Integer partitions and perfect crystals

I) Background (Lie algebras, characters, partitions)

References: Kac, Infinite dimensional Lie algebras

Carter, Lie algebras of finite and affine type

1) Lie algebras

def: A Lie algebra $\mathfrak{g}$ over a field $\mathbb{F}$ is a vector space with a bilinear operation $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, $(x,y) \mapsto [x,y]$

s.t. i) $[x,x] = 0$ for all $x \in \mathfrak{g}$

ii) $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$ for all $x,y,z \in \mathfrak{g}$

ex: A_{n-1} (or $sl_n(\mathbb{C})$) : Lie algebra of $n \times n$ matrices of trace 0, with bracket $[X,Y] = XY - YX$

n=2 sl_2 generated by

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$[h,e] = 2e, \quad [h,f] = -2f,$$

$$[e,f] = h$$

def: A representation (or module) of $\mathfrak{g}$ is a vector space V together with a linear map $\rho: \mathfrak{g} \to \mathfrak{gl}(V)$ s.t. $\rho([x,y]) = \rho(x)\rho(y) - \rho(y)\rho(x)$.

($\rho(x)(v)$ can be written $x \cdot v$)

ex: adjoint representation $\mathrm{ad}: \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ s.t.

$\mathrm{ad}(x)(y) = [x,y]$ for all $x, y \in \mathfrak{g}$.

def: A subspace W of a $\mathfrak{g}$-module V is a submodule of V if $x \cdot W \subset W$ $\forall x \in \mathfrak{g}$.

A $\mathfrak{g}$-module V is irreducible if it has no other submodule than 0 and V.

2) Kac – Moody algebras

def: Let I be a finite set of indices. A square matrix $A = (a_{ij})_{i,j \in I}$ with entries in $\mathbb{Z}$ is a generalised Cartan matrix if

- $a_{ii} = 2$ $\forall i \in I$
- $a_{ij} \leq 0$ if $i \neq j$
- $a_{ij} = 0$ iff $a_{ji} = 0$

Let $P^\vee$ be a free abelian group of rank $2|I| - \text{rank}A$ with $\mathbb{Z}$-basis $\{h_i : i \in I\} \cup \{d_s : s = 1, \dots, |I| - \text{rank}A\}$ and $\mathfrak{h} = \mathbb{F} \otimes_{\mathbb{Z}} P^\vee$ be the $\mathbb{F}$-linear space spanned by $P^\vee$.

$P^\vee$ is called the <u>dual weight lattice</u> and $\mathfrak{h}$ the <u>Cartan subalgebra</u>.

The weight lattice is $P = \{\lambda \in \mathfrak{h}^* : \lambda(P^\vee) \subset \mathbb{Z}\}$.

Let $\Pi^\vee := \{h_i \mid i \in I\}$ and choose a linearly indep. subset $\Pi = \{\alpha_i \mid i \in I\}$ of $\mathfrak{h}^*$ s.t. $\alpha_j(h_i) = a_{ij}$ for $i,j \in I$

$\alpha_j(d_s) = 0$ or 1 for all s

The elements of Π are the <u>simple roots</u>

——— $\Pi^\vee$ ——— coroots

$(A, \Pi, \Pi^\vee, P, P^\vee)$ is called a <u>Cartan datum</u>.

<u>Fundamental weights</u>: $\Lambda_i \in \mathfrak{h}^*$ s.t. $\Lambda_i(h_j) = \delta_{ij}$

$\Lambda_i(d_s) = 0$

$Q_+ = \bigoplus_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i$ positive <u>root lattice</u>

$r_i(\lambda) = \lambda - \lambda(h_i)\alpha_i$ simple reflection on $\mathfrak{h}^*$

Weyl group: subgroup of $GL(\mathfrak{h}^*)$ generated by all r_i ($i \in I$)

<u>def</u>: The Kac-Moody Lie algebra $\mathfrak{g}$ associated with a Cartan datum $(A, \Pi, \Pi^\vee, P, P^\vee)$ is the Lie algebra generated by e_i, f_i $(i \in I)$ and $h \in P^\vee$ subject to the relations:

1) $[h, h'] = 0 \quad \forall h, h' \in P^\vee$
2) $[e_i, f_j] = \delta_{ij} h_i$
3) $[h, e_i] = \alpha_i(h) e_i$
4) $[h, f_i] = -\alpha_i(h) f_i$

} for all $h \in P^\vee$, $i \in I$

5) $(\mathrm{ad}\, e_i)^{1-a_{ij}} e_j = 0$ for $i \neq j$
6) $(\mathrm{ad}\, f_i)^{1-a_{ij}} f_j = 0$ ———

<u>ex</u>: Cartan matrix (2)

$\mathfrak{h} = \mathbb{C} h \qquad \mathfrak{h}^* = \{\varphi : \mathfrak{h} \to \mathbb{C}\}$

$P^\vee = \mathbb{Z} h$, $\quad P = \{\lambda \in \mathfrak{h}^* \mid \lambda(h) \in \mathbb{Z}\}$

$\Pi^\vee = \{h\}$, $\quad \Pi = \{\alpha\}$ s.t. $\alpha(h) = 2$

1) $[h, h] = 0$
2) $[e, f] = h$
3) $[h, e] = 2e$
4) $[h, f] = -2f$

$\forall \alpha \in Q$, define $\mathfrak{g}_\alpha := \{x \in \mathfrak{g} \mid [h,x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}$

If $\alpha \neq 0$ and $\mathfrak{g}_\alpha \neq 0$, then α is a root of $\mathfrak{g}$ and $\mathfrak{g}_\alpha$ is the root space associated to α. Let Φ be the set of roots.

