Structure of quasi-crystal graphs and applications to the combinatorics of quasi-symmetric functions

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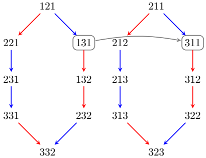
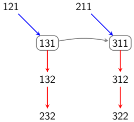
Outline

1. Plactic and Hypoplactic monoids
   - Tableaux
   - Crystal and quasi-crystal graphs
   - Symmetric and quasi-symmetric functions
2. Structure of quasi-crystal graphs
   - Quasi-arrays
   - Isomorphisms
   - Schur functions and Fundamental quasi-symmetric functions

Outline

Plactic Monoid	**Hypoplactic monoid**
Young tableaux, Schensted insertion	Quasi-ribbon tableaux, Krob–Thibon insertion
$2123 \longleftrightarrow \begin{array}{\|c\|c\|c\|}\hline 1&2&3\\ \hline 2 \\ \cline{1-1}\end{array}$	$2123 \longleftrightarrow \begin{array}{\|c\|c\|c\|}\cline{1-1} 1 \\ \hline 2&2&3\\ \hline\end{array}$
Crystal graphs	Quasi-crystal graphs
	
Symmetric functions s_λ	Quasi-symmetric functions F_α

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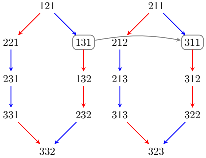	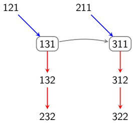
Symmetric functions s_λ	Quasi-symmetric functions F_α

Young tableaux and the plactic monoid

- We consider the alphabet $\mathcal{A} = \{1 < 2 < \ldots\}$ and the restriction $\mathcal{A}_n = \{1 < \ldots < n\}$.
- Given a partition λ, a semistandard **Young tableau** of shape λ is a filling of λ with letters from $\mathcal{A}$ such that the rows are weakly increasing and the columns are strictly increasing. In a **standard** tableau, each letter $i = 1, \ldots, |\lambda|$ appears exactly once.

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1	1	2	4	4
2	2			
3				

1	2	5	7	8
3	4			
6				

A semistandard and a standard Young tableaux, of shape $(5, 2, 1)$.

Young tableaux and the plactic monoid

- Schensted insertion: associates a word $w \in \mathcal{A}_n^*$ with a unique Young tableau $P_{\text{plac}}(w)$
- Plactic congruence in $\mathcal{A}^*$:

$$u \equiv_{\text{plac}} v \text{ iff } P_{\text{plac}}(u) = P_{\text{plac}}(v).$$

- The **plactic monoid** plac is the quotient of $\mathcal{A}^*$ by $\equiv_{\text{plac}}$. The **plactic monoid of rank** n plac_n is the quotient of $\mathcal{A}_n^*$ by the natural restriction of $\equiv_{\text{plac}}$.

Quasi-ribbon tableaux and the hypoplactic monoid

- Given a composition α, a **quasi-ribbon tableau** of shape α is an array of $|\alpha|$ cells, filled with letters from $\mathcal{A}$, with α_i cells on row i, such that the leftmost cell of the $(i+1)$-th row is below the rightmost cell of the i-th row, with the rows being weakly increasing and columns strictly increasing.

Quasi-ribbon tableaux and the hypoplactic monoid

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1	1	1	2	3		
				4		
				5	5	6

A quasi-ribbon tableau of shape $(5, 1, 3)$

Quasi-ribbon tableaux and the hypoplactic monoid

- Krob–Thibon insertion: associates a word $w \in \mathcal{A}_n^*$ with a unique quasi-ribbon tableau $P_{\mathrm{hypo}}(w)$.
- Hypoplactic congruence in $\mathcal{A}^*$:

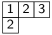
$$u \equiv_{\mathrm{hypo}} v \text{ iff } P_{\mathrm{hypo}}(u) = P_{\mathrm{hypo}}(v).$$

- The **hypoplactic monoid** hypo is the quotient of $\mathcal{A}^*$ by $\equiv_{\mathrm{hypo}}$. The **hypoplactic monoid of rank** n hypo_n is the quotient of $\mathcal{A}_n^*$ by the natural restriction of $\equiv_{\mathrm{hypo}}$.

