A Generalization of Conjugation of Integer PARTITIONS

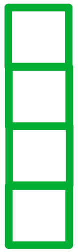
Moritz Gangl

SLC 92

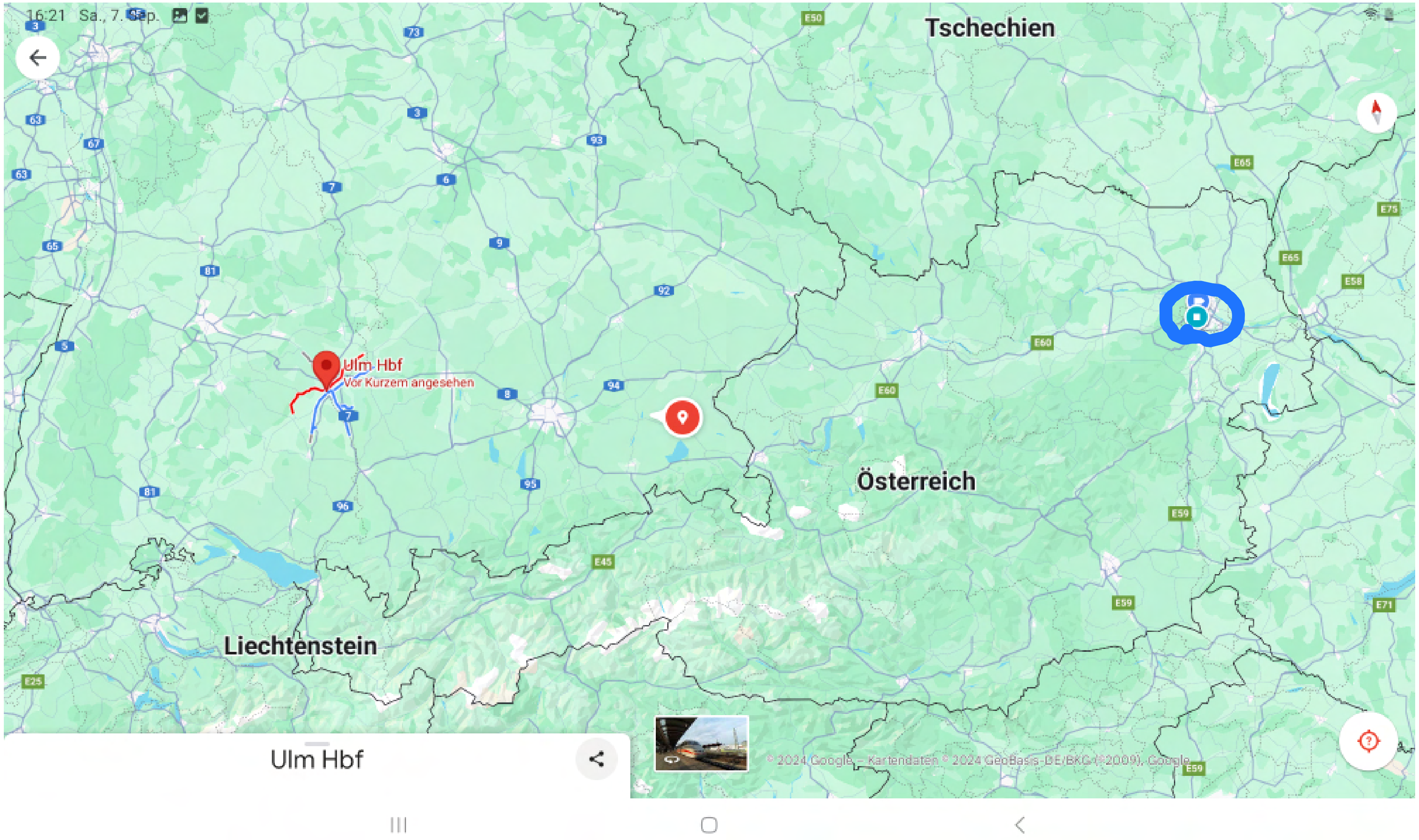
This is joint work with:
Seamus Albion, Theresia Eisenkölbl, Ilse Fischer,
Hans Höngesberg, Christian Krattenthaler & Martin Rubey

