The EGZ Theorem and a formula of Vladeta Jovovic

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Introduction

Theorem (EGZ Theorem (Erdős, Ginzburg and Ziv))

Each set of $2n-1$ integers contains some subset of n elements the sum of which is a multiple of n.

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THE ISRAEL MATHEMATICAL UNION

Theorem in the additive number theory

P. ERDÖS, A. GINZBURG AND A. ZIV, *Division of Mathematics, Technion–Israel Institute of Technology, Haifa*

THEOREM. *Each set of $2n-1$ integers contains some subset of n elements the sum of which is a multiple of n.*

Introduction

Question. Given a $(2n-1)$-member set of integers, can we count the number of n-member subsets such that its elements sum to a multiple of n?

Theorem (Conjectured by Vladeta Jovovic, Proved by Max Alekseyev)

The number of n-member subsets of $\{1,2,\ldots,2n-1\}$ *such that its elements sum to a multiple of n is*

$$s(n) = \frac{(-1)^n}{2n}\sum_{d|n}(-1)^d\phi\left(\frac{n}{d}\right)\binom{2d}{d}.$$

A145855	Number of n-element subsets of {1,2,...,2n-1} whose elements sum to a multiple of n.	+40 6

1, 1, 4, 9, 26, 76, 246, 809, 2704, 9226, 32066, 112716, 400024, 1432614, 5170604, 18784169, 68635478, 252085792, 930138522, 3446167834, 12815663844, 47820414962, 178987624514, 671825133644, 2528212128776, 9536894664376 (list; graph; refs; listen; history; text; internal format)

OFFSET 1,3

COMMENTS It is easy to see that {1,2,...,2n-1} can be replaced by any 2n-1 consecutive numbers and the results will be the same. Erdos, Ginzburg and Ziv proved that every set of 2n-1 numbers -- not necessarily consecutive -- contains a subset of n elements whose sum is a multiple of n.

LINKS Seiichi Manyama, Table of n, a(n) for n = 1..1669 (terms 1..200 from T. D. Noe)

Max Alekseyev, Proof of Jovovic's formula, 2008.

FORMULA a(n) = (1/(2*n))*Sum_{d|n} (-1)^(n+d)*phi(n/d)*binomial(2*d,d). Conjectured by Vladeta Jovovic, Oct 22 2008; proved by Max Alekseyev, Oct 23 2008 (see link).

Introduction

THE JOHN RIORDAN PRIZE

In 2015 the OEIS is offering a prize of \$1000 for the best solution to an open problem in the OEIS.

The On-Line Encyclopedia of Integer Sequences invites you to solve an open problem in an entry in the OEIS (https://oeis.org). After adding your solution to the OEIS entry, notify secretary@oeisf.org with subject line "Riordan Prize Nomination". (It is perfectly acceptable to nominate your own work.) The deadline for submission is December 1, 2015. The decision will be made by a special prize committee, and will be announced at the Joint Mathematics Meetings in Seattle in January 2016.

To find problems to work on, search in the OEIS for the words "conjecture", "empirical", "evidence suggests", "it would be nice", "would like", etc. Or find a formula or recurrence (with proof, of course) for a sequence that currently has no formula.

The prize is named after John Riordan (1903-1988; Bell Labs, 1926-1968), author of the classic books *An Introduction to Combinatorial Analysis* (1958) and *Combinatorial Identities* (1968), which were the source for hundreds of early entries in the OEIS.

Here is an example from 2008: Sequence A145855 gives the number of n-element subsets of $\{1, ..., n\}$ whose sum is a multiple of n $(1, 1, 4, 9, 26, 76, 246, \ldots$ for $n \geq 1)$. Vladeta Jovovic conjectured that the nth term is $\frac{1}{2n}\sum_{d|n}(-1)^{n+d}\phi(\frac{n}{d})\binom{2d}{d}$, and Max Alekseyev found a proof. (This is not a candidate for the prize, since only work carried out in 2015 is eligible.)