Kac-Moody Lie algebras are classified. From now on, we are interested in affine type. In that case, the center of $\mathfrak{g}$ is one-dimensional, generated by $c = c_0 h_0 + \dots + c_n h_n$ ($I = \{0, \dots, n\}$). The null root is $\delta = d_0 \alpha_0 + \dots + d_n \alpha_n$, where $c_i, d_i \in \mathbb{Z}_{\geq 0}$ for all i.

The affine weight lattice can be written

$$P = \mathbb{Z}\Lambda_0 \oplus \dots \oplus \mathbb{Z}\Lambda_n \oplus \mathbb{Z}\frac{1}{d_0}\delta .$$

$$P^+ = \{\lambda \in P \mid \lambda(h_i) \in \mathbb{Z}_{\geq 0} \ \forall i \in I\}$$ affine dominant integral weights

The level of $\lambda \in P^+$: $\lambda(c)$

Classical weights : $\overline{P}_+ = \mathbb{Z}_{\geq 0}\Lambda_0 \oplus \dots \oplus \mathbb{Z}_{\geq 0}\Lambda_n$

3) Characters

def: A $\mathfrak{g}$-module V is called a weight module if it admits a weight space decomposition $V = \bigoplus_{\mu \in \mathfrak{h}^*} V_\mu$,

$V_\mu = \{ v \in V \mid h \cdot v = \mu(h) v \quad \forall h \in \mathfrak{h} \}$.

$v \in V_\mu$ is called a <u>weight vector</u> of weight μ

If $V_\mu \neq 0$, μ is a weight of V and V_μ a weight space.

When $\dim V_\mu < \infty$ for all weights μ, the <u>character</u> of V is $\boxed{\mathrm{ch}\, V = \sum_{\mu \in \mathfrak{h}^*} \dim V_\mu \; e^\mu}$, where e^μ is a formal exponential with $e^\lambda e^\mu = e^{\lambda+\mu}$.

<u>def:</u> A weight λ is <u>higher</u> than μ if $\lambda - \mu$ can be written as a sum of positive roots.

Given a highest weight λ, there is an associated highest weight module $L(\lambda)$.

<u>Th (Weyl-Kac character formula)</u>:

$$\mathrm{ch}(L(\lambda)) = \frac{\sum_{w \in W} \mathrm{sgn}(w)\, e^{w(\lambda+\rho)-\rho}}{\prod_{\alpha \in \phi^+} (1-e^{-\alpha})^{\dim \mathfrak{g}_\alpha}}$$

where $\mathrm{sgn}(w) = (-1)^{\#\text{ reflections}}$

ϕ^+ set of positive roots

$\rho = \sum_{i \in I} \Lambda_i$, Weyl vector def: For a sequence $s = (s_0, \ldots, s_n)$ of non-negative integers, define the specialisation $F_s : \mathbb{Z}[[e^{-\alpha_0}, \ldots, e^{-\alpha_n}]] \to \mathbb{Z}[[q]]$

$$e^{-\alpha_i} \longmapsto q^{s_i}$$

principal specialisation for $s = (1, \ldots, 1)$, denoted F_1.

Lepowsky's product formula (1978):

$$F_1\left(e^{-\lambda} ch(L(\lambda))\right) = \frac{F_{(\lambda(h_0)+1, \ldots, \lambda(h_n)+1)}\left(D(\phi^\vee)\right)}{F_1\left(D(\phi)\right)},$$

where $D(\phi) = \prod_{\alpha \in \phi^+} \left(1 - e^{-\alpha}\right)^{\dim \mathfrak{g}_\alpha}$

ex: $A_1^{(1)}$ Kac-Moody Lie algebra with Cartan matrix $\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$

simple roots α_0, α_1, fundamental weights Λ_0, Λ_1

null root $\delta = \alpha_0 + \alpha_1$, central element $c = h_0 + h_1$

$P = \mathbb{Z}\Lambda_0 + \mathbb{Z}\Lambda_1 + \mathbb{Z}\delta$ affine weight lattice

$$\phi = \{\alpha + n\delta : \alpha = \alpha_0 \text{ or } \alpha_1 \text{ and } n \in \mathbb{Z}\}$$
$$\cup \{k\delta : k \in \mathbb{Z}^*\}$$

$$\Rightarrow \phi^+ = \{k\alpha_0 + k\alpha_1 : k > 0\} \cup \{k\alpha_0 + (k+1)\alpha_1 : k \geq 0\}$$
$$\cup \{(k+1)\alpha_0 + k\alpha_1 : k \geq 0\}.$$

Here $\phi^\vee = \phi$ because $\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}$ symmetric.

And $D(\phi) = \prod_{k=1}^{\infty} \left(1-e^{-k\alpha_0-k\alpha_1}\right)\left(1-e^{-(k-1)\alpha_0-k\alpha_1}\right)\left(1-e^{-k\alpha_0-(k-1)\alpha_1}\right)$

$$F_1(D(\phi)) = \prod_{k=1}^{\infty} (1-q^{2k})(1-q^{2k-1})(1-q^{2k-1})$$

$$= (q;q)_\infty (q;q^2)_\infty$$

where $(a;q)_n = \prod_{k=0}^{n-1} (1-aq^k)$, $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$

For $\lambda = \Lambda_0$, $F_{(2,1)}(D(\phi)) = \prod_{k=1}^{\infty} (1-q^{3k})(1-q^{3k-2})(1-q^{3k-1})$
$$= (q;q)_\infty$$

By Lepowsky's formula, $F_1\left(e^{-\Lambda_0} ch(L(\Lambda_0))\right) = \dfrac{(q;q)_\infty}{(q;q)_\infty (q;q^2)_\infty} = \dfrac{1}{(q;q^2)_\infty}$.