Emergence of the Project

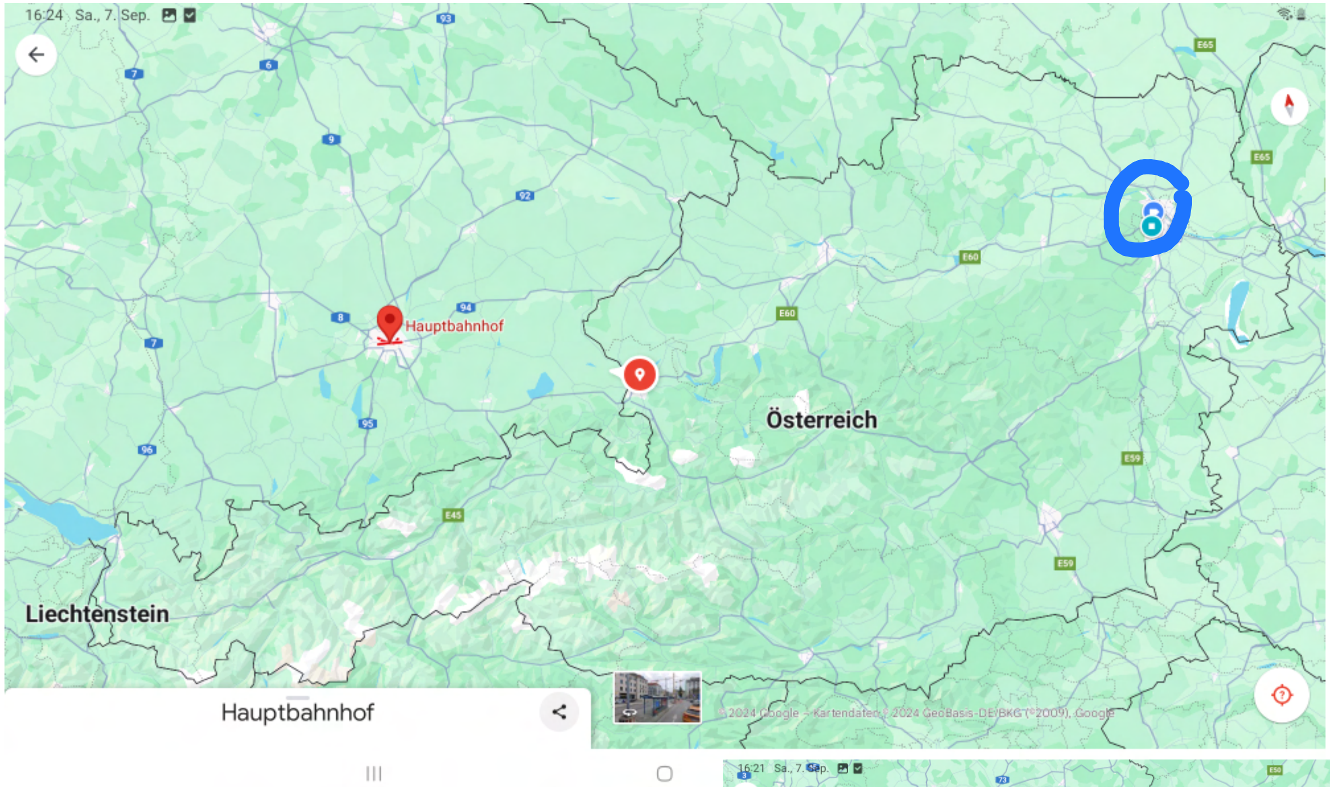
Emergence of the Project

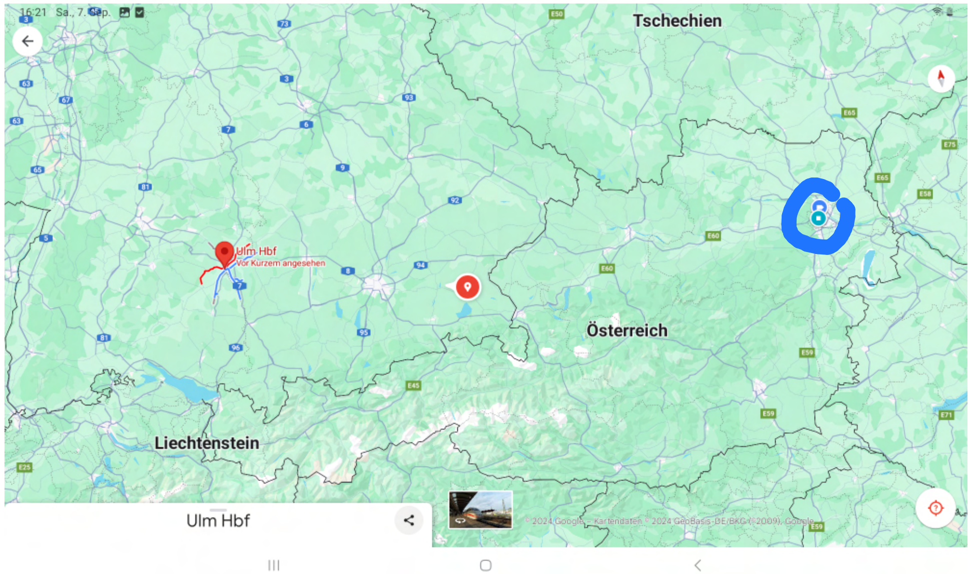


Emergence of the Project



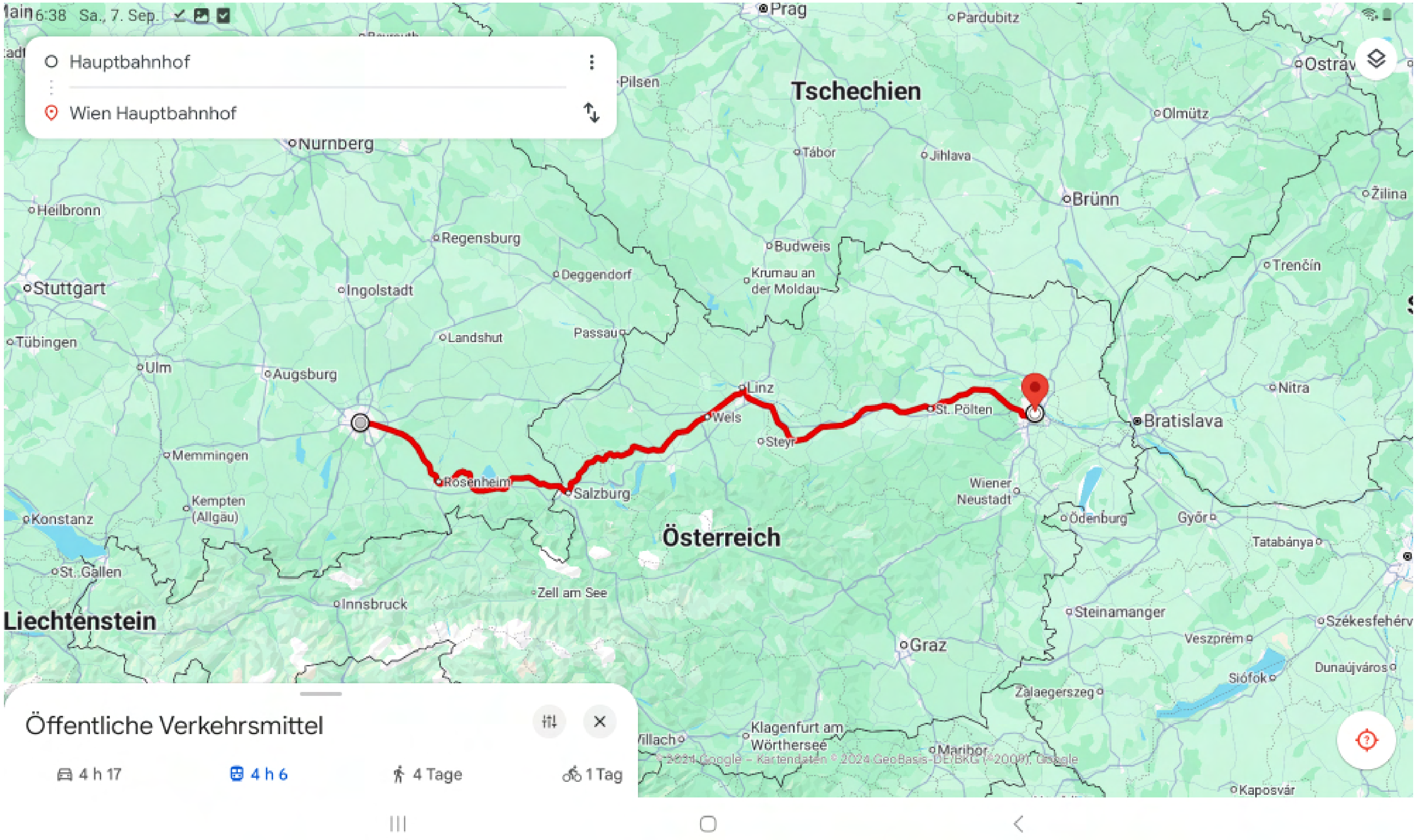
Emergence of the Project





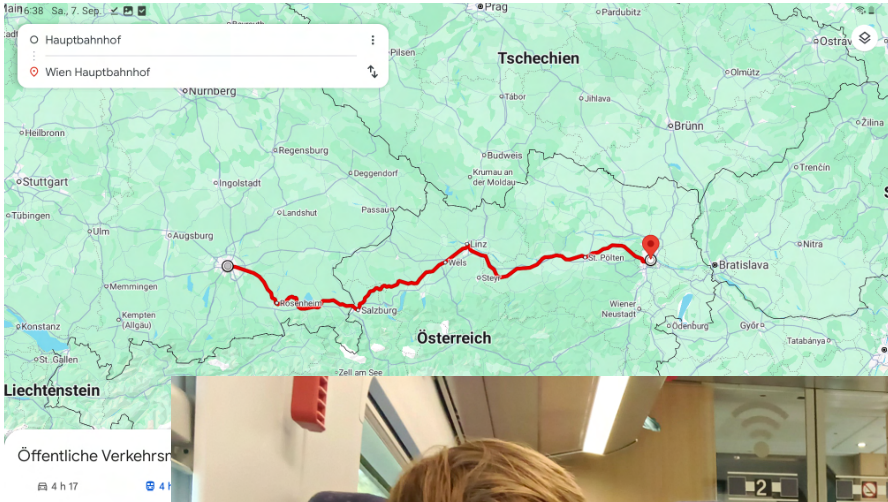
Emergence of the Project

Emergence of the Project



Emergence of the Project

Emergence of the Project



Emergence of the Project

With the help of https://www.findstat.org/ I found two statistics r_s and c_s, for any $s \in \mathbb{N}$, s.t. (r_s, c_s) has a joint symmetric distribution on the set of all integer partitions of a given size.

Emergence of the Project

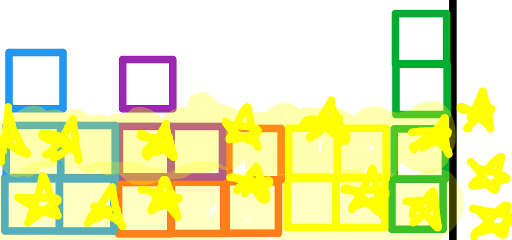
With the help of https://www.findstat.org I found two statistics r_s and c_s, for any $s \in \mathbb{N}$, s.t. (r_s, c_s) has a joint symmetric distribution on the set of all integer partitions of a given size.

But Martin... what are r_s & c_s?

Basic Definitions

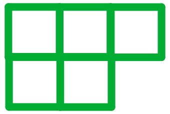
- A partion λ of $n \in \mathbb{N}$ is a weakly decreasing sequence of positive integers that add up to n
  - write $\lambda \vdash n = |\lambda|$ size of λ
  - $\ell(\lambda)$ length of λ
  - λ' the conjugate
  - z a cell in Ferrers diagram of λ then
    - ~ $\mathrm{leg}(z)$ = # cells in same column, strictly below z
    - ~ $\mathrm{arm}(z)$ = # cells in same row, strictly to the right of z

(English convention & Matrix coordinates for cells!!!!)