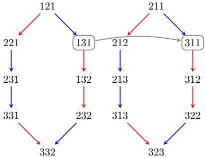
Outline

Plactic Monoid

Young tableaux,
Schensted insertion

2123 $\longleftrightarrow$ 

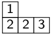
Crystal graphs



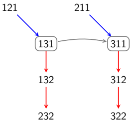
Symmetric functions
s_λ

Hypoplactic monoid

Quasi-ribbon tableaux,
Krob–Thibon insertion

2123 $\longleftrightarrow$ 

Quasi-crystal graphs



Quasi-symmetric functions
F_α

Crystal graphs

- Given a word $w \in \mathcal{A}^*$, the **Kashiwara operators** $\tilde{f}_i$, $\tilde{e}_i$ are partial functions computed as follows:
  - Consider only the letters i and $i+1$.
  - Replace each letter i with $+$ and each $i+1$ with $-$. Cancel all pairs $-+$.
  - $\tilde{f}_i$ (resp. $\tilde{e}_i$) changes the rightmost $+$ to $-$ (resp. the leftmost $-$ to $+$), if possible. Otherwise, it is undefined.

$$\begin{matrix} 1 & 1 & 2 & 2 & 1 & 1 & 3 & 2 & 4 \\ + & + & - & - & + & + & & - & \end{matrix} \xrightarrow{\tilde{f}_1} \begin{matrix} 1 & \mathbf{2} & 2 & 2 & 1 & 1 & 3 & 2 & 4 \\ + & - & - & - & + & + & & - & \end{matrix}$$

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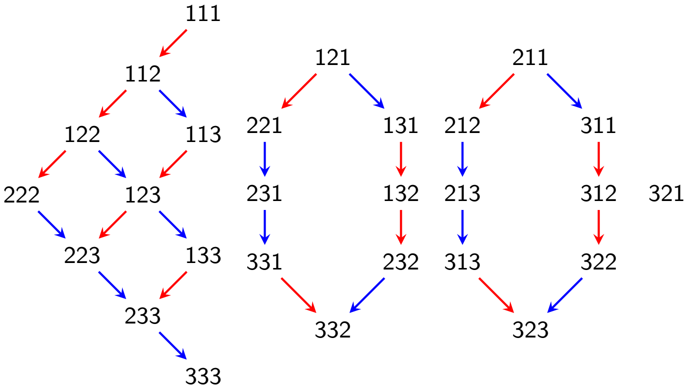
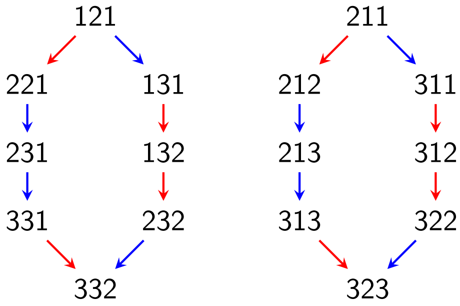
Crystal graphs

- The **crystal graph** $\Gamma(\text{plac})$ is the directed labelled graph with vertex set $\mathcal{A}^*$, having an edge $u \xrightarrow{i} v$ iff $\tilde{f}_i(u) = v$.
- $\Gamma(\text{plac}_n)$ is the subgraph induced by $\mathcal{A}_n^*$.



Crystal graphs

- Another characterization for the plactic monoid: $u \equiv_{\text{plac}} v$ iff there is a crystal isomorphism sending u to v.



