SLC 92
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Combinatorics on resolving subcategories for gentle trees

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[joint work in progress with
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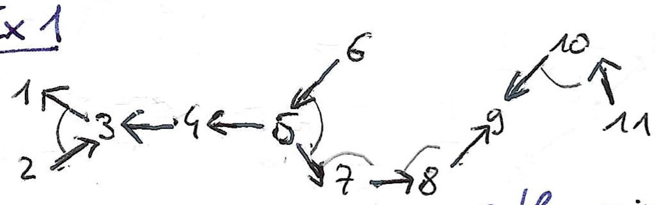
I/ Algebraic context
II/ Geometric model
III/ Resolving subcat 1
IV/ Res-relation & Resolving poset
V/ Resolving subcat 2
VI/ To go farther

Notes

I/ Algebraic context

Ex 1

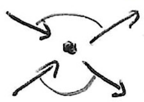


(Q,R) gentle tree (= gentle quiver + tree)

Q = Oriented tree

$R = \{ \to\!\!\curvearrowright\!\!\to \}$

such that at each vertex of Q we have a subconfiguration of



$\mathbb{K}$ = algebraically closed field

$\mathrm{rep}_{\mathbb{K}}(Q,R)$ = category of representations of (Q,R)

= assignment of

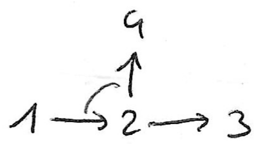
1) a $\mathbb{K}$-vector space to each vertex of Q
2) a $\mathbb{K}$-linear map to each arrow of Q
3) Compositions in R vanish

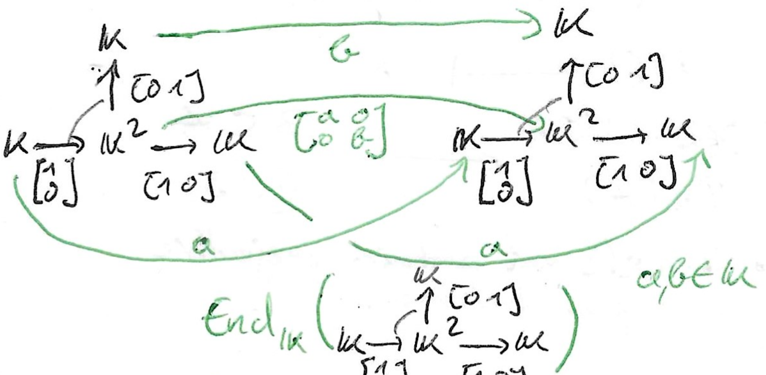
Thm 2 $\mathrm{rep}_{\mathbb{K}}(Q,R) \underset{\text{eq. of cat.}}{\simeq} \mathbb{K}Q/\langle R\rangle\text{-mod}$

$\mathbb{K}Q$ = path algebra

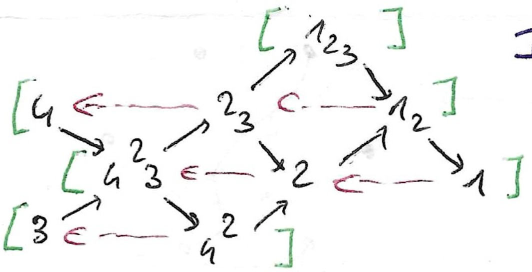
$\langle R\rangle$ = (bisided) ideal generated by R

Ex 3





To describe $\mathrm{rep}_{\mathbb{K}}(Q,R)$, we calculate the Auslander-Reiten quiver of (Q,R).



Indecomposable representations + Irreducible morphisms [projective injective]

[Butler-Ringel '87]

Indec. repo in $\mathrm{rep}_{\mathbb{K}}(Q,R)$ $\xleftrightarrow{\sim}$ Reduced adm. walks in (Q,R)

[Crawley-Boevey '89]

Maps between indecomposable representations.