Basic Definitions

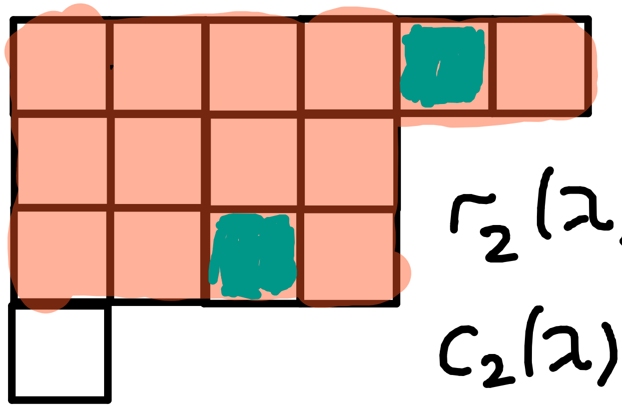
- $\lambda \vdash n$, $s \in \mathbb{N}$ then
  - $r_s(\lambda) := \#$ parts of λ divisible by s
  - $c_s(\lambda) := \#$ of cells z in λ s.t. $\mathrm{leg}(z) = 0$ & $\mathrm{arm}(z) + 1$ is divisible by s.



Basic Definitions

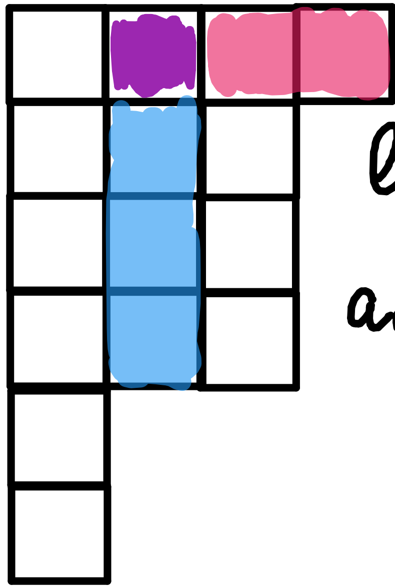
- $\lambda \vdash n$, $s \in \mathbb{N}$ then
  - $r_s(\lambda) := \#$ parts of λ divisible by s
  - $c_s(\lambda) := \#$ of cells z in λ s.t. $\mathrm{leg}(z) = 0$ & $\mathrm{arm}(z)+1$ is divisible by s.

Example: $\lambda = (6,4,4,1)$ & $\lambda' = (4,3,3,3,1,1)$



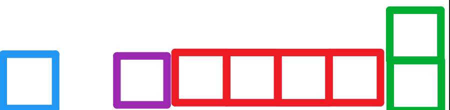
$r_2(\lambda) = 3$

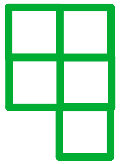
$c_2(\lambda) = 2$

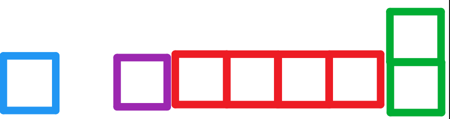


$\mathrm{leg}(1,2) = 3$

$\mathrm{arm}(1,2) = 2$





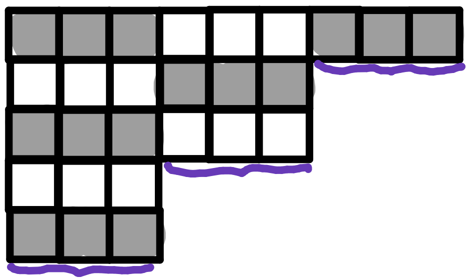
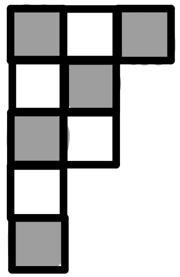


Main goal: Show that

$$\sum_{\lambda \vdash n} R^{r_s(\lambda)} C^{c_s(\lambda)}$$

is symmetric in R and C.

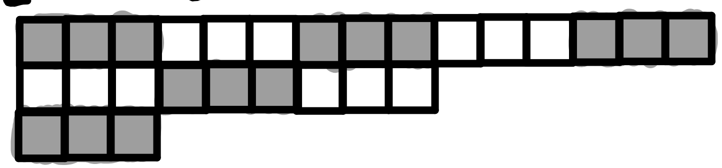
Toy example: Consider only set of $\lambda \vdash n$ s.t. all parts of λ are divisible by s.
Then:

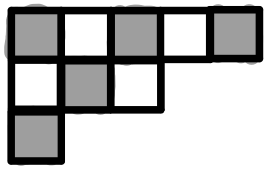
$\lambda =$  $\longmapsto$ $\lambda\downarrow_3 =$ 

$r_3(\lambda) = \ell(\lambda) = 5$, $c_3(\lambda) = 3$

$\ell(\lambda\downarrow_3) = r_3(\lambda)$, $(\lambda\downarrow_3)_1 = c_3(\lambda)$

$[\lambda\downarrow_3]'\uparrow_3 =$



$\longleftarrow$ $[\lambda\downarrow_3]' =$ 

$r_3([\lambda\downarrow_3]'\uparrow_3) = c_3(\lambda)$,

$c_3([\lambda\downarrow_3]'\uparrow_3) = r_3(\lambda)$

$\ell([\lambda\downarrow_3]') = c_3(\lambda)$, $[\lambda\downarrow_3]'_1 = r_3(\lambda)$

Def: $\lambda\downarrow_s := (\lfloor \lambda_1/s \rfloor, \lfloor \lambda_2/s \rfloor, \dots)$

$\lambda\uparrow_s := (s\cdot\lambda_1, s\cdot\lambda_2, \dots)$

Toy example: Consider only set of $\lambda \vdash n$ s.t. all parts of λ are divisible by s.

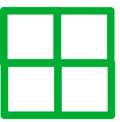
Solution: $\lambda \longmapsto [\lambda\downarrow_s]'\uparrow_s$

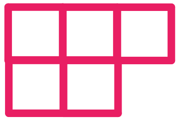
Lem: The generating function w.r.t. $R^{r_s(\lambda)} C^{c_s(\lambda)} q^{|\lambda|}$ of all partitions with all parts divisible by s is given by

$$1 + \sum_{k\geq 1} R^k \frac{CQ^k}{(CQ,Q)_k},$$

where $Q = q^s$.

Proof: Divide every part of λ by s, then $R^k \frac{CQ^k}{(CQ,Q)_k}$ is GF of partitions of length k.
(Consider λ' ...)



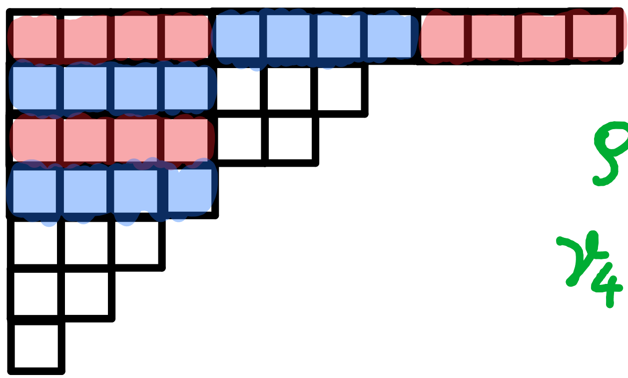


If not all parts of λ are divisible by s, then some parts are $\neq 0$ mod s.

Def: The remainder sequence of λ modulo s is the sequence $\rho_s(\lambda) = (\rho_1, \ldots, \rho_m)$ of non-zero remainders of the parts of λ when dividing by s and reading λ from left to right.
The row position sequence $\gamma_s(\lambda) = (\gamma_1, \ldots, \gamma_m)$ is the sequence of indices $1 \leq \gamma_1 < \gamma_2 < \ldots < \gamma_m$ s.t. λ_{γ_i} has non zero remainde after division by s.

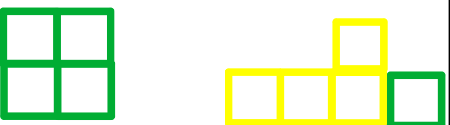
Example:

$\lambda = (12, 7, 6, 4, 3, 2, 1)$



$\rho_4(\lambda) = (3, 2, 3, 2, 1)$

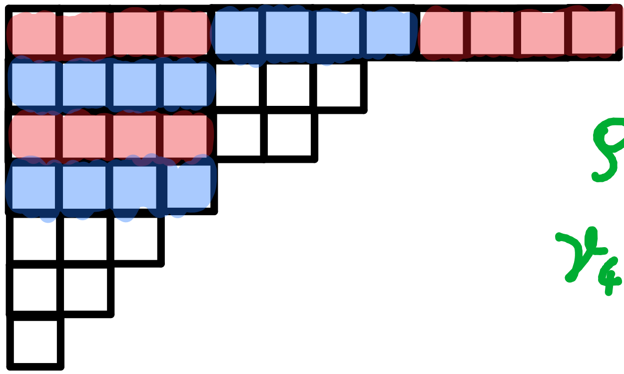
$\gamma_4(\lambda) = (2, 3, 5, 6, 7)$



Def: $\lambda \vdash n$ with $\rho_s(\lambda) = (\rho_1, \dots, \rho_m)$ and $\gamma_s(\lambda) = (\gamma_1, \dots, \gamma_m)$. Then $\Delta_s \lambda$ is the partition obtained by deleting the last ρ_m cells in the γ_m-th row of λ.