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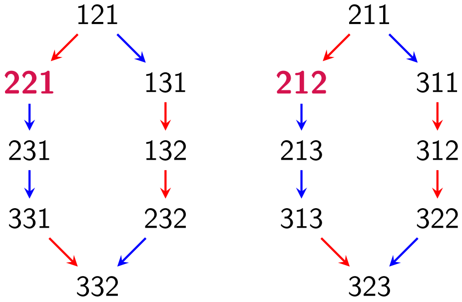

$$221 \equiv_{\text{plac}} 212 \text{ but } 221 \not\equiv_{\text{plac}} 311.$$

Quasi-crystal graphs

- The **quasi-Kashiwara operators** $\ddot{f}_i$, $\ddot{e}_i$ are defined on $w \in \mathcal{A}^*$ as follows:
  - If w has $(i+1)i$ as a subword, the operators are undefined.
  - Otherwise, $\ddot{f}_i(w) = \tilde{f}_i(w)$ and $\ddot{e}_i(w) = \tilde{e}_i(w)$.

$$\ddot{f}_1(112211324) = \perp$$

$$\ddot{f}_1(111122) = 111222$$

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$$\ddot{f}_1(112211324) = \bot$$
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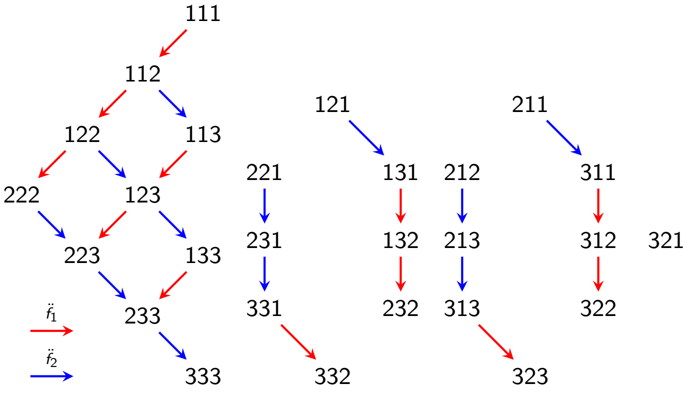
Quasi-crystal graphs

- The **quasi-crystal graph** $\Gamma(\mathrm{hypo})$ and its subgraph $\Gamma(\mathrm{hypo}_n)$ are defined similarly, considering the quasi-Kashiwara operators.



Quasi-crystal graphs

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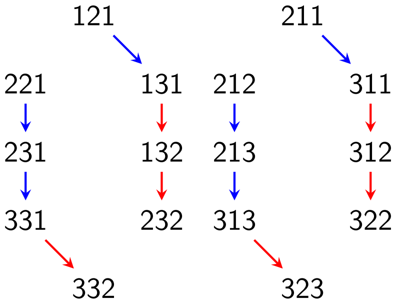

- $131 \equiv_{\mathrm{hypo}} 311$ but $131 \not\equiv_{\mathrm{hypo}} 231$, because their components are not isomorphic as labelled directed graphs.

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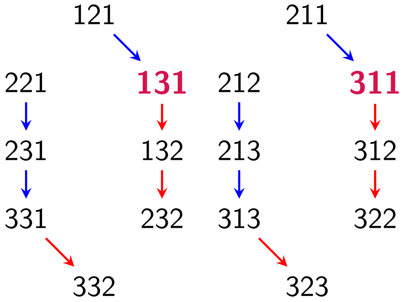

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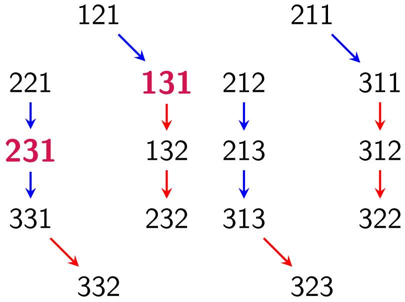