Sketch of Alekseyev's Proof. We notice

$$s(n) = \sum_{\ell} \text{coeff. of } x^{\ell n} y^n \text{ in } \prod_{k=1}^{2n-1} (1 + yx^k).$$

Orthogonality of roots of unity $\Rightarrow$

$$G(q) = \sum_n g(n) q^n$$

$$\Downarrow$$

$$\sum_{n \equiv h \pmod{H}} g(n) q^n = \frac{1}{H} \sum_{\ell=1}^{H} e^{-\frac{2\pi i h \ell}{H}} G\left(e^{\frac{2\pi i \ell}{H}} q\right).$$

Sketch of Alekseyev's Proof (continued). Then

$$s(n) = -1 + \frac{1}{n^2} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \prod_{k=1}^{2n-1} \left(1 + \exp\left(\frac{2\pi \mathrm{i}(i+kj)}{n}\right)\right).$$

Further, we may compute that if $d = \gcd(n, j)$ and $n' = n/d$,

$$\prod_{k=1}^{2n-1} \left(1 + \exp\left(\frac{2\pi \mathrm{i}(i+kj)}{n}\right)\right)$$

$$= (-1)^{n+d} 2^{2n} \left(\prod_{t=0}^{n'-1} \cos\left(\frac{2\pi(i+td)}{2n}\right)\right)^{2d} \left(1 + \exp\left(\frac{2\pi \mathrm{i} i}{n}\right)\right)^{-1}.$$

The rest requires heavy computation involving trigonometric functions. □

Any simple proofs? Or generalizations?

Main Result

Theorem (C.)

Let $u > v$ be two positive integers and c be an integer. Let n be a positive integer. If $s(u, v; c; n)$ counts the number of subsets of $\{1, 2, \ldots, un-1\}$ of vn elements whose sum is congruent to c modulo n, then

$$s(u,v;c;n) = \begin{cases} \dfrac{u-v}{un}\displaystyle\sum_{d\mid n}\mu\left(\frac{n}{d\cdot\gcd\left(c,\frac{n}{d}\right)}\right)\frac{\phi\left(\frac{n}{d}\right)}{\phi\left(\frac{n}{d\cdot\gcd\left(c,\frac{n}{d}\right)}\right)}\binom{ud}{vd} & \text{if } n \text{ is odd or } n \text{ and } v \text{ are both even,} \\ \dfrac{u-v}{un}\displaystyle\sum_{d\mid n}(-1)^d\mu\left(\frac{n}{d\cdot\gcd\left(c,\frac{n}{d}\right)}\right)\frac{\phi\left(\frac{n}{d}\right)}{\phi\left(\frac{n}{d\cdot\gcd\left(c,\frac{n}{d}\right)}\right)}\binom{ud}{vd} & \text{if } n \text{ is even and } v \text{ is odd,} \end{cases}$$

where $\mu(\cdot)$ is the Möbius function and $\phi(\cdot)$ is Euler's totient function. We also adopt the convention that $\gcd(0, n) = n$.

Examples

Table: 2-, 3- and 4-Member subsets of $\{1,2,3,4,5\}$

$n = 3,\ u = 2,\ v = 1$

3-mem.	Σ mod 3	3-mem.	Σ mod 3
$\{1,2,3\}$	0	$\{1,4,5\}$	1
$\{1,2,4\}$	1	$\{2,3,4\}$	0
$\{1,2,5\}$	2	$\{2,3,5\}$	1
$\{1,3,4\}$	2	$\{2,4,5\}$	2
$\{1,3,5\}$	0	$\{3,4,5\}$	0

$n = 2,\ u = 3,\ v = 1$

2-mem.	Σ mod 2	2-mem.	Σ mod 2
$\{1,2\}$	1	$\{2,4\}$	0
$\{1,3\}$	0	$\{2,5\}$	1
$\{1,4\}$	1	$\{3,4\}$	1
$\{1,5\}$	0	$\{3,5\}$	0
$\{2,3\}$	1	$\{4,5\}$	1

Examples

Table: 2-, 3- and 4-Member subsets of $\{1, 2, 3, 4, 5\}$ (continued)