Example:

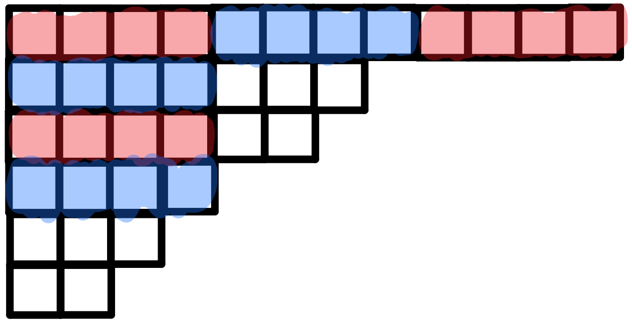
$\lambda = (12,7,6,4,3,2,1)$



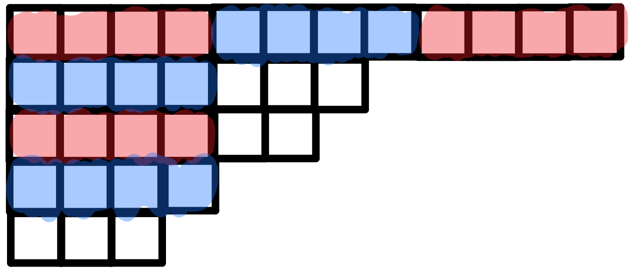
$\rho_4(\lambda) = (3,2,3,2,1)$

$\gamma_4(\lambda) = (2,3,5,6,7)$

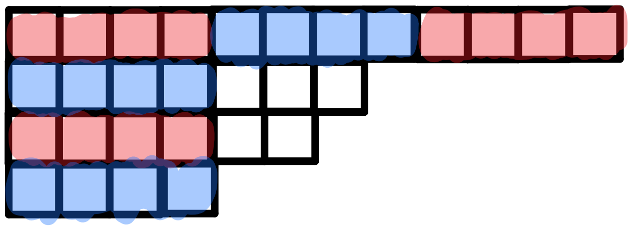
$\Delta_4 \lambda =$



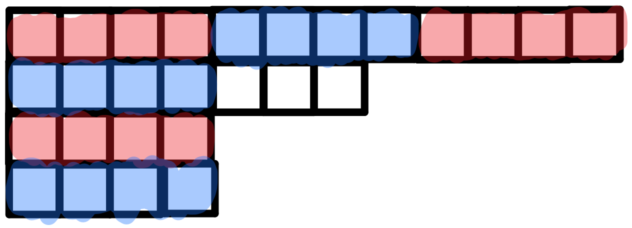
$\Delta_4^2 \lambda =$

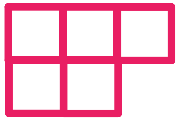


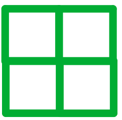
$\Delta_4^3 \lambda =$

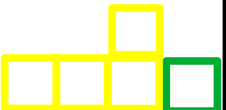


$\Delta_4^4 \lambda =$





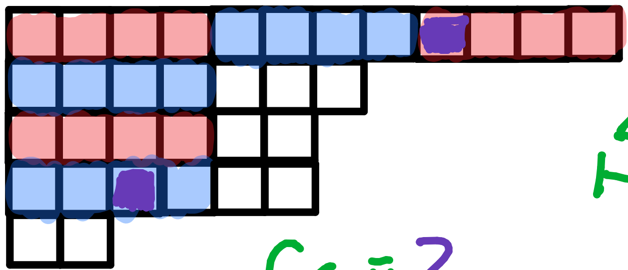




Lem: $\lambda \vdash n$ with $\varrho_s(\lambda) = (\varrho_1, \dots, \varrho_m)$ and $\gamma_s(\lambda) = (\gamma_1, \dots, \gamma_m)$. Then

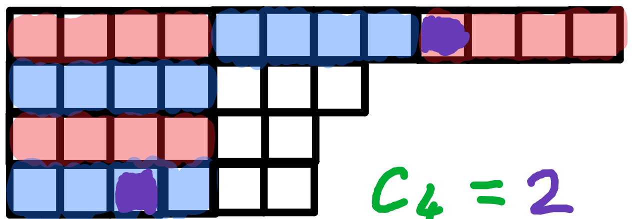
$$c_s(\Delta_s \lambda) = \begin{cases} c_s(\lambda) & m = 1 \;\&\; \gamma_1 = 1 \\ c_s(\lambda) & \gamma_{m-1} = \gamma_m - 1 \;\&\; \varrho_{m-1} \geq \varrho_m \\ c_s(\lambda) + 1 & \text{else} \end{cases}$$

Example:



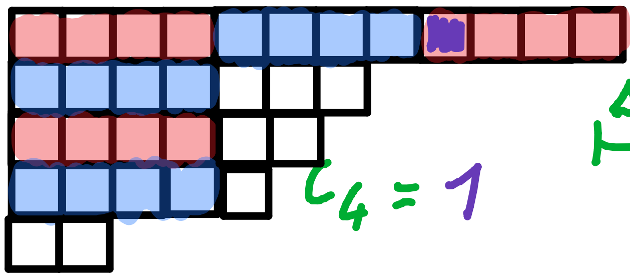
$c_4 = 2$

$\xrightarrow{\Delta_4}$



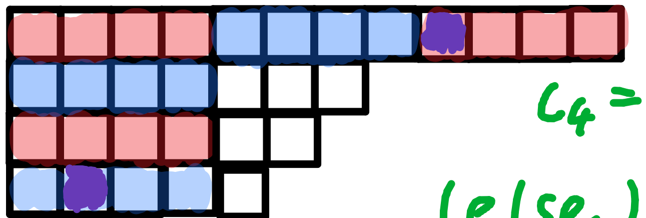
$c_4 = 2$

$(\gamma_4 = \gamma_{m-1} \;\&\; \varrho_4 \geq \varrho_5)$



$c_4 = 1$

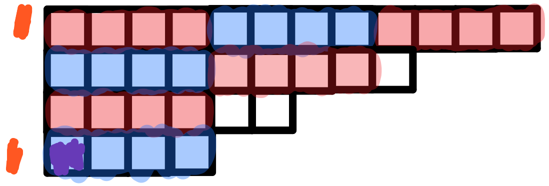
$\xrightarrow{\Delta_4}$



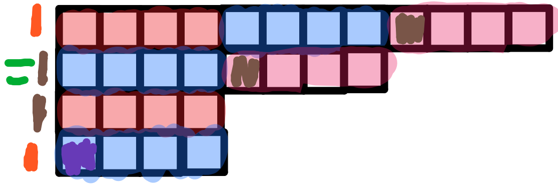
$c_4 = 2$

(else)

Second toy example: Consider only $\lambda \vdash n$ with strictly increasing remainder sequence. Then always $c_s(\Delta_s^k \lambda) = c_s(\lambda) + k$.

$\lambda =$ 

$\xrightarrow{k=2}$

$\Delta_4^2 \lambda =$ 

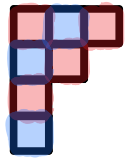
$\rho_4(\lambda) = (1, 2, 3), c_4(\lambda) = 1,$

$r_4(\lambda) = 2$

$c_4(\Delta_4^2 \lambda) = 1 + 2, \; r_4(\lambda) = 2 + 2$

$\searrow k=2$

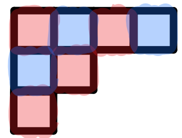
$\downarrow$

$[\Delta_4^2 \lambda]\downarrow_4 = \lambda\downarrow_4 =$ 

$(\lambda\downarrow_4)_1 = 1 + 2$

$\ell(\lambda\downarrow_4) = 2 + 2$

$\downarrow$

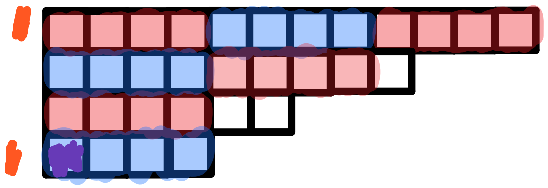
$[\lambda\downarrow_4]' =$ 

$([\lambda\downarrow_4]')_1 = \ell(\lambda\downarrow_4) = 4$

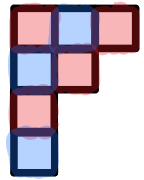
$\ell([\lambda\downarrow_4]') = (\lambda\downarrow_4)_1 = 3$

Blow up & insert remainders again! $\xleftarrow{k=2}$

Second toy example: Consider only $\lambda \vdash n$ with strictly increasing remainder sequence. Then always $c_s(\Delta_s^k \lambda) = c_s(\lambda) + k$.