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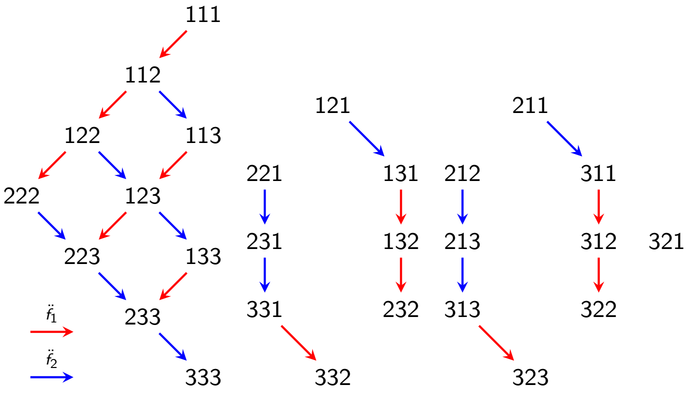
Crystal and quasi-crystal graphs

- If the quasi-Kashiwara operators $\ddot{e}_i$, $\ddot{f}_i$ are defined, then the Kashiwara operators $\tilde{e}_i$, $\tilde{f}_i$ are defined. Thus, $\Gamma(\text{hypo}_n)$ is a subgraph of $\Gamma(\text{plac}_n)$.



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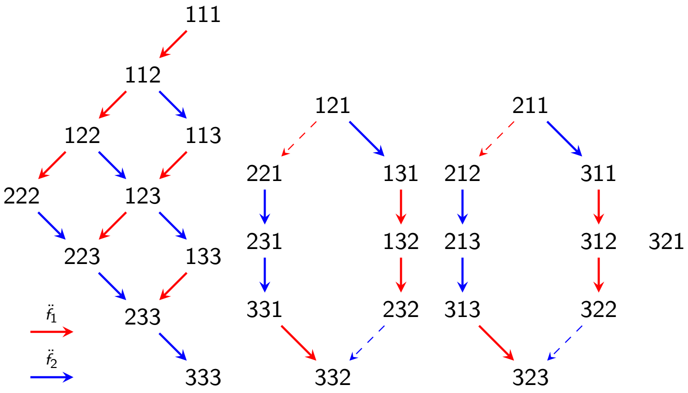
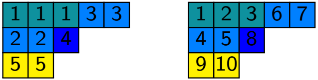


Crystal and quasi-crystal graphs

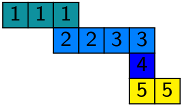
- The **minimal parsing** of a Young tableau T is the set of maximal horizontal bands. The minimal parsing of T has type α if the i-th maximal horizontal band has α_i cells.



Both tableaux have minimal parsing of type $(3, 4, 1, 2)$.

- Given a semistandard Young tableau T with minimal parsing of type α, $P_{\text{hypo}}(u)$ has shape α, for any word u such that $P_{\text{plac}}(u) = T$. The component of $\Gamma(\text{hypo})$ that contains u consists of the words of $\Gamma(\text{plac})$ whose corresponding Young tableau has the same minimal parsing as T.

$P_{\text{hypo}}(5522411133) =$



Crystal and quasi-crystal graphs

- The **descent composition** of a standard Young tableau is

$$\alpha = (i_1, i_2 - i_1, \ldots, i_k - i_{k-1}, n - i_k)$$

  where $\{i_1 < \ldots < i_k\}$ is its descent set.
- The descent composition of T coincides with the type of its minimal parsing.