Resolving subcategories of $\mathrm{rep}_{\mathbb{K}}(Q,R)$

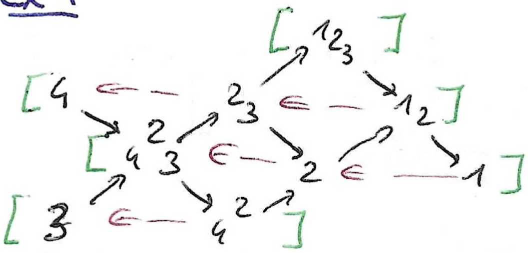
= • contain $\mathrm{proj}(Q,R)$
• are closed under Ext^i $(i \geq 0)$
• closed under kernels of epimorphisms.

↳ We can reduce ourselves to check Ext^1-closure between indecomposable reps, and kernels of $M \twoheadrightarrow N$ where $N \in \mathrm{ind}(Q,R)$

Goal: Describe explicitly all of them (+ ...)

For $\mathcal{X} \subseteq \mathrm{ind}(Q,R)$, write $\mathrm{Res}(\mathcal{X})$ for the resolving closure of $\mathcal{X}$ = the smallest resolving subcategory containing $\mathcal{X}$.

Ex 4



If $\mathcal{R} \subseteq \mathrm{rep}(Q,R)$ is resolving, then:
• $1, 2 \in \mathcal{R} \Rightarrow {}^{1}{}_{2} \in \mathcal{R}$
• $1 \in \mathcal{R} \Rightarrow {}^{2}{}_{3} \in \mathcal{R}$
etc...

II/ Geometric model

Thm 5 [Baur - Coelho Simoes '18, Opper - Plamondon - Schroll '18]

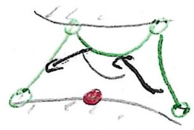
$(Q,R) \overset{\sim}{\longmapsto} (\Sigma, \mathcal{M}, \Delta^\circ)$

gentle quiver ⟶ Marked oriented surface endowed with a dissection

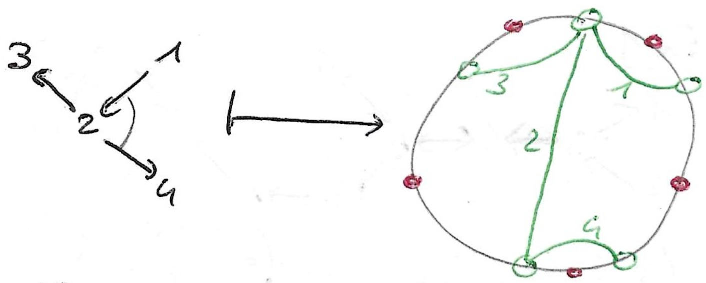
Here: gentle tree $\overset{\sim}{\longleftrightarrow}$ Disc

$Q \ni \bullet \overset{\sim}{\longmapsto}$ 

$\bullet \leftarrow \bullet \longmapsto$ 

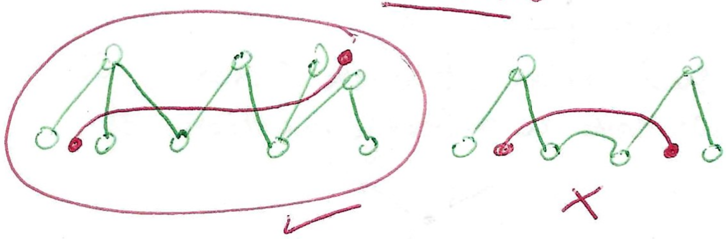
$\bullet \leftarrow \bullet \leftarrow \bullet \longmapsto$ 

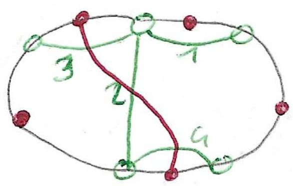
Ex 6

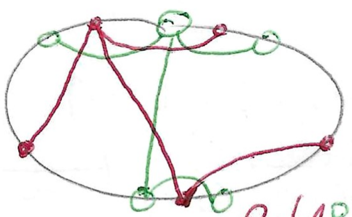


$\mathrm{ind}(Q,R) \overset{\sim}{\longrightarrow} \mathcal{A} = \mathcal{A}(\Sigma, \mathcal{M}, \Delta^\circ)$