4) Connection with partitions

def: A partition of a positive integer n is a sequence of positive integers, called parts, whose sum is n.

$p_{m \bmod N}(n,k)$ = # partitions of n with k parts all $\equiv m \bmod N$

$d_{m \bmod N}(n,k)$ = ——— distinct parts ———

Prop: $$\sum_{n,k \geq 0} d_{m \bmod N}(n,k)\, q^n x^k = (-xq^m; q^N)_\infty$$

$$\sum_{n,k \geq 0} p_{m \bmod N}(n,k)\, q^n x^k = \frac{1}{(xq^m; q^N)_\infty}$$

Rogers - Ramanujan : $\sum_{n\geq 0} \frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}$

The number of partitions of n where parts differ by at least 2 (and 1 is not a part) equals the number of partitions of n into parts $\equiv \pm 1 \bmod 5$ $\pm 2 \bmod 5$.

RR-type identity : for all n, partitions of n satisfying some congruence conditions are equinumerous with partitions of n satisfying some congruence conditions.

Lepowsky - Milne (1978) :

$$F_1\left(e^{-3\Lambda_0} ch(L(3\Lambda_0))\right) = \frac{F_{(4,1)} D(\phi)}{F_1 D(\phi)} = \frac{(q^5;q^5)_\infty (q;q^5)_\infty (q^4;q^5)_\infty}{(q;q)_\infty (q;q^2)_\infty}$$

$$= \frac{1}{(q;q^2)_\infty} \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}$$

$$F_1\left(e^{-(\Lambda_0+2\Lambda_1)} ch(L(\Lambda_0+2\Lambda_1))\right) = \frac{1}{(q;q^2)_\infty} \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty}$$

Lepowsky - Wilson (1984) : interpretation of the sum-side by constructing a basis of $L(3\Lambda_0)$ and $L(\Lambda_0+2\Lambda_1)$ using vertex operators.

Idea: Find another expression of the character using crystals, and deduce partition identities

II) Perfect crystals

Reference : Hong – Kang , Introduction to quantum groups and crystal bases

def: A crystal B is a non-empty set with maps

$wt : B \to P$

$e_i : B \to B \cup \{0\}$

$f_i : B \to B \cup \{0\}$

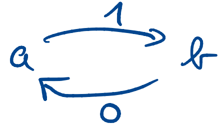
$\varepsilon_i : B \to \mathbb{Z}$

$\varphi_i : B \to \mathbb{Z}$

st:
- $f_i(b) = b'$ iff $e_i(b') = b$
- if $f_i(b) \neq 0$ then $wt(f_i(b)) = wt(b) - \alpha_i$
- if $e_i(b) \neq 0$ then $wt(e_i(b)) = wt(b) + \alpha_i$
- $\varphi_i(b) = \varepsilon_i(b) + wt(b)(h_i)$
- $\varepsilon_i(b) = \max\{k \geq 0 : e_i^k(b) \neq 0\}$
- $\varphi_i(b) = \max\{k \geq 0 : f_i^{(k)}(b) \neq 0\}$

B will be represented by a graph with vertex set B and an edge $b \xrightarrow{i} b'$ iff $f_i(b) = b'$.
(if $f_i(b) = 0$, we do not draw an edge)

ex: $A_1^{(1)}$ B_1 :



$f_1(a) = b$
$f_0(a) = 0$

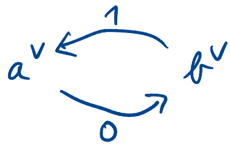
$\varphi_0(a) = 0$, $\varphi_1(a) = 1$
$\varphi_0(b) = 1$, $\varphi_1(b) = 0$

$$\mathrm{wt}(b) = \mathrm{wt}(a) - \alpha_1$$
$$\mathrm{wt}(a) = \mathrm{wt}(b) - \alpha_0$$
$$\mathrm{wt}(a) = \mathrm{wt}(a) - \alpha_0 - \alpha_1$$

$\delta = \alpha_0 + \alpha_1$ vanishes ✓

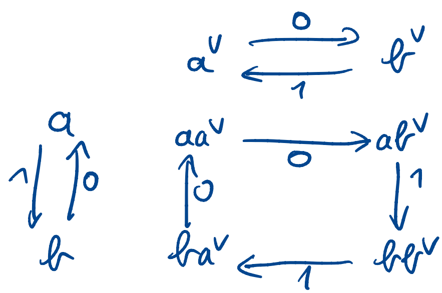
def/prop : The dual $B^\vee$ of a crystal B, obtained by reversing the edges, is also a crystal, with $\mathrm{wt}(b) = -\mathrm{wt}(b^\vee)$

ex: $B_1^\vee$



def/prop: If B_1 and B_2 are crystals, then $B_1 \otimes B_2$ is also a crystal, with
- $\mathrm{wt}(b_1 \otimes b_2) = \mathrm{wt}(b_1) + \mathrm{wt}(b_2)$
- $$f_i(b_1 \otimes b_2) = \begin{cases} f_i(b_1) \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2) \\ b_1 \otimes f_i(b_2) & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2) \end{cases}$$

ex: $A_1^{(1)}$ $B_1 \otimes B_1^\vee$



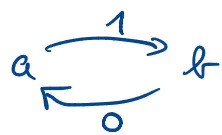
2) Perfect crystals

(Partial) def: For a positive integer ℓ, a finite crystal B is a <u>perfect crystal of level ℓ</u> if:

