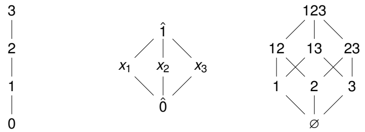


Posets

All posets are *finite*, *ranked*, and have the *greatest element* $\hat{1}$ and the *least element* $\hat{0}$.

Important examples of posets:

- C_n for $n \geq 0$ is the *chain* of rank n;
- $A_2(j)$ for $j \geq 1$ is the *antichain* of j elements, with $\hat{0}$ and $\hat{1}$ added;
- B_n for $n \geq 1$ is the *Boolean lattice* of rank n.



A *k-Dedekind map on P* is map $f\colon P \to [0,k]$ satisfying $f(\hat{0}) = 0$, $f(\hat{1}) = k$, $x \lessdot x' \Rightarrow f(x') - f(x) \in \{0,1\}$ (*Boolean growth rule*).

Alternating sign matrices

An *alternating sign matrix (ASM)* is a square matrix with entries in $\{0, 1, -1\}$ such that in each row and each column the non-zero entries alternate and sum up to 1.

Every permutation matrix is an ASM, and there are many other examples.

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 1 & -1 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Alternating sign matrices

The counting sequence is $1, 1, 2, 7, 42, 429, 7436, 218348, \ldots$ (*Robbins numbers*).

Theorem (ASM enumeration)

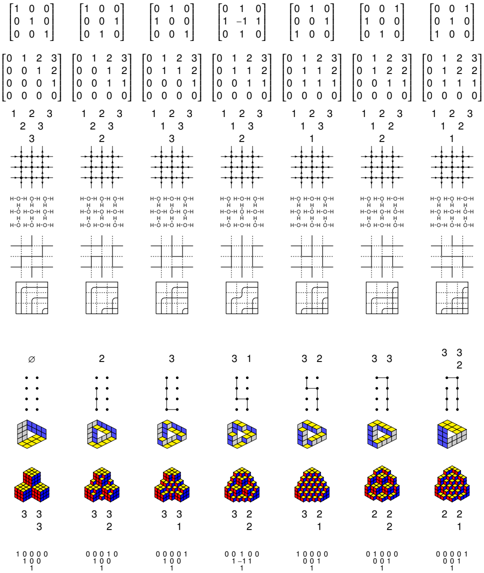
The number of $n \times n$ alternating sign matrices is

$$|\,\mathrm{ASM}_n\,| = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}.$$

ASMs were introduced by Robbins and Rumsey in the 1980s when studying their λ-*determinant*, and they formed the conjecture with Mills.

The conjecture was proved by Zeilberger in the early 1990's. Other significant proofs are due to Kuperberg and Fischer.

Robbins number also count...



Corner sum matrix

Take a real matrix A of size $k \times n$ with rank $\min\{k, n\}$. For $0 \leq i \leq k$, $0 \leq j \leq n$, let $f(i, j)$ denote the *rank of the submatrix* of A consisting of the i bottommost rows and j leftmost columns.

The resulting matrix is a map $f \colon C_k \times C_n \to \mathbb{N}$ satisfying:

- $f(i, 0) = 0$ for $i = 0, \ldots, k$, $f(0, j) = 0$ for $j = 0, \ldots, n$,
- $f(k, n) = \min\{k, n\}$, and
- if $(i, j) \lessdot (i', j')$ in $C_k \times C_n$, then $f(i', j') - f(i, j) \in \{0, 1\}$.

We take the above to be the definition of a *corner sum matrix*.

Bijection with rectangular ASMs

For CSM f, take $a_{i,j} = f(i,j) - f(i,j-1) - f(i-1,j) + f(i-1,j-1)$.

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 3 & 4 & 5 \\ 0 & 1 & 1 & 2 & 3 & 3 & 4 \\ 0 & 0 & 1 & 2 & 2 & 3 & 3 \\ 0 & 0 & 0 & 1 & 2 & 2 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 1 & 0 & -1 & 1 & 0 \\ 1 & -1 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

- all the entries are 0, 1, or -1,
- in each row and each column the non-zero entries alternate,
- the leftmost non-zero entry in every row and the bottommost non-zero entry in every column is 1,
- if $k \leq n$, the rightmost non-zero entry in every row is 1, and
- if $k \geq n$, the topmost non-zero entry in every column is 1.

This is a *rectangular/truncated ASM*.

Monotone triangles (MT)

For an ASM of size $k \times n$ and $1 \leq i \leq k$, take the sum of the i bottom rows and write down the positions of 1s.