$\lambda =$ 

$\rho_4(\lambda) = (1, 2, 3)$, $c_4(\lambda) = 1$, $r_4(\lambda) = 2$

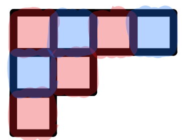
$\overset{k=2}{\longmapsto}$ $\lambda\downarrow_4 =$ 

$(\lambda\downarrow_4)_1 = 1 + 2$

$\ell(\lambda\downarrow_4) = 2 + 2$

$\downarrow$

Blow up & insert remainders again! $\overset{k=2}{\longleftarrow}$

$[\lambda\downarrow_4]' =$ 

$([\lambda\downarrow_4]')_1 = \ell(\lambda\downarrow_4) = 4$

$\ell([\lambda\downarrow_4]') = (\lambda\downarrow_4)_1 = 3$

Problem: Where to insert remainders again?

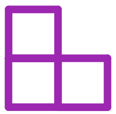
Problem: Where to insert remainders again?

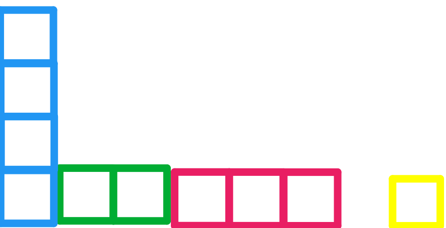
Solution:

Def: Let $\nu_s(\lambda) = (\nu_1, \dots, \nu_m)$ be the row position sequence of λ, then

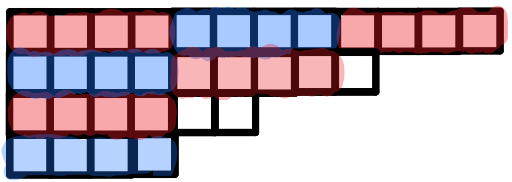
$$\nu'_s(\lambda) = (\lceil \lambda_{\nu_1}/s \rceil, \dots, \lceil \lambda_{\nu_m}/s \rceil)$$

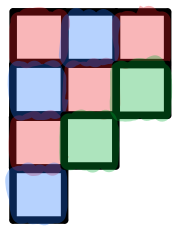
is the column position sequence of λ and the remainder diagram $\nu_s^+(\lambda)$ is obtained from the Ferrers diagram of $\lambda\downarrow_s$ by adding the cells $(\nu_1, \nu'_1), \dots, (\nu_m, \nu'_m)$ as green colored cells.

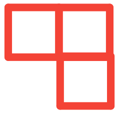




Example:

$\lambda =$ 

$\nu_4(\lambda) =$ 

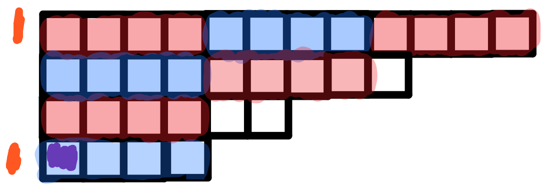
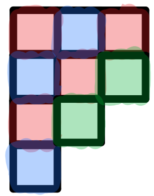


Lem: $\lambda \vdash n$ with strictly increasing remainder sequence, then

$r_s(\lambda) = \#\text{rows of } \nu_s^+(\lambda) - \#\text{green cells}$

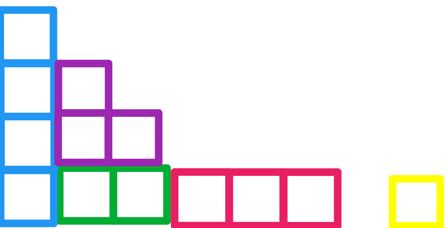
$c_s(\lambda) = \#\text{columns of } \nu_s^+(\lambda) - \#\text{green cells}$

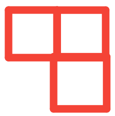
Example: $\lambda = (12, 9, 6, 4)$, $\rho_4(\lambda) = (1, 2)$

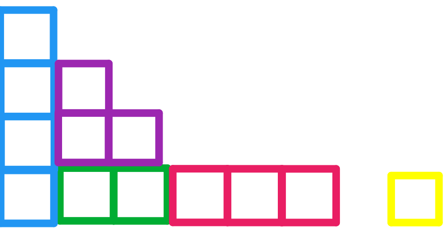
$\lambda =$   $\nu_4^+(\lambda) =$ 

$r_4(\lambda) = 2$, $c_4(\lambda) = 1$

$(\nu_4^+(\lambda))_1 = 2 + c_4(\lambda)$, $\ell(\nu_4^+(\lambda)) = 2 + r_4(\lambda)$





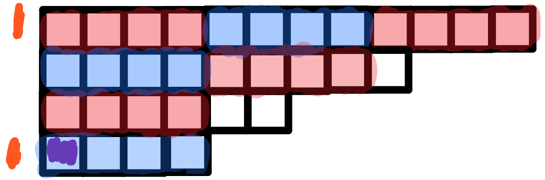
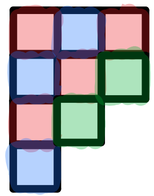


Lem: $\lambda \vdash n$ with strictly increasing remainder sequence, then

$r_s(\lambda) = \#\text{rows of } \nu_s^+(\lambda) - \#\text{green cells}$

$c_s(\lambda) = \#\text{columns of } \nu_s^+(\lambda) - \#\text{green cells}$

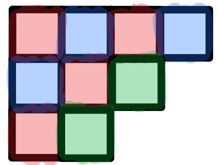
Example: $\lambda = (12, 9, 6, 4)$, $\rho_4(\lambda) = (1, 2)$

$\lambda =$  $\longmapsto$ $\nu_4^+(\lambda) =$ 

$r_4(\lambda) = 2$, $c_4(\lambda) = 1$

$(\nu_4^+(\lambda))_1 = 2 + c_4(\lambda)$, $\ell(\nu_4^+(\lambda)) = 2 + r_4(\lambda)$

$\downarrow \prime$

$[\nu_4^+(\lambda)]' =$ 

$\ell([\nu_4^+(\lambda)]') = 2 + c_4(\lambda)$,

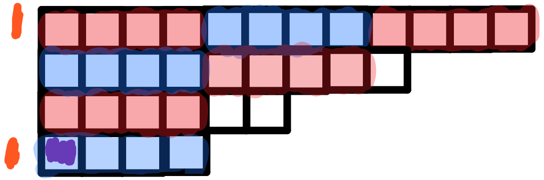
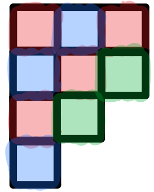
$([\nu_4^+(\lambda)]')_1 = 2 + r_4(\lambda)$

Lem: $\lambda \vdash n$ with strictly increasing remainder sequence, then

$r_s(\lambda) = \#\text{rows of } \nu_s^+(\lambda) - \#\text{green cells}$

$c_s(\lambda) = \#\text{columns of } \nu_s^+(\lambda) - \#\text{green cells}$

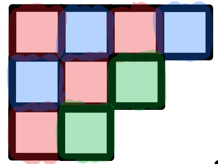
Example: $\lambda = (12, 9, 6, 4)$, $\rho_4(\lambda) = (1, 2)$

$\lambda =$  $\longmapsto$ $\nu_4^+(\lambda) =$ 

$r_4(\lambda) = 2$, $c_4(\lambda) = 1$

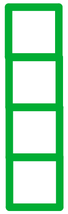
$(\nu_4^+(\lambda))_1 = 2 + c_4(\lambda)$, $\ell(\nu_4^+(\lambda)) = 2 + r_4(\lambda)$

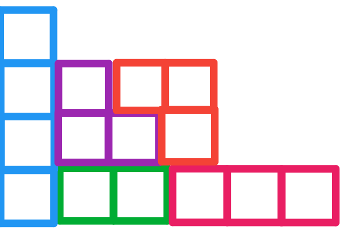
$\downarrow$

$[\nu_4^+(\lambda)]' =$ 

$\ell([\nu_4^+(\lambda)]') = 2 + c_4(\lambda)$, $([\nu_4^+(\lambda)]')_1 = 2 + r_4(\lambda)$

Def: $[\nu_s^+(\lambda)] \xleftarrow{}_s \rho_s(\lambda)$ is obtained from s-blow up of non green cells & inserting $\rho_s(\lambda)$ in order into the green cells.