- some properties omitted here
- $B \otimes B$ connected
- $\sum_{i=0}^{n} \varepsilon_i(b)\Lambda_i(c) \geq \ell \quad \forall b \in B$
- $\forall \lambda \in \overline{P_\ell^+} := \{\mu \in P_+ \mid \mu(c) = \ell\}$, there exist unique b^λ and b_λ in B s.t. $\varepsilon(b^\lambda) = \lambda$ and $\varphi(b_\lambda) = \lambda$

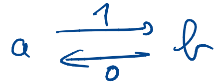
where $\varepsilon(b) = \sum_{i=0}^{n} \varepsilon_i(b)\Lambda_i$ and $\varphi(b) = \sum_{i=0}^{n} \varphi_i(b)\Lambda_i$

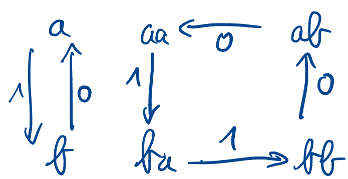
<u>ex:</u> $A_1^{(1)}$ $\quad B_1$:



$\varepsilon_0(a) = 1, \ \varepsilon_1(a) = 0$
$\varepsilon_0(b) = 0, \ \varepsilon_1(b) = 1$

- $B_1 \otimes B_1$ connected





- $\varepsilon_0(a)\Lambda_0(c) + \varepsilon_1(a)\Lambda_1(c) = 1$

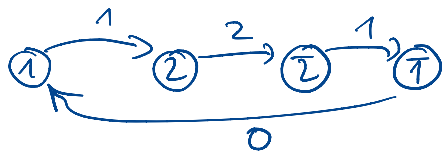
  $\varepsilon_0(b)\Lambda_0(c) + \varepsilon_1(b)\Lambda_1(c) = 1$

- $\varepsilon(a) = \Lambda_0$, $\varepsilon(b) = \Lambda_1$, $\varphi(a) = \Lambda_1$, $\varphi(b) = \Lambda_0$

$\Rightarrow$ $b^{\Lambda_0} = a$, $b_{\Lambda_0} = b$, $b^{\Lambda_1} = b$, $b_{\Lambda_1} = a$

$B_1 \otimes B_1^\vee$ is also a perfect crystal of level 1.

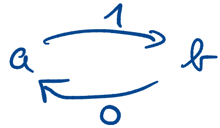
<u>Counter-ex</u>: $C_2^{(1)}$



not perfect because $\varphi(1) = \Lambda_1 = \varphi(\bar{2})$

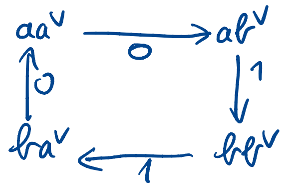
<u>def</u>: For $\lambda \in \overline{P}_\ell^+$, let $\lambda_0 = \lambda$, $g_0 = b_\lambda$,
and for all $k \geq 0$, $\lambda_{k+1} = \varepsilon(b_{\lambda_k})$, $g_{k+1} = b_{\lambda_{k+1}}$
The <u>ground state path of weight λ</u> is $p_\lambda := \cdots \otimes g_k \otimes \cdots \otimes g_1 \otimes g_0$.
It is always periodic of period at most $|B|$.
A <u>λ-path</u> is a path $p = \cdots \otimes p_k \otimes \cdots \otimes p_1 \otimes p_0$ s.t.
$p_k = g_k$ for k large enough. Let $\mathcal{P}(\lambda)$ be the set of λ-paths.

<u>ex</u>: • $A_1^{(1)}$, B_1, $\lambda = \Lambda_0$:



$\lambda_0 = \Lambda_0$, $b_{\lambda_0} = b$, $\lambda_1 = \varepsilon(b) = \Lambda_1$, $g_1 = b_{\Lambda_1} = a$,
$\lambda_2 = \varepsilon(a) = \Lambda_0$, $g_2 = b_{\Lambda_0} = b$ $\boxed{\cdots \otimes a \otimes b \otimes a \otimes b}$

- $A_1^{(1)}$, $B_1 \otimes B_1^\vee$, $\lambda = \Lambda_0$:



$\lambda_0 = \Lambda_0$, $g_0 = b_{\Lambda_0} = aa^\vee$

$\lambda_1 = \varepsilon(aa^\vee) = \Lambda_0$, $g_1 = b_{\Lambda_0} = aa^\vee$

$$\boxed{\cdots \otimes aa^\vee \otimes aa^\vee}$$

For some applications, it is easier to consider crystals with constant ground state paths.

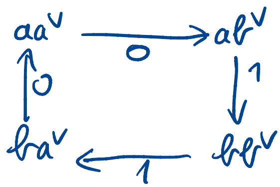
We will use perfect crystals to obtain other character formulas.

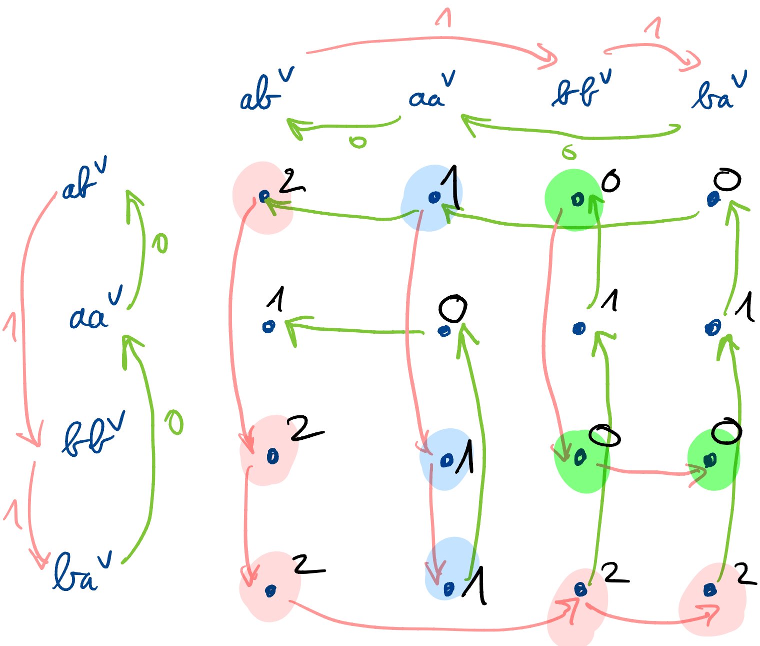
3) Energy functions and character formulas