For

$$\begin{bmatrix} 0 & 1 & 0 & -1 & 1 & 0 \\ 1 & -1 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

and its transpose, we get

$$\begin{array}{ccccccccc} 1 & & 2 & & 3 & & 5 & & 6 \\ & 1 & & 3 & & 4 & & 6 & \\ & & 2 & & 3 & & 5 & & \\ & & & 3 & & 4 & & & \\ & & & & 3 & & & & \end{array} \quad \text{and} \quad \begin{array}{ccccccccc} 1 & & 2 & & 3 & & 4 & & 5 \\ & 1 & & 2 & & 3 & & 5 & \\ & & 1 & & 2 & & 4 & & \\ & & 1 & & 3 & & 5 & & \\ & & & 3 & & 5 & & & \\ & & & & 4 & & & & \end{array}$$

Or: record the ascents in CSM.

Interlacing sets

We say that the sets $\{s_1 < s_2 < \cdots < s_p\}$ and $\{t_1 < t_2 < \cdots < t_q\}$ *interlace* when either $p = q - 1$ and

$$t_1 \leq s_1 \leq t_2 \leq s_2 \leq \cdots \leq s_{q-1} \leq t_q$$

or $p = q$ and

$$t_1 \leq s_1 \leq t_2 \leq s_2 \leq \cdots \leq t_q \leq s_q$$

Consecutive rows of MTs interlace:

$$\begin{array}{ccccccccc}
1 & & 2 & & 3 & & 5 & & 6 \\
& 1 & & 3 & & 4 & & 6 & \\
& & 2 & & 3 & & 5 & & \\
& & & 3 & & 4 & & & \\
& & & & 3 & & & &
\end{array}
\qquad
\begin{array}{ccccccccc}
1 & & 2 & & 3 & & 4 & & 5 \\
& 1 & & 2 & & 3 & & 5 & \\
& & 1 & & 2 & & 4 & & \\
& & 1 & & 3 & & 5 & & \\
& & & 3 & & 5 & & & \\
& & & & 4 & & & &
\end{array}$$

Quilts of alternating sign matrices

Take a real matrix A of size $k \times n$ with rank $\min\{k, n\}$. For $I \subseteq [k]$, $J \subseteq [n]$, let $f(I, J)$ denote the *rank of the submatrix* of A consisting of rows in I and columns in J.

We get a map $f\colon B_k \times B_n \to \mathbb{N}$ satisfying:

- $f(I, \emptyset) = 0$ for all $I \in B_k$, $f(\emptyset, J) = 0$ for all $J \in B_n$,
- $f([k], [n]) = \min\{k, n\}$, and
- if $(I, J) \lessdot (I', J')$ in $B_k \times B_n$, then $f(I', J') - f(I, J) \in \{0, 1\}$.

Quilts of alternating sign matrices

Let P and Q be finite ranked posets with least and greatest elements.

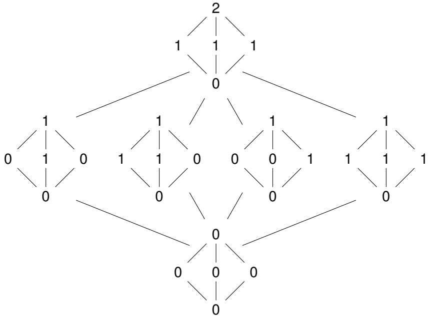
A *quilt of alternating sign matrices of type* (P,Q) (or *ASM quilt* or just *quilt*) is a map $f\colon P \times Q \to \mathbb{N}$ satisfying:

- $f(x,\hat{0}_Q) = 0$ for all $x \in P$, $f(\hat{0}_P, y) = 0$ for all $y \in Q$,
- $f(\hat{1}_P, \hat{1}_Q) = \min\{\operatorname{rank} P,\ \operatorname{rank} Q\}$, and
- if $(x,y) \lessdot (x',y')$ in $P \times Q$, then $f(x',y') - f(x,y) \in \{0,1\}$.

The set of all quilts of type (P,Q) will be denoted by $\mathrm{Quilts}(P,Q)$.

Example of a quilt

The following is an example of a quilt of type $(A_2(4), A_2(3))$:



Quilts of alternating sign matrices

For a quilt f of type (P, Q) and any pair of *maximal chains*

$$\hat{0}_P = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_{k-1} \lessdot x_k = \hat{1}_P, \quad \hat{0}_Q = y_0 \lessdot y_1 \lessdot \cdots \lessdot y_{n-1} \lessdot y_n = \hat{1}_Q$$

in P and Q, the map

$$(i, j) \mapsto f(x_i, y_j)$$

is a CSM of size $k \times n$.