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1	2	3	6	7
4	5	8		
9	10			

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$$\text{DesSet}(T) = \{3, 7, 8\}$$
$$\text{DesCom}(T) = (3, 4, 1, 2)$$

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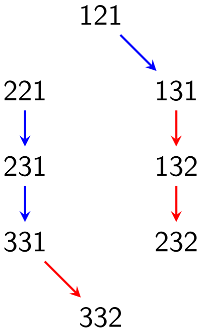
Crystal and quasi-crystal graphs

- Certain components of quasi-crystal graphs seem to be isomorphic as unlabelled directed graphs:



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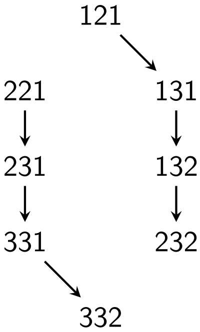

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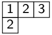
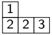
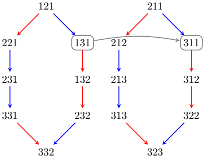
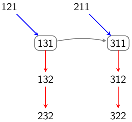
2123 $\longleftrightarrow$ 	2123 $\longleftrightarrow$ 
Crystal graphs 	Quasi-crystal graphs 
Symmetric functions s_λ	**Quasi-symmetric functions** F_α

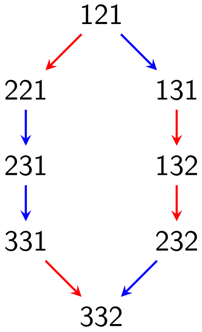
Symmetric and quasi-symmetric functions

- Given a partition λ, the **Schur function** is defined as

$$s_\lambda(\mathbf{x}) = \sum_{T \in \mathrm{SSYT}_n(\lambda)} \mathbf{x}^T$$

- Schur functions form a basis for the ring of symmetric functions. They also appear as characters of components of $\Gamma(\mathrm{plac}_n)$ where the associated Young tableau has shape λ.

Symmetric and quasi-symmetric functions



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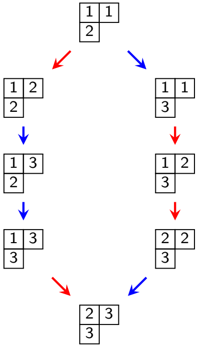
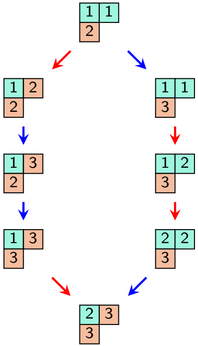


$$s_{21}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_1 x_2 x_3 + x_2^2 x_3 + x_1 x_2^2 + x_1 x_2 x_3 + x_1 x_3^2 + x_2 x_3^2$$

Symmetric and quasi-symmetric functions



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Symmetric and quasi-symmetric functions

- Given a composition $\alpha = (\alpha_1, \ldots, \alpha_k)$, the **monomial quasi-symmetric function** is $M_\alpha(\mathbf{x}) = \sum_{i_1 < \ldots < i_k} x_{i_1}^{\alpha_1} \ldots x_{i_k}^{\alpha_k}$.
- The **fundamental quasi-symmetric function** is defined as $F_\alpha = \sum_{\alpha \preceq \beta} M_\beta$.
- If $\alpha \preceq \beta$, then M_β corresponds to a unique quasi-ribbon tableau of shape α with β_j letters i_j. Thus,

$$F_\alpha(\mathbf{x}) = \sum_{T \in \mathrm{QRT}_n(\alpha)} \mathbf{x}^T$$

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$$F_{13} = M_{13} + M_{121} + M_{112} + M_{1111}$$

$$\begin{array}{|c|} \hline 1 \\ \hline \end{array}\!\!\begin{array}{l} \\ \end{array} \quad \begin{array}{l} 1 \\ 2\,2\,2 \end{array} \qquad \begin{array}{l} 1 \\ 2\,2\,3 \end{array} \qquad \begin{array}{l} 1 \\ 2\,3\,3 \end{array} \qquad \begin{array}{l} 1 \\ 2\,3\,4 \end{array}$$