Accordions



${}_{4}{}^{2}{}_{3} \longmapsto$ 

$\mathrm{proj}(Q,R) \longmapsto$ 

$\mathrm{Prj}(\Delta^\circ)$

(isomorphic dissections)

In the following, we will use $\mathrm{Prj}(\Delta^\circ)$

Notations:

$\mathrm{ind}(Q,R) \overset{\sim}{\longrightarrow} \mathcal{A}$

$X \longmapsto \gamma(X)$

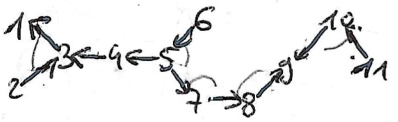
$M(\delta) \longleftarrow \delta$

$\mathcal{X} \subseteq \mathrm{ind}(Q,R) \longleftrightarrow \mathcal{B} \subseteq \mathcal{A}$

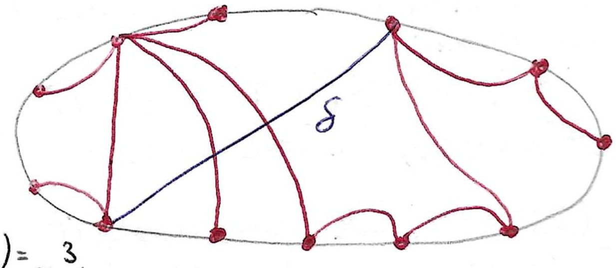
$\mathrm{Res}(\mathcal{X})$ $\qquad$ $\mathcal{R}(\mathcal{B})$

• $\mathcal{X}$ will not contain projective representations.

III / Resolving subcategories generated by one indecomposable representations

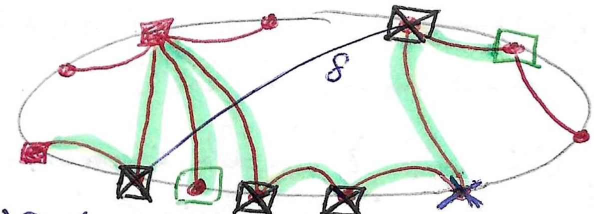
Ex 7 $(Q,R) =$ 

Proj



$\mathcal{N}(\delta) = {}_2{}^3{}_4{}_5$

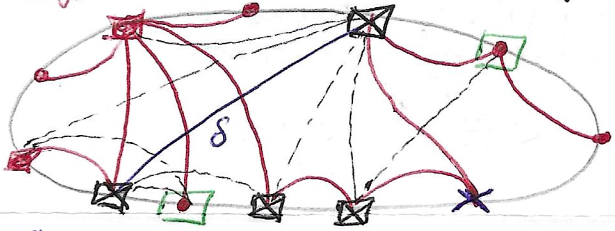
We will construct $\mathcal{R}(\delta)$



1) Determine the neighboring projective accordions $N_{proj}(\delta)$ (Ext^i + proj resolution)

2) Add coloration to vertices attained by $N_{proj}(\delta)$

3) Prop [DS '24+] $\mathcal{R}(\delta) = \mathrm{Proj} \cup \{ \ldots, \ldots, \ldots \} \cup \{\delta\}$



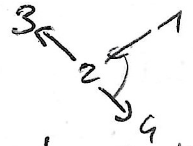
IV / Res-relation and Resolving poset

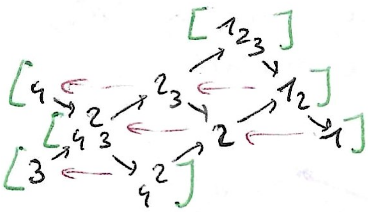
Thm [DS '24+]
The Res-relation $\trianglelefteq$ defined on $\mathrm{ind}(Q,R) \setminus \mathrm{proj}$ by
$$X \trianglelefteq Y \iff \mathrm{Res}(X) \subseteq \mathrm{Res}(Y)$$
is an order relation.