<u>def</u>: Let B be a perfect crystal. An <u>energy</u> function on B is a map $H : B \otimes B \to \mathbb{Z}$ s.t. for all $i \in I$ and $b_1, b_2 \in B$:

$$H(e_i(b_1 \otimes b_2)) = \begin{cases} H(b_1 \otimes b_2) & \text{if } i \neq 0 \\ H(b_1 \otimes b_2) + 1 & \text{if } i = 0 \text{ and } \varphi_0(b_1) \geq \varepsilon_0(b_2) \\ H(b_1 \otimes b_2) - 1 & \text{if } i = 0 \text{ and } \varphi_0(b_1) < \varepsilon_0(b_2) \end{cases}$$

ex: $B_1 \otimes B_1^\vee$





$$H = \begin{array}{c} \\ ab^\vee \\ aa^\vee \\ bb^\vee \\ ba^\vee \end{array} \begin{array}{c} \begin{array}{cccc} ab^\vee & aa^\vee & bb^\vee & ba^\vee \end{array} \\ \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 2 & 1 & 0 & 0 \\ 2 & 1 & 2 & 2 \end{pmatrix} \end{array}$$

$(KMN)^2$ character formula (Kang - Kashiwara - Misra - Miwa - Nakashima - Nakayashiki):

Let $\lambda \in \overline{P}_\ell^+$, B perfect crystal of level ℓ, H energy function on B, $g = \cdots \otimes g_k \otimes \cdots \otimes g_0$ ground-state path.

For all $p = \cdots \otimes p_k \otimes \cdots \otimes p_0 \in \mathcal{P}(\lambda)$ define

$$\operatorname{wt} p = \lambda + \sum_{k=0}^{\infty} \left((\operatorname{wt}(p_k) - \operatorname{wt}(g_k)) - \frac{\delta}{d_0} \sum_{\ell=k}^{\infty} H(p_{\ell+1} \otimes p_\ell) - H(g_{\ell+1} \otimes g_\ell) \right)$$

Then $\operatorname{ch}(L(\lambda)) = \sum_{p \in \mathcal{P}(\lambda)} e^{\operatorname{wt} p}$.

Cor: If the ground state path is constant $\cdots g\otimes\cdots\otimes g$ and $H(g\otimes g)=0$, then $\mathrm{wt}\, g = 0$ and

$$\mathrm{wt}\, p = \lambda + \sum_{k=0}^{\infty}\left(\mathrm{wt}(p_k) - \frac{\delta}{d_0}\sum_{\ell=k}^{\infty} H(p_{\ell+1}\otimes p_\ell)\right)$$

Primc (1998) computed $F_1(e^{-\Lambda_0} ch(L(\Lambda_0)))$ using the KMN^2 character formula for $B_1\otimes B_1^\vee$ in $A_1^{(1)}$ and deduced a Rogers–Ramanujan type partition identity, which we will prove in the next section using a more general method.

III) Connection with partitions and applications

D.–Konan (2022): combinatorial version of the KMN^2 character formula.

def: $\mathcal{C}$ set of colours, $c_g\in\mathcal{C}$

$\succ$ binary relation on coloured integers $\mathbb{Z}_{\mathcal{C}} := \{k_c : k\in\mathbb{Z}, c\in\mathcal{C}\}$

A grounded partition with ground c_g and relation $\succ$ is a finite sequence $(\pi_0,\ldots,\pi_s)$ of coloured integers s.t.

1) $\forall i\in\{0,\ldots,s-1\}$, $\pi_i\succ\pi_{i+1}$

2) $\pi_s = 0_{c_g}$

3) $\pi_{s-1} \neq 0_{c_g}$.

Let $\mathcal{P}^{>}_{c_g}$ denote the set of such grounded partitions.

For $\pi \in \mathcal{P}^{>}_{c_g}$, let $C(\pi) = c_0 \cdots c_s$ where c_i is the colour of π_i $\forall i$

<u>ex/def:</u> B perfect crystal of level ℓ, $\lambda \in \overline{P_\ell^+}$ s.t. ground state path is $\cdots \otimes g \otimes \cdots \otimes g$, H energy function s.t. $H(g \otimes g) = 0$

Define $\mathcal{C}_B := \{ c_b : b \in B \}$ and the two binary relations $\geq$ and $\gg$

- $k_{c_b} \geq k'_{c_{b'}}$ iff $k - k' = H(b' \otimes b)$
- $k_{c_b} \gg k'_{c_{b'}}$ iff $k - k' \geq H(b' \otimes b)$.