We can think of quilts as encoding *collections of alternating sign matrices*, one for each pair of maximal chains in the two posets, appropriately “pieced” together.

The motivation for the definition came from Billey–Stark work on *Fubini words*, which was in turn motivated by Pawlowski–Rhoades.

Quilt lattice

Theorem

Let P, Q be finite ranked posets with least and greatest elements. The poset $\mathrm{Quilts}(P, Q)$ *is a distributive lattice ranked by*

$$\operatorname{quiltrank} f = \sum_{x \in P,\, y \in Q} f(x, y) - \sum_{x \in P,\, y \in Q} f_{\hat{0}}(x, y),$$

where $f_{\hat{0}}(x, y) = \max\{0, \operatorname{rank} x + \operatorname{rank} y - \max\{n, k\}\}$ *is the least element of* $\mathrm{Quilts}(P, Q)$*. The greatest element of* $\mathrm{Quilts}(P, Q)$ *is* $f_{\hat{1}}(x, y) = \min\{\operatorname{rank} x,\ \operatorname{rank} y\}$*.*

Quilt lattice

Theorem

- *If* φ *is an (involutive) antiautomorphism of* P *and* $\operatorname{rank} P \geq \operatorname{rank} Q$*, then*

  $$\Phi\colon \operatorname{Quilts}(P,Q) \to \operatorname{Quilts}(P,Q), \text{ where } \Phi f(x,y) = \operatorname{rank} y - f(\varphi(x), y)$$

  is an (involutive) antiautomorphism of the lattice $\operatorname{Quilts}(P,Q)$*.*
- *Given an involutive antiautomorphism* $\varphi\colon P \to P$, $\operatorname{rank} P \geq 2$*, there is a faithful action of the dihedral group* D_4 *acting on* $\operatorname{Quilts}(P,P)$*.*

Enumeration of quilts

Enumerating quilts of type (P, C_1) is equivalent to enumerating antichains in P.

For example, the number of antichains in B_n is known only up to $n = 9$.

Theorem

Computing $|\mathrm{Quilts}(P, Q)|$ for general P and Q is a #P-complete problem.

Antichain quilts

A set $S \subseteq P$ is *convex* if $x, y \in S$ implies $[x, y] \subseteq S$.

We say that S is a *cut set* if it intersects every maximal chain in P.

Denote the *number of antichains* in S by $\alpha_P(S)$.

Theorem

We have

$$|\mathrm{Quilts}(P, A_2(j))| = \sum_C \alpha_P(C)^j,$$

where the sum is over all subsets C of $P \setminus \{\hat{0}_P, \hat{1}_P\}$ that are convex cut sets of P. In particular, as j goes to infinity, we have

$$|\mathrm{Quilts}(P, A_2(j))| \sim \alpha(P \setminus \{\hat{0}_P, \hat{1}_P\})^j.$$

Antichain quilts

Corollary

For arbitrary integers $j \geq 1$ and $k \geq 2$, we have

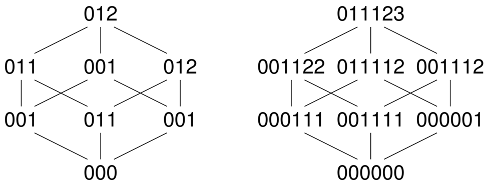
$$|\operatorname{Quilts}(C_k, A_2(j))| = \sum_{i=2}^{k} (k+1-i) i^j$$

$$= \frac{k^{j+2} + \sum_{l=1}^{j} \binom{j+2}{l} \left(l \operatorname{Ber}_{l-1} - (l-1) \operatorname{Ber}_l \right) k^{j+2-l}}{(j+1)(j+2)} + (\operatorname{Ber}_j - \operatorname{Ber}_{j+1} - 1) k.$$

Here Ber_n is the n-the *Bernoulli number*.

Chain quilts

The following are two quilts of type (B_3, C_2) and (B_3, C_5).