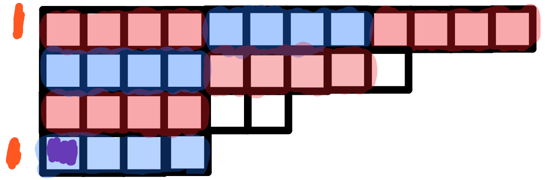
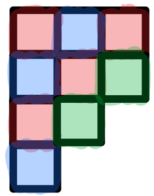


Lem: $\lambda \vdash n$ with strictly increasing remainder sequence, then

$r_s(\lambda) = \#\text{rows of } \nu_s^+(\lambda) - \#\text{green cells}$

$c_s(\lambda) = \#\text{columns of } \nu_s^+(\lambda) - \#\text{green cells}$

Example: $\lambda = (12, 9, 6, 4)$, $\rho_4(\lambda) = (1, 2)$

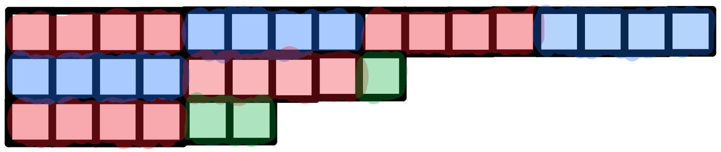
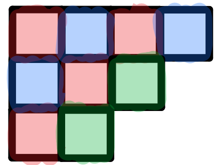
$\lambda =$  $\longmapsto$ $\nu_4^+(\lambda) =$ 

$r_4(\lambda) = 2$, $c_4(\lambda) = 1$

$(\nu_4^+(\lambda))_1 = 2 + c_4(\lambda)$, $\ell(\nu_4^+(\lambda)) = 2 + r_4(\lambda)$

$\downarrow$

$[\nu_4^+(\lambda)]' \leftarrow_4 \rho_4(\lambda)$

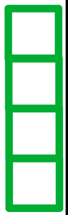
 $\longleftarrow$ $[\nu_4^+(\lambda)]' =$ 

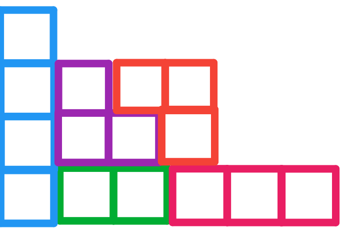
$\ell([\nu_4^+(\lambda)]') = 2 + c_4(\lambda)$,

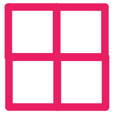
$([\nu_4^+(\lambda)]')_1 = 2 + r_4(\lambda)$

$r_4([\nu_4^+(\lambda)]' \leftarrow_4 \rho_4(\lambda)) = c_4(\lambda) = 1$

$c_4([\nu_4^+(\lambda)]' \leftarrow_4 \rho_4(\lambda)) = r_4(\lambda) = 2$







Second toy example: Consider only $\lambda \vdash n$ with strictly increasing remainder sequence.

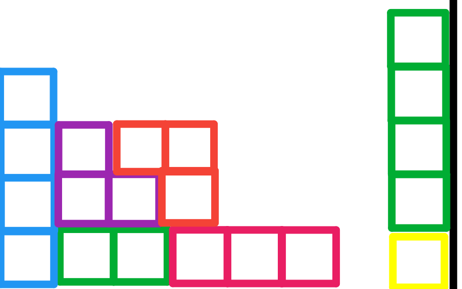
Solution: $\lambda \longmapsto [\nu_s(\lambda)]' \longleftarrow_s \rho_s(\lambda)$

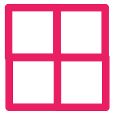
Lem: $\rho = (\rho_1, \dots, \rho_m)$ strictly increasing vector of integers between 1 and $s-1$. The generating function w.r.t. $R^{r_s(\lambda)} C^{c_s(\lambda)} q^{|\lambda|}$ of all partitions λ with $\rho_s(\lambda) = \rho$ is given by

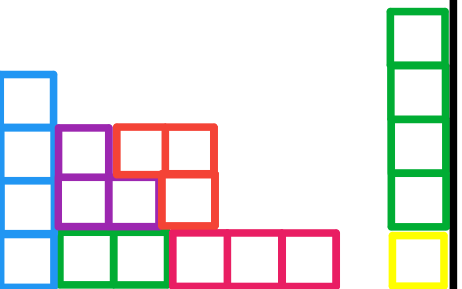
$$q^{|\rho|} \sum_{1 \le \nu_1 < \nu_2 < \dots < \nu_m} Q^{|\nu| - m} \left(R^{\nu_m - m} + \sum_{k \ge 1} \frac{C Q^k}{(CQ;Q)_k} R^{\max(\nu_m - m, k - m)} \right),$$

where as before $Q = q^s$.

Proof: Apply Δ_s^m to λ and delete s cells in each row above ν_i after deleting ρ_i in each step. This gives q^{ρ_i} & $Q^{\nu_i - 1}$. Now apply previous lemma.







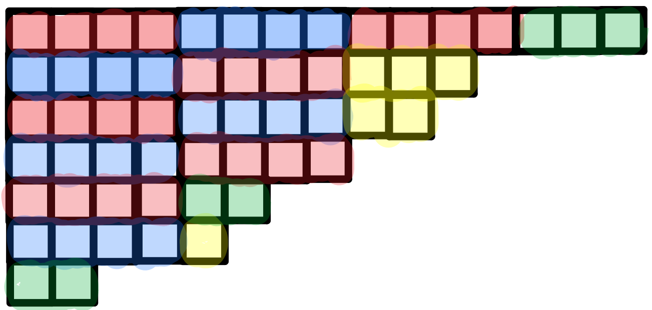
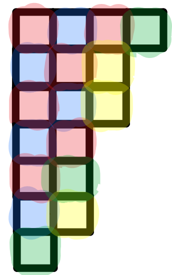
General case: Consider all $\lambda \vdash n$, i.e. $\rho_s(\lambda)$ is not necessarily strictly increasing anymore.

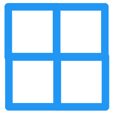
Def: $\lambda \vdash n$ with remainder sequence $\rho_s(\lambda)=(\rho_1,\dots,\rho_m)$, row position sequence $\nu_s(\lambda)=(\nu_1,\dots,\nu_m)$ and column position sequence $\nu'_s(\lambda)=(\nu'_1,\dots,\nu'_m)$. The *(extended) remainder diagram $\nu_s^+(\lambda)$* is obtained from the Ferrers diagram of $\lambda\downarrow_s$ by adding the cell $(\nu_{m+1-i}, \nu'_{m+1-i})$ coloured in

- yellow if $c_s(\Delta_s^i \lambda) = c_s(\Delta_s^{i-1}\lambda)$,
- green if $c_s(\Delta_s^i \lambda) = c_s(\Delta_s^{i-1}\lambda)+1$ or $i=m$,

to the Ferrers diagram.

Example:

$\lambda =$  $\nu_4^+(\lambda) =$ 

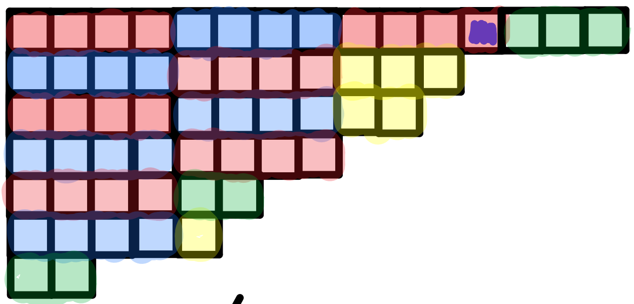
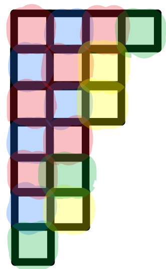


Lem: $\lambda \vdash n$ then

$$r_s(\lambda) = \#\text{rows in } \nu_s^+(\lambda) - \#\text{green cells} - \#\text{yellow cells}$$

$$c_s(\lambda) = \#\text{columns in } \nu_s^+(\lambda) - \#\text{green cells}$$

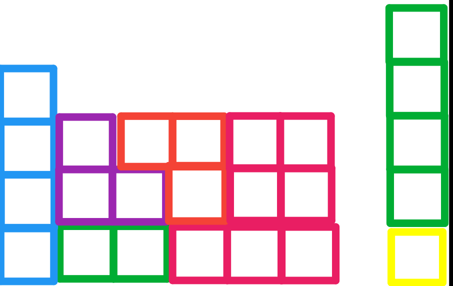
Example: $\lambda = (15, 11, 10, 8, 6, 5, 2)$, $\rho_4(\lambda) = (3, 3, 2, 2, 1, 2)$