<u>ex:</u> $A_1^{(1)}$, $B_1 \otimes B_1^\vee$, Λ_0

constant ground state path $\cdots \otimes aa^\vee \otimes \cdots \otimes aa^\vee$

energy H:

	$ab^\vee$	$aa^\vee$	$bb^\vee$	$ba^\vee$
$ab^\vee$	2	1	0	0
$aa^\vee$	1	0	1	1
$bb^\vee$	2	1	0	0
$ba^\vee$	2	1	2	2

$(1_{bb^\vee}, 1_{ab^\vee}, 0_{aa^\vee}) \in \mathcal{P}^{>}_{c_{aa^\vee}}$

$(2_{bb^\vee}, 2_{ab^\vee}, 0_{aa^\vee}) \in \mathcal{P}^{\gg}_{c_{aa^\vee}}$

Th (D.-Konan 2022):

Let $\lambda \in \bar{P}_\ell^+$, $\mathcal{B}$ perfect crystal of level ℓ, H energy function on $\mathcal{B}$, $\cdots \otimes g \otimes \cdots \otimes g$ ground-state path.

Let $q = e^{-\delta/d_0}$, $c_b = e^{wt(b)}$ for all $b \in \mathcal{B}$. Then

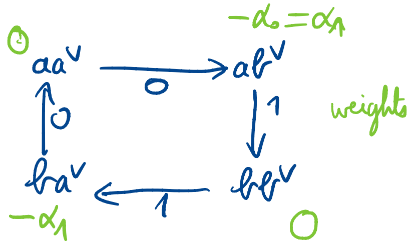
$$e^{-\lambda} ch(L(\lambda)) = \sum_{\pi \in \mathcal{P}^{\gtrdot}_{c_g}} C(\pi)\, q^{|\pi|}$$

$|\pi|$ = sum of parts of π

$$= (q;q)_\infty \sum_{\pi \in \mathcal{P}^{\gg}_{c_g}} C(\pi)\, q^{|\pi|}.$$

By taking the principal specialisation and equating with the Lepowsky product formula, we obtain RR-type partition identities.

<u>ex:</u> $A_1^{(1)}$, $\mathcal{B}_1 \otimes \mathcal{B}_1^\vee$, $\lambda = \Lambda_0$



$$e^{-\Lambda_0} ch(L(\Lambda_0)) = (q;q)_\infty \sum_{\pi \in \mathcal{P}^{\gg}_{c_{aa^\vee}}} C(\pi)\, q^{|\pi|}$$

Principal specialisation: $q = e^{-\delta} = e^{-\alpha_0 - \alpha_1} \longmapsto q^2$

$c_{aa^\vee} = c_{bb^\vee} = 1 \longmapsto 1$

$c_{ab^\vee} = e^{\alpha_1} \longmapsto q^{-1}$

$c_{ba^\vee} = e^{-\alpha_1} \longmapsto q$

$$F_1\left(e^{-\Lambda_0} ch(L(\Lambda_0))\right) \overset{\text{Lepowsky}}{=} \frac{1}{(q;q^2)_\infty}$$

$$\overset{\text{DK}}{=} (q^2;q^2)_\infty \sum_{\pi \in \mathcal{P}_{C_g}^{\gg}} q^{2|\pi| + \#C_{ba^\vee} - \#C_{ab^\vee}}$$

$$\Rightarrow \text{Prime}: \sum_{\pi \in \mathcal{P}_{C_g}^{\gg}} q^{2|\pi| + \#C_{ba^\vee} - \#C_{ab^\vee}} = \frac{1}{(q;q)_\infty}$$

$$H = \begin{array}{c} \\ ab^\vee \\ aa^\vee \\ bb^\vee \\ ba^\vee \end{array} \begin{array}{c} \begin{array}{cccc} ab^\vee & aa^\vee & bb^\vee & ba^\vee \end{array} \\ \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 2 & 1 & 0 & 0 \\ 2 & 1 & 2 & 2 \end{pmatrix} \end{array} \xrightarrow{F_1} \begin{array}{c} \\ ab^\vee \\ aa^\vee \\ bb^\vee \\ ba^\vee \end{array} \begin{array}{c} \begin{array}{cccc} ab^\vee & aa^\vee & bb^\vee & ba^\vee \end{array} \\ \begin{pmatrix} 4 & 3 & 1 & 2 \\ 1 & 0 & 2 & 3 \\ 3 & 2 & 0 & 1 \\ 2 & 1 & 3 & 4 \end{pmatrix} \end{array} = H'$$

Th: The number of partitions of n into parts coloured $ab^\vee$, $aa^\vee$, $bb^\vee$ and $ba^\vee$ s.t.

- parts coloured $aa^\vee$ and $bb^\vee$ are even
- parts coloured $ab^\vee$ and $ba^\vee$ are odd
- the minimal difference between parts coloured c_1 and c_2 is given by the entry (c_1, c_2) in the matrix H' (starting from the smallest part)

equals the number of partitions of n.

D.-Hardiman-Konan (2023): $A_1^{(1)}$ all levels $\ell \geq 1$

Benedek Dombos (2024+) : $G_2^{(2)}$ level 1

<u>Other approach</u>: Study grounded partitions combinatorially (keeping track of the colours if possible) to obtain nice expressions for characters.

Often, we obtain infinite products or sums of infinite products without having to perform the principal specialisation.

This was done in a series of papers D.-Konan (2022-2024+)

It includes a version of the combinatorial character formula for non-constant ground state paths too, using <u>multigrounded partitions</u>.