Proof: Quiver rep tools!

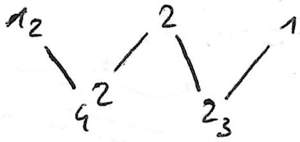
Rk:
- By construction, resolving subcategories correspond to some ideals of $(\mathrm{ind}\setminus\mathrm{proj}, \trianglelefteq)$
- $\trianglelefteq$ does not keep the extensions informations

$\hookrightarrow$ Relationship maps

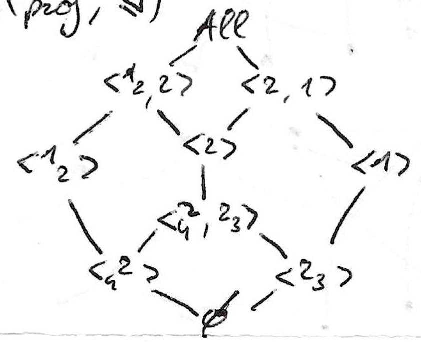
Ex 9 $(Q,R) =$ 



$(\mathrm{ind}\setminus\mathrm{proj}, \trianglelefteq)$



$\mathcal{J}(\mathrm{ind}\setminus\mathrm{proj}, \trianglelefteq)$



Who is the intruder?

Def 10: $(P,\leq)$ finite poset
- $\text{¥} = (\text{¥}^k : P^k \longrightarrow \mathcal{P}(P))_{1\leq k\leq n}$ Relationship map (of degree $n \in \mathbb{N}^*$)
- $I \in \mathcal{J}(P,\leq)$ ¥-closed $\iff$ $\forall k$, $\forall x_1, \ldots, x_k \in I$, $\text{¥}^k(x_1,\ldots,x_k) \subseteq I$
- $I^{\text{¥}}$ ¥-closure of I
- $(\mathcal{J}_{\text{¥}}(P,\leq), \subseteq)$ (complete) lattice of ¥-closed ideals of $\mathcal{J}(P,\leq)$
- ¥ ideally increasing $\iff$ $\forall k$ $\forall u_1,\ldots,u_k, v_1,\ldots,v_k \in P$, $u_i \geq v_i$: $\langle \text{¥}^k(v_1,\ldots,v_k)\rangle \subseteq \langle \text{¥}^k(u_1,\ldots,u_k)\rangle$

In our setting:
- $(P,\leq) = (\mathrm{ind}\setminus\mathrm{proj}, \trianglelefteq)$
- $\mathcal{E}$ relationship map of degree 2
  - $\mathcal{E}^1(x) = \emptyset$
  - $\mathcal{E}^2(x,y) = \{x,y\} \cup \{z \mid z \in \mathrm{Ext}^1(x,y) \cup \mathrm{Ext}^1(y,x)\}$
- Prop [DS '24+]
  $\mathcal{J}_{\mathcal{E}}(P,\leq) \underset{\text{poset}}{\cong} \mathrm{Res}(Q,R)$ = res. cat. ordered by inclusion — Resolving poset
- Prop [DS '24+]
  $\mathcal{E}$ is ideally increasing.

- $\mathcal{Y}(P,\leq) \xrightarrow{\sim}$ Antichains in $(P,\leq)$
  $I \longmapsto$ max. el. in I
- Prop [DS '24+] Υ ideally increasing
  $\mathcal{Y}$ antichain of $(P,\leq)$

$\langle\mathcal{Y}\rangle$ Υ-closed $\iff$
$\forall k, \forall a_1,\dots,a_k \in \mathcal{Y}, \Upsilon^k(a_1,\dots,a_k) \subseteq \langle\mathcal{Y}\rangle$