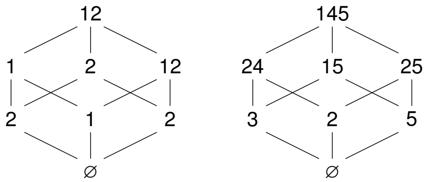


For a quilt f of type (P, C_n) and $x \in P$, define *jumps of f at x* by

$$J_f(x) = \{i \in [n] \colon f(x,i) - f(x,i-1) = 1\} \subseteq [n].$$

Monotone triangle form of a chain quilt

This gives us a represenation of a quilt as a map $P \to B_n$, $x \mapsto J_f(x)$:



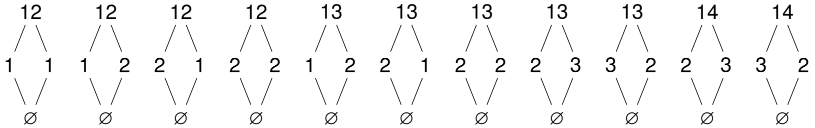
We will call this the *monotone triangle (MT) form* of the quilt f.

For all $x, y \in P$ with $x \lessdot y$, the sets $S = J_f(x)$ and $T = J_f(y)$ are interlacing. When $n \leq \operatorname{rank} P$, $J_f(\hat{1}_P) = [n]$. When $n \geq \operatorname{rank} P$, we have $|J_f(x)| = \operatorname{rank} x$ for all $x \in P$.

Essential and standard quilts

A chain quilt is *m-fundamental* if its MT form contains precisely the elements $1, \ldots, m$. It is *standard* if its MT form contains exactly one of each of $1, \ldots, b(P)$, where $b(P) = \sum_{x \in P} \operatorname{rank} x$.

For example, for $P = B_2$, we have $b(P) = 4$, and there are four 2-fundamental, five 3-fundamental, and two 4-fundamental (standard) quilts:



We denote the set of m-fundamental quilts by $F_m(P)$, and the set of standard quilts by $S(P)$.

Enumeration of chain quilts

Theorem

For a fixed poset P, the number of chain quilts of type (P, C_n), $n \geq \operatorname{rank} P$, is given by a polynomial in n, namely

$$|\operatorname{Quilts}(P, C_n)| = \sum_{m=k}^{b(P)} |F_m(P)| \binom{n}{m}.$$

In particular,

$$|\operatorname{Quilts}(P, C_n)| \sim \frac{|S(P)|}{b(P)!} \cdot n^{b(P)}.$$

Corollary

The number of rectangular ASMs of size $k \times n$, $k \leq n$, is a polynomial in n of degree $\binom{k+1}{2}$ with leading coefficient $\frac{\prod_{i=0}^{k-1}(2i)!}{\prod_{i=0}^{k-1}(k+i)!}$.

Number of chain quilts with given top set

Define $\mathrm{MT}_P(a_1, \ldots, a_k)$ as the set of quilts $f \in \mathrm{Quilts}(P, C_n)$ for which $J_f(\hat{1}_P) = \{a_1, \ldots, a_k\}$.

Theorem

For a finite poset P of rank k with least and greatest elements, we have

$$|\mathrm{MT}_P(a_1, \ldots, a_k)| = \sum_{f \in F(P)} \prod_{i=2}^{k} \binom{a_i - a_{i-1} - 1}{J_f(\hat{1}_P)_i - J_f(\hat{1}_P)_{i-1} - 1},$$

where T_j denotes the j-th largest element of the set T.

Boolean quilts

Theorem

There exist positive numbers A_P and B_P so that if $n \geq \operatorname{rank} P$, we have

$$A_P^{\binom{n}{\lfloor n/2 \rfloor}} \leq |\operatorname{Quilts}(P, B_n)| \leq B_P^{\binom{n}{\lfloor n/2 \rfloor}}.$$

For example,

$$2^{\binom{k}{\lfloor k/2 \rfloor}\binom{n}{\lfloor n/2 \rfloor}} \leq |\operatorname{Quilts}(B_k, B_n)| \leq 2^{k2^{k-1}(1+c\ln n/\sqrt{n})\binom{n}{\lfloor n/2 \rfloor}}$$

for $n \geq 2k$ and some constant $c > 0$.

Future work

- are there other explicit formulas for $|\mathrm{Quilts}(P_n, P_n)|$ for families of posets P_n?
- statistics on quilts
- study the quilt polytope
- does the polynomial $|\mathrm{MT}_P(a_1, \ldots, a_k)|$ have some of the properties of $|\mathrm{MT}_{C_k}(a_1, \ldots, a_k)|$?
- Boolean quilts: improve bounds; study *representable* quilts of type (B_k, B_n); find $\lim_{n\to\infty} \log|\mathrm{Quilts}(P, B_n)|/\binom{n}{\lfloor n/2 \rfloor}$