$n = 2, \ u = 3, \ v = 2$

4-mem.	Σ mod 2
$\{1, 2, 3, 4\}$	0
$\{1, 2, 3, 5\}$	1
$\{1, 2, 4, 5\}$	0
$\{1, 3, 4, 5\}$	1
$\{2, 3, 4, 5\}$	0

Corollaries

Corollary

Let $u > v$ be two positive integers. The number of subsets of $\{1, 2, \ldots, un-1\}$ of vn elements which sum to a multiple of n equals

$$\begin{cases} \dfrac{u-v}{un} \displaystyle\sum_{d|n} \phi\left(\frac{n}{d}\right)\binom{ud}{vd} & \text{if } n \text{ is odd or } n \text{ and } v \text{ are both even,} \\ \dfrac{u-v}{un} \displaystyle\sum_{d|n} (-1)^d \phi\left(\frac{n}{d}\right)\binom{ud}{vd} & \text{if } n \text{ is even and } v \text{ is odd.} \end{cases}$$

Corollary

Let $u > v$ be two positive integers. The number of subsets of $\{1, 2, \ldots, un-1\}$ of vn elements which sum to one more than a multiple of n equals

$$\begin{cases} \dfrac{u-v}{un} \displaystyle\sum_{d|n} \mu\left(\frac{n}{d}\right)\binom{ud}{vd} & \text{if } n \text{ is odd or } n \text{ and } v \text{ are both even,} \\ \dfrac{u-v}{un} \displaystyle\sum_{d|n} (-1)^d \mu\left(\frac{n}{d}\right)\binom{ud}{vd} & \text{if } n \text{ is even and } v \text{ is odd.} \end{cases}$$

Proof

Distinct partition $\to$ Restricted partition $\Leftrightarrow$ q-Binomial coefficient

Given a vn-element subset $\{x_1, x_2, \ldots, x_{vn}\}$ of $\{1, 2, \ldots, un-1\}$, we assume

$$\begin{array}{ccccccccccc} 1 & \leq & x_1 & < & x_2 & < & \cdots & < & x_{vn} & \leq & un-1 \\ & & | & & | & & & & | & & \\ & & 1 & & 2 & & & & vn & & \\ \hline 0 & \leq & x_1' & \leq & x_2' & \leq & \cdots & \leq & x_{vn}' & \leq & (u-v)n-1 \end{array}$$

We are led to a partition with largest part $\leq (u-v)n-1$ and number of parts $\leq vn$. Its generating function is

$$R(q) = \sum_{m=0}^{vn((u-v)n-1)} r(m)q^m = \begin{bmatrix} un-1 \\ vn \end{bmatrix}_q .$$

Proof

Notice that

$$\begin{aligned} 1+2+\cdots+vn &= \frac{vn(vn+1)}{2} \\ &\equiv \begin{cases} 0 \pmod{n} & \text{if } n \text{ is odd or } n \text{ and } v \text{ are both even,} \\ \frac{n}{2} \pmod{n} & \text{if } n \text{ is even and } v \text{ is odd.} \end{cases} \end{aligned}$$

Thus

$$\sum_{i=1}^{vn} x_i \equiv c \pmod{n}$$

$$\Updownarrow$$

$$\sum_{i=1}^{vn} x_i' \equiv \begin{cases} c \pmod{n} & \text{if } n \text{ is odd or } n \text{ and } v \text{ are both even,} \\ \frac{n}{2} + c \pmod{n} & \text{if } n \text{ is even and } v \text{ is odd.} \end{cases}$$