Characters for all level 1 irreducible highest weight modules of:

- $A_n^{(1)}$ (recovers Kac-Peterson character formula)
- $C_n^{(1)}$ (proves the Capparelli-Meurman-Primc-Primc conjecture)
- $A_{2n}^{(2)}$ and $D_{n+1}^{(2)}$ (recovers character formulas of Wakimoto)
- $A_{2n-1}^{(2)}$, $B_n^{(1)}$, $D_n^{(1)}$ (new)

setting $q = e^{-\frac{\delta}{2}}$, $c_i = e^{\alpha_i + \dots + \alpha_{n-1} + \frac{\alpha_n}{2}}$ (where $\delta = \alpha_0 + \alpha_1 + 2\alpha_2 + \dots + 2\alpha_{n-1} + \alpha_n$),

$$e^{-\Lambda_0} ch(L(\Lambda_0)) = \frac{(q^2;q^4)_\infty}{2} \left(\prod_{k=1}^{n} (-c_k q;q^2)_\infty (-c_k^{-1} q;q^2)_\infty + \prod_{k=1}^{n} (c_k q;q^2)_\infty (c_k^{-1} q;q^2)_\infty \right)$$

The case of non-constant ground-state path

<u>def:</u> $\mathcal{C}$ set of colours, $\succ$ binary relation on $\mathbb{Z}_{\mathcal{C}}$.

Suppose there exist colours $c_{g_0}, \dots, c_{g_{t-1}}$ in $\mathcal{C}$ and unique coloured integers $u_{c_{g_0}}^{(0)}, \dots, u_{c_{g_{t-1}}}^{(t-1)}$ s.t.

- $u^{(0)} + \dots + u^{(t-1)} = 0$
- $u_{c_{g_0}}^{(0)} \succ u_{c_{g_1}}^{(1)} \succ \dots \succ u_{c_{g_{t-1}}}^{(t-1)} \succ u_{c_{g_0}}^{(0)}$

Then a <u>multigrounded partition</u> with ground $(c_{g_0}, \dots, c_{g_{t-1}})$ and relation $\succ$ is a finite sequence $\pi = (\pi_0, \dots, \pi_{s-1}, u_{c_{g_0}}^{(0)}, \dots, u_{c_{g_{t-1}}}^{(t-1)})$ of coloured integers s.t.

- $\pi_i \succ \pi_{i+1} \ \forall i$
- $(\pi_{s-t}, \dots, \pi_{s-1}) \neq (u_{c_{g_0}}^{(0)}, \dots, u_{c_{g_{t-1}}}^{(t-1)})$

The set of multigrounded partitions (with number of parts divisible by t) is denoted $\mathcal{P}^{\succ}_{c_{g_0}, \dots, c_{g_{t-1}}}$ (t)

<u>ex:</u> $\mathcal{C} = \{1, 2, 3\}$, $\quad \Pi = \begin{pmatrix} 2 & 2 & 2 \\ 0 & 0 & 2 \\ -2 & 0 & 2 \end{pmatrix}$

$k_c \succ k'_{c'}$ iff $k - k' \geq \Pi_{cc'}$

ground $(g_0, g_1) = (1, 3)$

Then $(u^{(0)}, u^{(1)}) = (1, -1)$ is the unique pair s.t. $u^{(0)} + u^{(1)} = 0$, $u_{c_1}^{(0)} \succ u_{c_3}^{(1)} \succ u_{c_1}^{(0)}$.

$(3_{c_3}, 3_{c_2}, 3_{c_1}, -1_{c_3}, 1_{c_1}, -1_{c_3})$ multigrounded partition

B perfect crystal of level ℓ, $\lambda \in \overline{P}_\ell^+$, ground state path

$p_\lambda = \cdots \otimes g_{k+1} \otimes g_k \otimes \cdots \otimes g_0$ of period t, H energy on B,

Set $H_\lambda(b \otimes b') := H(b \otimes b') - \frac{1}{t} \sum_{k=0}^{t-1} H(g_{k+1} \otimes g_k)$.

thus $\sum_{k=0}^{t-1} H_\lambda(g_{k+1} \otimes g_k) = 0$.

let $D \in \mathbb{Z}_{>0}$ s.t. • $DH_\lambda(B \otimes B) \subset \mathbb{Z}$

• $\frac{1}{t} \sum_{k=0}^{t-1} (k+1) DH_\lambda(g_{k+1} \otimes g_k) \in \mathbb{Z}$

Define $>$ and $\gg$ on $\mathbb{Z}_{\mathcal{C}_b}$ as:

- $k_{c_b} > k'_{c_{b'}}$ iff $k - k' = DH_\lambda(b \otimes b')$
- $k_{c_b} \gg k'_{c_{b'}}$ iff $k - k' \geq DH_\lambda(b \otimes b')$

${}^d_t P^{\gg}_{c_{g_0}, \dots, c_{g_{t-1}}} :=$ set of $\pi = (\pi_0, \dots, \pi_{s-1}, u^{(0)}_{c_{g_0}}, \dots, u^{(t-1)}_{c_{g_{t-1}}}) \in P^{\gg}_{c_{g_0}, \dots, g_{t-1}}$ s.t.

- $t \mid s$
- for all k, $\pi_k - \pi_{k+1} - DH_\lambda(p_{k+1} \otimes p_k) \in d\mathbb{Z}_{\geq 0}$ where $c(\pi_k) = c_{p_k}$

  $\pi_s = u^{(0)}_{c_{g_0}}$

Th (D.-Konan 2022):

All notation as before, $\cdots \otimes g_0 \otimes g_{t-1} \otimes \cdots \otimes g_0$ ground-state path.