We can explicit an algorithm such that, given any antichain $\mathcal{X}$ in $(P,\leq)$, returns the antichain $\mathcal{Y}$ such that:

$$\langle\mathcal{X}\rangle^{\Upsilon} = \langle\mathcal{Y}\rangle$$

Algo: Υ ideally increasing

(1) Input $\mathcal{X}_0 = \mathcal{X}$ and $K_0 = \langle\mathcal{X}_0\rangle$

(2) i^{th} iteration: $\mathcal{X}_i$, $K_i = \langle\mathcal{X}_i\rangle$

(3) If $\exists k, \exists a_1,\dots,a_k \in \mathcal{X}_i \mid \Upsilon^k(a_1,\dots,a_k) \not\subseteq K_i$
Then construct the antichain $\mathcal{Y}_{i+1}$ such that
$$\langle\mathcal{Y}_{i+1}\rangle = \langle\mathcal{Y}_i \cup \Upsilon^k(a_1,\dots,a_k)\rangle$$

(4) otherwise, we are done.

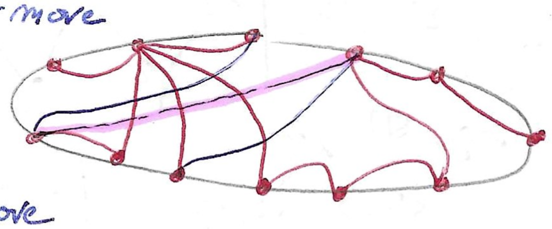
Cor [DS '24+] We can apply it to $(\mathrm{ind}\setminus\mathrm{proj}, \trianglelefteq)$ equipped with $\underline{\varepsilon}$ (or to its equivalent in the geometric model)

V/ All the resolving subcategories

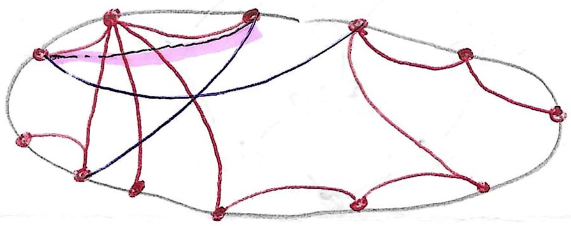
Back to $(\mathrm{ind}\setminus\mathrm{proj}, \trianglelefteq)$ with $\underline{\varepsilon}$

We can geometrically and explicitly express Step (3) of the algorithme

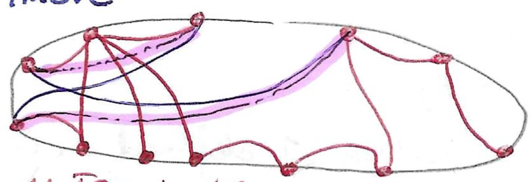
CoZ-move



V-move



X-move



Thm 11 [DS '24+]

$\forall \mathcal{X} \subseteq \mathrm{ind}\setminus\mathrm{proj}(Q,\Bbbk)$

$\mathrm{Res}(\mathcal{X}) = \mathrm{Res}(\mathcal{Y}) = \mathrm{add}(\langle\mathcal{Y}\rangle \cup \mathrm{proj})$

where $\mathcal{Y}$ is the antichain of $(\mathrm{ind}\setminus\mathrm{proj}, \trianglelefteq)$ obtained by applying the algorithm to $\mathcal{X}'$, the antichain such that $\langle\mathcal{X}'\rangle = \langle\mathcal{X}\rangle$.

"$\mathrm{Res}(\mathcal{X}) = \bigcup_{y \in \mathcal{Y}} \mathrm{Res}(y)$"

VI/ To go further

- Extract a tilting rep from each resolving subcat.
  [Auslander-Reiten '91]
  $\mathrm{Res}(Q,\Bbbk) \xrightarrow{\sim}$ Tilting reps
  ↳ Using the antichain obtained via the algorithm + ...
- Extend results to any gentle quiver (with some unavoidable hypothesis ...)
  ↳ Bricks ($\mathrm{End}_{\Bbbk}(x) \cong \Bbbk$ for all $x \in \mathrm{ind}$)
  The Res-relation is still an order
  It should therefore work similarly

↳ In general, the Res-relation is no longer an order
↳ Use the universal cover ...

- Study more posets with rel. maps.

Any question ?