Proof

Thus

$$s(u,v;c;n) = \begin{cases} \displaystyle\sum_{\substack{m=0\\ m\equiv c \pmod n}}^{vn((u-v)n-1)} r(m) & \text{if } n \text{ is odd or } n \text{ and } v \text{ are both even,}\\ \displaystyle\sum_{\substack{m=0\\ m\equiv \frac{n}{2}+c \pmod n}}^{vn((u-v)n-1)} r(m) & \text{if } n \text{ is even and } v \text{ is odd} \end{cases}$$

$$= \begin{cases} \displaystyle\frac{1}{n}\sum_{\ell=1}^{n} e^{-\frac{2\pi i c\ell}{n}} R\big(e^{\frac{2\pi i\ell}{n}}\big) & \text{if } n \text{ is odd or } n \text{ and } v \text{ are both even,}\\ \displaystyle\frac{1}{n}\sum_{\ell=1}^{n} (-1)^{\ell} e^{-\frac{2\pi i c\ell}{n}} R\big(e^{\frac{2\pi i\ell}{n}}\big) & \text{if } n \text{ is even and } v \text{ is odd.} \end{cases}$$

Proof

Lemma

Let ℓ be a positive integer. Then

$$R\left(e^{\frac{2\pi i\ell}{n}}\right) = \frac{u-v}{u}\binom{u\cdot\gcd(\ell,n)}{v\cdot\gcd(\ell,n)}.$$

Proof. Recall that

$$R(q) = \begin{bmatrix} un-1 \\ vn \end{bmatrix}_q = \prod_{k=1}^{(u-v)n-1} \frac{1-q^{vn+k}}{1-q^k}.$$

Hence,

$$R\left(e^{\frac{2\pi i\ell}{n}}\right) = \prod_{k=1}^{(u-v)n-1} \lim_{q\to e^{\frac{2\pi i\ell}{n}}} \frac{1-q^{vn+k}}{1-q^k},$$

where

$$\lim_{q\to e^{\frac{2\pi i\ell}{n}}} \frac{1-q^{vn+k}}{1-q^k} = \begin{cases} 1 & \text{if } \left(e^{\frac{2\pi i\ell}{n}}\right)^k \neq 1 \\ \frac{vn+k}{k} & \text{if } \left(e^{\frac{2\pi i\ell}{n}}\right)^k = 1 \quad \Longleftrightarrow \quad n \text{ divides } k\ell \end{cases}$$

Proof

Proof (continued). For convenience, we write $n = n'd$ and $\ell = \ell' d$ where $d = \gcd(\ell, n)$. Then $n \mid k\ell$ is equivalent to $n' \mid k$. Hence,

$$\prod_{k=1}^{(u-v)n-1} \lim_{q\to e^{\frac{2\pi i\ell}{n}}} \frac{1-q^{vn+k}}{1-q^k} = \prod_{\substack{k=1\\ k\equiv 0 \pmod{n'}}}^{(u-v)n-1} \frac{vn+k}{k} \overset{(k=k'n')}{=} \prod_{k'=1}^{(u-v)d-1} \frac{vn+k'n'}{k'n'}$$

$$= \prod_{k'=1}^{(u-v)d-1} \frac{vd+k'}{k'} = \binom{ud-1}{(u-v)d-1} = \frac{u-v}{u}\binom{ud}{vd}.$$

Consequently, we have that

$$R(e^{\frac{2\pi i\ell}{n}}) = \frac{u-v}{u}\binom{ud}{vd} = \frac{u-v}{u}\binom{u\cdot\gcd(\ell,n)}{v\cdot\gcd(\ell,n)},$$

which is our desired result. □

Proof

Case 1. If n is odd or n and v are both even, then

$$\begin{aligned} s(u,v;c;n) &= \frac{1}{n}\sum_{\ell=1}^{n} e^{-\frac{2\pi i c\ell}{n}} R\left(e^{\frac{2\pi i\ell}{n}}\right) \\ &= \frac{u-v}{un}\sum_{\ell=1}^{n} e^{-\frac{2\pi i c\ell}{n}} \binom{u\cdot\gcd(\ell,n)}{v\cdot\gcd(\ell,n)} \\ &= \frac{u-v}{un}\sum_{d\mid n}\binom{ud}{vd} \sum_{\substack{\ell=1\\ \gcd(\ell,n)=d}}^{n} e^{-\frac{2\pi i c\ell}{n}} \\ &= \frac{u-v}{un}\sum_{d\mid n}\binom{ud}{vd} \sum_{\substack{\ell'=1\\ \gcd\left(\ell',\frac{n}{d}\right)=1}}^{\frac{n}{d}} \exp\left(-\frac{2\pi i c\ell'}{n/d}\right). \end{aligned}$$