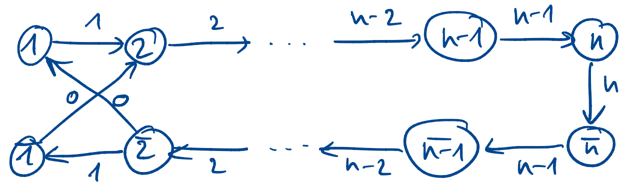
let $q = e^{-\delta/d_0 D}$, $c_b = e^{wt(b)}$ for all $b \in B$. Then

$$e^{-\lambda} ch(L(\lambda)) = \sum_{\pi \in {}_tP^{>}_{c_{g_0}\cdots c_{g_{t-1}}}} C(\pi)\, q^{|\pi|}$$

$$= (q^d;q^d)_\infty \sum_{\pi \in {}^d_tP^{\gg}_{c_{g_0}\cdots c_{g_{t-1}}}} C(\pi)\, q^{|\pi|}.$$

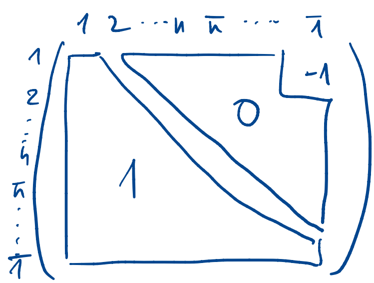
ex: $L(\Lambda_0)$ in $A^{(2)}_{2n-1}$ $(n \geq 3)$

B



ground state path $p_{\Lambda_0} = \cdots \otimes 1 \otimes \bar{1} \otimes 1 \otimes \bar{1}$

energy $H =$

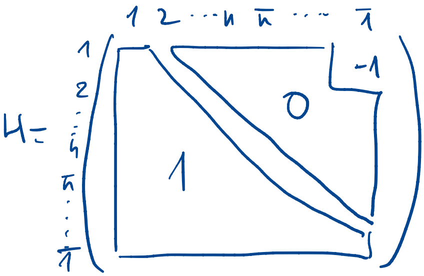


- $H(1 \otimes \bar{1}) + H(\bar{1} \otimes 1) = 0 \Rightarrow H_{\Lambda_0} = H$
- For $D = 2$ and $d = 2$, $e^{-\Lambda_0} ch(L(\Lambda_0)) = (q^2;q^2)_\infty \sum_{\pi \in {}^2_2P^{\gg}_{c_{\bar{1}} c_1}} C(\pi)\, q^{|\pi|}$

where $q = e^{-\delta/2}$, $c_i = e^{\alpha_i + \cdots + \alpha_{n-1} + \frac{\alpha_n}{2}}$.

We need the g.f. for ${}^{2}_{2}\mathcal{P}^{\gg}_{c_{\bar{1}}c_1}$, the set of multigrounded partitions $(\pi_0, \dots, \pi_{2s-1}, -1_{c_{\bar{1}}}, 1_{c_1})$ with relation $\gg$ having an even number of parts, s.t. $\forall k \in \{0, \dots, 2s-1\}$,

$$\pi_k - \pi_{k+1} - 2H(p_{k+1} \otimes p_k) \in 2\mathbb{Z}_{\geq 0} \qquad (*)$$



Because of H, $(*)$ and the fact that $\pi_{2s} = -1_{c_{\bar{1}}}$, the parts are all odd.

$\gg$ corresponds to the partial order on coloured odd integers:

$$\begin{matrix} -1_{c_{\bar{1}}} \\ 1_{c_1} \end{matrix} \ll 1_{c_2} \ll \dots \ll 1_{c_n} \ll 1_{c_{\bar{n}}} \ll \dots \ll 1_{c_{\bar{2}}} \ll \begin{matrix} 1_{c_{\bar{1}}} \\ 3_{c_1} \end{matrix} \ll 3_{c_2} \ll \dots$$

Only parts coloured c_1 and $c_{\bar{1}}$ can appear several times, in sequences $\dots \ll (2k-1)_{c_{\bar{1}}} \ll (2k+1)_{c_1} \ll (2k-1)_{c_{\bar{1}}} \ll \dots$

Such a sequence for $k \geq 1$ is generated by $\dfrac{(1+c_{\bar{1}}q^{2k-1})(1+c_1 q^{2k+1})}{(1-c_{\bar{1}}c_1 q^{4k})}$.

For $k=0$, we can have a single 1_{c_1}.

So, ignoring the cond. that the number of parts is even, the g.f. for ${}^{2}_{2}\mathcal{P}^{\gg}_{c_{\bar{1}}c_1}$ is

$$G(c_1, c_{\bar{1}}, \dots, c_n, c_{\bar{n}}; q) = (1+c_1 q) \frac{(-c_1 q^3, c_1 q, c_2 q, c_{\bar{2}} q, \dots, c_n q, c_{\bar{n}} q; q^2)_\infty}{(c_{\bar{1}} c_1 q^4; q^4)_\infty}.$$

Remark: $\sum_{n,k\geq 0} a_{n,k}\, x^k q^n + \sum_{n,k\geq 0} a_{n,k} (-x)^k q^n = 2 \sum_{n,k\geq 0} a_{n,2k}\, x^{2k} q^n$.

Hence $\sum_{\pi \in {}_2^2\mathcal{P}_{c_1 \bar{c}_1}^{>>}} C(\pi)\, q^{|\pi|} = \frac{1}{2}\Big(G(c_1, \bar{c}_1, \dots, c_n, \bar{c}_n ; q) + G(-c_1, -\bar{c}_1, \dots, -c_n, -\bar{c}_n; q) \Big)$

$$= \frac{1}{2(\bar{c}_1 c_1 q^4; q^4)_\infty} \Big((c_1 q, \bar{c}_1 q, \dots, c_n q, \bar{c}_n q; q^2)_\infty + (-c_1 q, -\bar{c}_1 q, \dots, -c_n q, -\bar{c}_n q; q^2)_\infty \Big)$$

by DK

$$= \frac{e^{-\Lambda_0} ch(L(\Lambda_0))}{(q^2;q^2)_\infty}, \quad q = e^{-\delta/2}, \quad c_i = e^{\alpha_i + \dots + \alpha_{n-1} + \frac{\alpha_n}{2}}.$$

$\square$