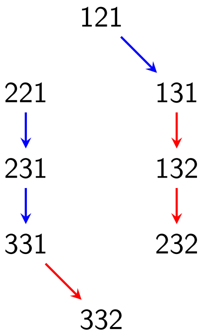
Symmetric and quasi-symmetric functions

- The functions F_α form a basis for the ring of quasi-symmetric functions, and they also appear as characters of components of $\Gamma(\mathsf{hypo}_n)$ where the associated quasi-ribbon tableau has shape α.
- Since $\Gamma(\mathsf{hypo}_n)$ is a subgraph of $\Gamma(\mathsf{plac}_n)$, we get the following decomposition by Gessel ('19)

$$s_\lambda = \sum_{T \in \mathrm{SYT}(\lambda)} F_{\mathrm{DesComp}(T)}$$

- Based on this decomposition, Maas-Gariépy ('23) independently introduced the notion of quasi-crystal graphs, as components of crystal graphs corresponding to fundamental quasi-symmetric functions.

Symmetric and quasi-symmetric functions



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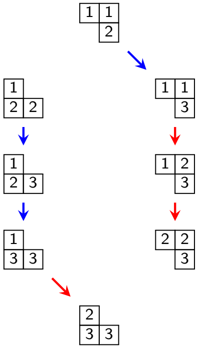

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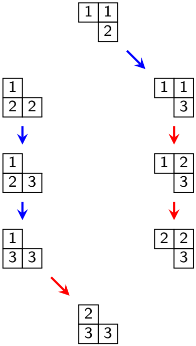

$$F_{21}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_1 x_2 x_3 + x_2^2 x_3$$

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$$s_{21} = F_{21} + F_{12}$$

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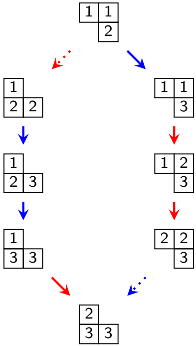

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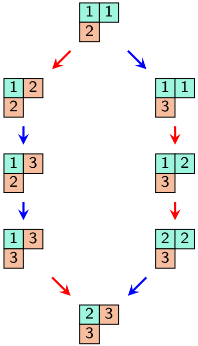

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Quasi-arrays

- A **quasi-array** of size m is a triangular array of $\frac{m(m+1)}{2}$ cells, filled with letters from $\mathcal{A}$, such that:
    1. the first row is weakly increasing;
    2. each diagonal from upper right to lower left is an increasing sequence of consecutive letters of $\mathcal{A}$.
- $\mathcal{QA}$ denotes the set of quasi-arrays, and $\mathcal{QA}_n$ denotes the set of quasi-arrays where the rightmost letter of the first row is at most n.

$Q =$

1	2	2	4	5	7
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4	6	7	9		
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- It follows from the definition that the rows are weakly increasing and the columns are strictly increasing.
- Thus, choosing a quasi-ribbon shape α, a composition of m, results in a quasi-ribbon tableau, denoted $\mathfrak{r}(Q, \alpha)$.

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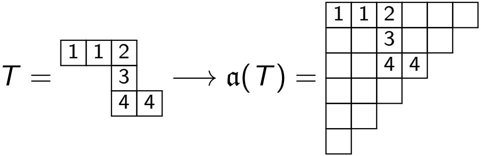

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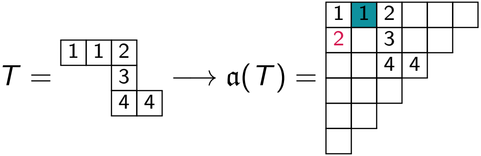

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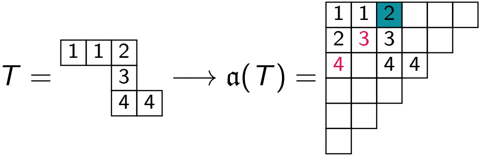

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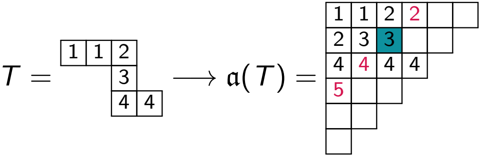