Proof

Recall the evaluation of Ramanujan's sum

$$c_q(m) := \sum_{\substack{\ell=1 \\ \gcd(\ell,q)=1}}^{q} \exp\left(\frac{2\pi i m\ell}{q}\right) = \mu\left(\frac{q}{\gcd(m,q)}\right) \frac{\phi(q)}{\phi\left(\frac{q}{\gcd(m,q)}\right)}.$$

Thus

$$s(u,v;c;n) = \cdots$$

$$= \frac{u-v}{un} \sum_{d|n} \binom{ud}{vd} \sum_{\substack{\ell'=1 \\ \gcd\left(\ell',\frac{n}{d}\right)=1}}^{\frac{n}{d}} \exp\left(-\frac{2\pi i c\ell'}{n/d}\right)$$

$$= \frac{u-v}{un} \sum_{d|n} \mu\left(\frac{n}{d\cdot\gcd\left(c,\frac{n}{d}\right)}\right) \frac{\phi\left(\frac{n}{d}\right)}{\phi\left(\frac{n}{d\cdot\gcd\left(c,\frac{n}{d}\right)}\right)} \binom{ud}{vd}.$$

Proof

Case 2. If n is even and v is odd, then

$$\begin{aligned}
s(u,v;c;n) &= \frac{1}{n}\sum_{\ell=1}^{n}(-1)^{\ell}e^{-\frac{2\pi i c\ell}{n}}R\left(e^{\frac{2\pi i\ell}{n}}\right)\\
&= \frac{u-v}{un}\sum_{\ell=1}^{n}(-1)^{\ell}e^{-\frac{2\pi i c\ell}{n}}\binom{u\cdot\gcd(\ell,n)}{v\cdot\gcd(\ell,n)}\\
&= \frac{u-v}{un}\sum_{\ell=1}^{n}(-1)^{\gcd(\ell,n)}e^{-\frac{2\pi i c\ell}{n}}\binom{u\cdot\gcd(\ell,n)}{v\cdot\gcd(\ell,n)}\\
&= \frac{u-v}{un}\sum_{d\mid n}(-1)^{d}\binom{ud}{vd}\sum_{\substack{\ell=1\\ \gcd(\ell,n)=d}}^{n}e^{-\frac{2\pi i c\ell}{n}}\\
&= \frac{u-v}{un}\sum_{d\mid n}(-1)^{d}\mu\left(\frac{n}{d\cdot\gcd\left(c,\frac{n}{d}\right)}\right)\frac{\phi\left(\frac{n}{d}\right)}{\phi\left(\frac{n}{d\cdot\gcd\left(c,\frac{n}{d}\right)}\right)}\binom{ud}{vd}.
\end{aligned}$$

Closing Remarks

If we replace the set $\{1, 2, \ldots, un-1\}$ by an arbitrary set of $un-1$ consecutive integers, say $\{a+1, a+2, \ldots, a+un-1\}$, the formula in the main result still holds. In fact, if $\{x_1, x_2, \ldots, x_{vn}\}$ (with $x_1 < x_2 < \cdots < x_{vn}$) is a subset, we may put $x_i = x_i' + a + i$ for each $1 \leq i \leq vn$ and then carry out the same procedure.

Closing Remarks

For a general set $\mathcal{S}$ of $2n-1$ integers in the EGZ Theorem, we know that the number of such n-member subsets is

$$-1+\frac{1}{n^2}\sum_{j=1}^{n}\sum_{k=1}^{n}\prod_{s\in\mathcal{S}}\left(1+\exp\left(\frac{2\pi\mathrm{i}(j+ks)}{n}\right)\right).$$

However, there is no obvious approach that can simplify this expression.

Open Question. Can we find other examples of $\mathcal{S}$ that give nice closed forms of the number of the desired n-member subsets, or even may lead to parallel extensions like our main result?

Thank You!