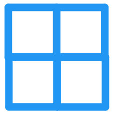
$\lambda =$  $\longmapsto \nu_4^+(\lambda) =$ 

$c_4(\lambda) = 1$, $r_4(\lambda) = 1$

$\ell(\nu_4^+(\lambda)) = 1 + 3 + 3$

$(\nu_4^+(\lambda))_1 = 1 + 3$



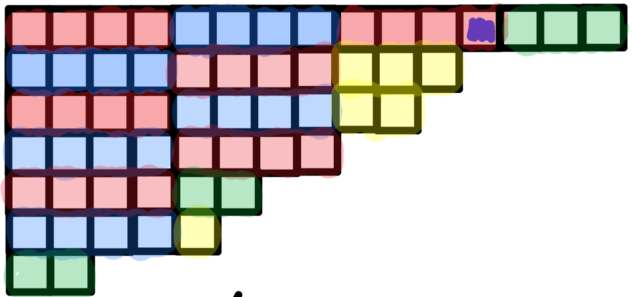
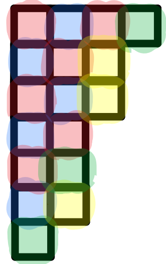


Lem: $\lambda \vdash n$ then

$r_s(\lambda) = \#$ rows in $\nu_s^+(\lambda) - \#$ green cells $- \#$ yellow cells

$c_s(\lambda) = \#$ columns in $\nu_s^+(\lambda) - \#$ green cells

Example: $\lambda = (15, 11, 10, 8, 6, 5, 2)$, $\rho_4(\lambda) = (3, 3, 2, 2, 1, 2)$

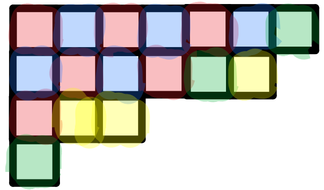
$\lambda =$  $\longmapsto \nu_4^+(\lambda) =$ 

$c_4(\lambda) = 1$, $r_4(\lambda) = 1$

$\ell(\nu_4^+(\lambda)) = 1 + 3 + 3$

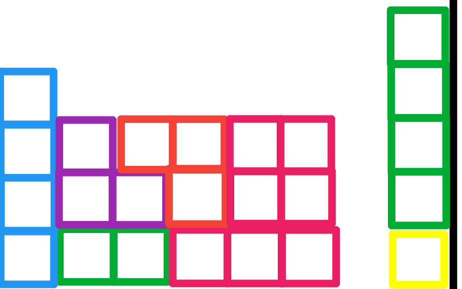
$(\nu_4^+(\lambda))_1 = 1 + 3$

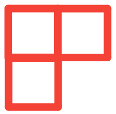
$\downarrow$

$[\nu_4^+(\lambda)]' =$ 

$\ell([\nu_4^+(\lambda)]') = 1 + 3$

$([\nu_4^+(\lambda)]')_1 = 1 + 3 + 3$



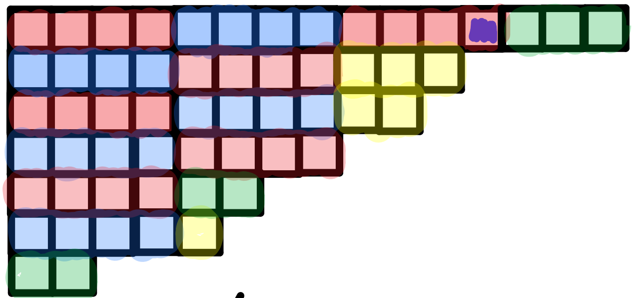
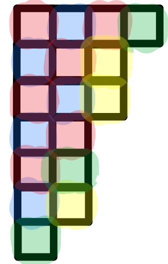


Lem: $\lambda \vdash n$ then

$r_s(\lambda) = \#\text{rows in } \nu_s^+(\lambda) - \#\text{green cells} - \#\text{yellow cells}$

$c_s(\lambda) = \#\text{columns in } \nu_s^+(\lambda) - \#\text{green cells}$

Example: $\lambda=(15,11,10,8,6,5,2)$, $\rho_4(\lambda)=(3,3,2,2,1,2)$

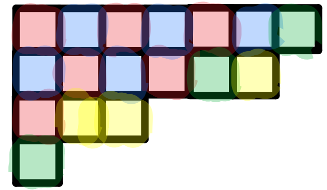
$\lambda =$  $\longmapsto \nu_4^+(\lambda) =$ 

$c_4(\lambda)=1, r_4(\lambda)=1$

$\ell(\nu_4^+(\lambda)) = 1+3+3$

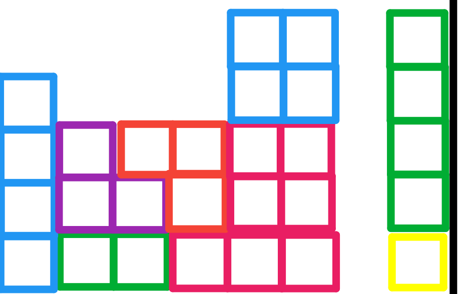
$(\nu_4^+(\lambda))_1 = 1+3$

$\downarrow$

Blowup & insertion? $\longleftarrow$ $[\nu_4^+(\lambda)]' =$ 

$\ell([\nu_4^+(\lambda)]') = 1+3$

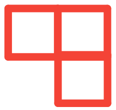
$([\nu_4^+(\lambda)]')_1 = 1+3+3$

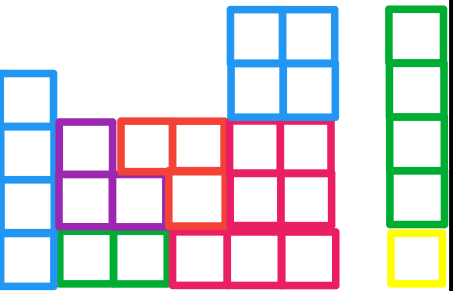


Lem: $\lambda \vdash n$ then

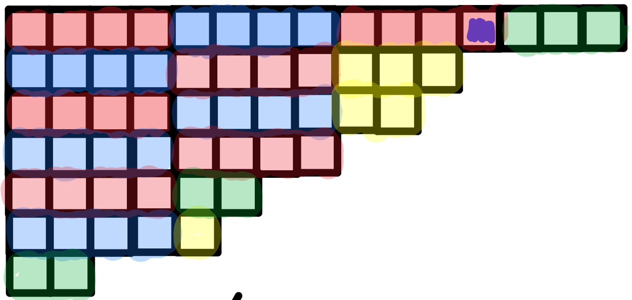
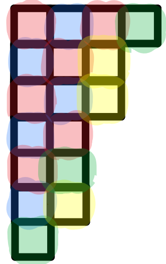
$$r_s(\lambda) = \#\text{rows in } \nu_s^+(\lambda) - \#\text{green cells} - \#\text{yellow cells}$$

$$c_s(\lambda) = \#\text{columns in } \nu_s^+(\lambda) - \#\text{green cells}$$





Example: $\lambda = (15, 11, 10, 8, 6, 5, 2)$, $\rho_4(\lambda) = (3, 3, 2, 2, 1, 2)$

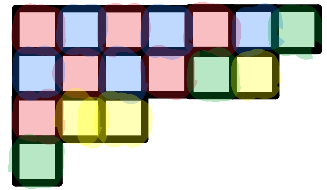
$\lambda =$  $\longmapsto \nu_4^+(\lambda) =$ 

$c_4(\lambda) = 1$, $r_4(\lambda) = 1$

$\ell(\nu_4^+(\lambda)) = 1 + 3 + 3$

$(\nu_4^+(\lambda))_1 = 1 + 3$

$\downarrow$

$[\nu_4^+(\lambda)]' =$ 

$\ell([\nu_4^+(\lambda)]') = 1 + 3$

$([\nu_4^+(\lambda)]')_1 = 1 + 3 + 3$

No! $\longleftarrow$

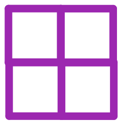
$\rightsquigarrow$ Problem....

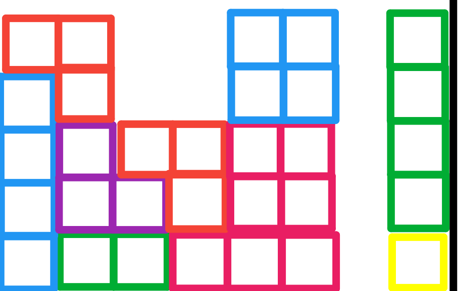
Solution:

Lem: $\rho = (\rho_1, \ldots, \rho_m)$ vector of integers between 1 and $s-1$. The generating function w.r.t. $R^{r_s(\lambda)} C^{c_s(\lambda)} q^{|\lambda|}$ of all partitions λ with remainder sequence $\rho_s(\lambda) = \rho$ is given by

$$q^{|\rho|} \sum_{1 \leq \gamma_1 < \ldots < \gamma_m} Q^{d(\rho,\gamma)\cdot(\gamma - 1)} \left(R^{\gamma_m - m} + \sum_{k \geq 1} \frac{C Q^k}{(CQ;Q)_k} R^{\max(\gamma_m - m,\, k-m)} \right),$$

where $d(\rho,\gamma) = (d_1, \ldots, d_m)$ with $d_j = 1$ unless $j > 1$, $\rho_{j-1} \geq \rho_j$ and $\gamma_j = \gamma_{j-1} + 1$.





Solution:

Lem: $\rho=(\rho_1,\dots,\rho_m)$ vector of integers between 1 and $s-1$. The generating function w.r.t. $R^{r_s(\lambda)}C^{c_s(\lambda)}q^{|\lambda|}$ of all partitions λ with remainder sequence $\rho_s(\lambda)=\rho$ is given by

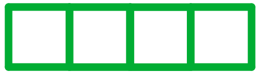
$$q^{|\rho|}\sum_{1\le\nu_1<\dots<\nu_m} Q^{d(\rho,\nu)\cdot(\nu-1)}\left(R^{\nu_m-m}+\sum_{k\ge1}\frac{CQ^k}{(CQ;Q)_k}R^{\max(\nu_m-m,\,k-m)}\right),$$

where $d(\rho,\nu)=(d_1,\dots,d_m)$ with $d_j=1$ unless $j>1$, $\rho_{j-1}\ge\rho_j$ and $\nu_j=\nu_{j-1}+1$.

Lem: $\rho=(\rho_1,\dots,\rho_m)$ vector of integers between 1 and $s-1$. Then for fixed ν_m we have

$$\sum_{1\le\nu_1<\dots<\nu_m} Q^{d(\rho,\nu)\cdot(\nu-1)}=Q^{-\mathrm{wmaj}(\rho)}\sum_{1\le\nu_1<\nu_2<\dots<\nu_m} Q^{|\nu|-m},$$

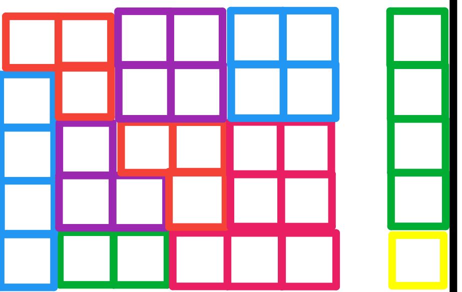
where $\mathrm{wmaj}(\rho)=\sum_{j:\rho_j\ge\rho_{j+1}} j$.



Solution:

$$q^{|\varrho|}\sum_{1\le\gamma_1<\dots<\gamma_m} Q^{d(\varrho,\gamma_\ell)\cdot(\gamma_\ell-1)}\left(R^{\gamma_m-m}+\sum_{k\ge1}\frac{CQ^k}{(CQ;Q)_k}R^{\max(\gamma_m-m,k-m)}\right)$$

$$= q^{|\varrho|}Q^{-\mathrm{wmaj}(\varrho)}\sum_{1\le\gamma_1<\gamma_2<\dots<\gamma_m} Q^{|\gamma|-m}\left(R^{\gamma_m-m}+\sum_{k\ge1}\frac{CQ^k}{(CQ;Q)_k}R^{\max(\gamma_m-m,k-m)}\right)$$

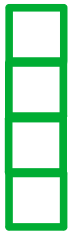


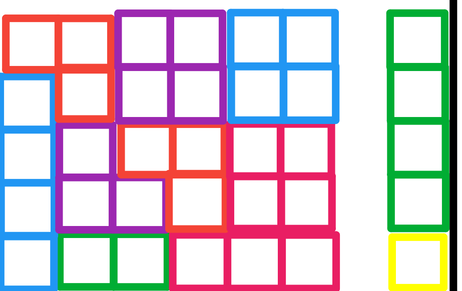
Solution:

$$\overbrace{q^{|\rho|}\sum_{1\le\gamma_1<\dots<\gamma_m} Q^{d(\rho,\gamma)\cdot(\gamma-1)}\left(R^{\gamma_m-m}+\sum_{k\ge1}\frac{CQ^k}{(CQ;Q)_k}R^{\max(\gamma_m-m,\,k-m)}\right)}^{\text{arbitrary remainders (green \& yellow cells in } \nu_s^+(\lambda))}$$

$$=\underbrace{q^{|\rho|}Q^{-\mathrm{wmaj}(\rho)}\sum_{1\le\gamma_1<\gamma_2<\dots<\gamma_m} Q^{|\gamma|-m}\left(R^{\gamma_m-m}+\sum_{k\ge1}\frac{CQ^k}{(CQ;Q)_k}R^{\max(\gamma_m-m,\,k-m)}\right)}_{\text{strictly increasing remainders (green cells only in } \nu_s^+(\lambda))}$$

times $Q^{-\mathrm{wmaj}(\rho)}$ as prefactor.



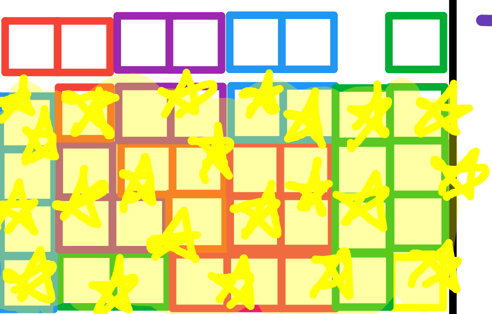


Solution:

$$\overbrace{q^{|\varrho|}\sum_{1\le\gamma_1<\dots<\gamma_m} Q^{d(\varrho,\gamma)\cdot(\gamma-1)}\left(R^{\gamma_m-m}+\sum_{k\ge1}\frac{CQ^k}{(CQ;Q)_k}R^{\max(\gamma_m-m,\,k-m)}\right)}^{\text{arbitrary remainders (green \& yellow cells in } \nu_s^+(\lambda))}$$

$$=\underbrace{q^{|\varrho|}Q^{-\mathrm{wmaj}(\varrho)}\sum_{1\le\gamma_1<\gamma_2<\dots<\gamma_m} Q^{|\gamma|-m}\left(R^{\gamma_m-m}+\sum_{k\ge1}\frac{CQ^k}{(CQ;Q)_k}R^{\max(\gamma_m-m,\,k-m)}\right)}_{\text{strictly increasing remainders (green cells only in } \nu_s^+(\lambda))}$$

times $Q^{-\mathrm{wmaj}(\varrho)}$ as prefactor.



key observation: The Lemma can be translated into a combinatorial revertible algorithm! Let us denote this procedure by F.

Solution:

$$\lambda \longmapsto \left[F^{-1}\left(\left[F(\nu_s(\lambda))\right]'\right)\right] \longleftarrow\!\!\mid_s \rho_s(\lambda)$$

Thm: $\rho = (\rho_1, \dots, \rho_m)$ vector of integers between 1 and $s-1$. The generating function w.r.t. $q^{|\lambda|}$ of all partitions λ with $\rho_s(\lambda) = \rho$ and $(r_s(\lambda), c_s(\lambda)) = (r, c)$ is given by

$$q^{|\rho|} Q^{-\mathrm{wmaj}(\rho) + \binom{m}{2} + r + c} \left(\frac{[r+c+m-1]_Q!}{[r]_Q!\,[c]_Q!\,[m-1]_Q!} + Q^{m-1} \frac{[r+c+m-2]_Q!}{[r-1]_Q!\,[c-1]_Q!\,[m]_Q!} \right).$$

Note that this is clearly symmetric in r & c. This has a calculation based & a combinatorial proof too.

HIGH
SCORE

Thank you very much for your attention and the opportunity to give this talk!

Have a nice day!