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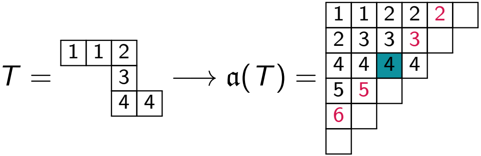

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$$T = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & \\ \hline & & 3 & \\ \hline & & 4 & 4 \\ \hline \end{array} \longrightarrow \mathfrak{a}(T) = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 2 & 2 & 2 & 2 \\ \hline 2 & 3 & 3 & 3 & 3 & \\ \hline 4 & 4 & 4 & 4 & & \\ \hline 5 & 5 & 5 & & & \\ \hline 6 & 6 & & & & \\ \hline 7 & & & & & \\ \hline \end{array}$$

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Quasi-array graphs

- Given Q a quasi-array of size m, we define partial operators $\ddot{d}_k$ and $\ddot{c}_k$, for $k = 1, \ldots, m$ the following way
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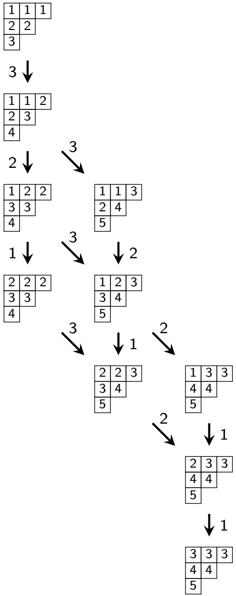
Quasi-array graphs

- The **quasi-array graph** $\Delta(\mathcal{QA})$ is the directed labelled graph whose vertex set is $\mathcal{QA}$, having an edge $Q_1 \xrightarrow{k} Q_2$ iff $\ddot{d}_k(Q_1) = Q_2$.



Connection to quasi-crystal graphs

- Let Q be a quasi-array of size m and let α be a composition of m. Suppose that the k-th diagonal of Q intersects $T = \mathfrak{r}(Q, \alpha)$ at a cell filled with ℓ. Then, $\ddot{d}_k$ is defined on Q iff $\ddot{f}_\ell$ is defined on T and changes the letter ℓ on the k-th diagonal. In this case, we have

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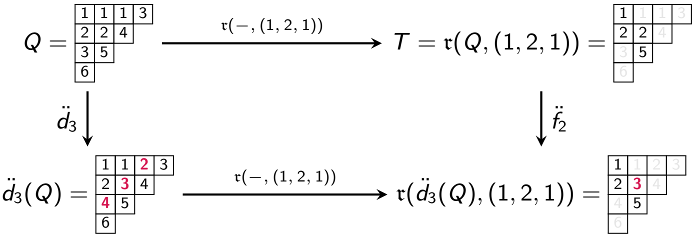

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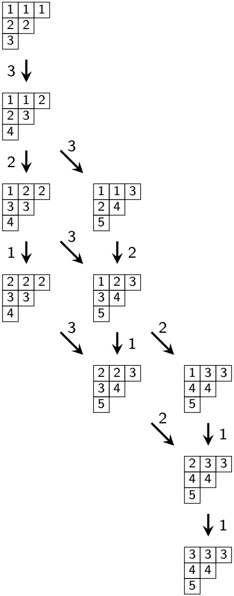

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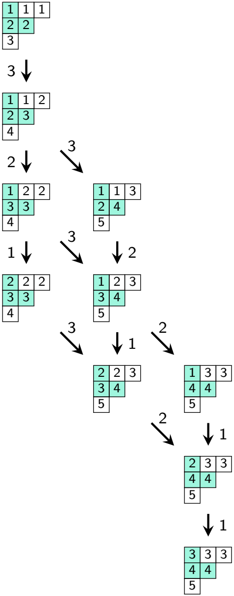

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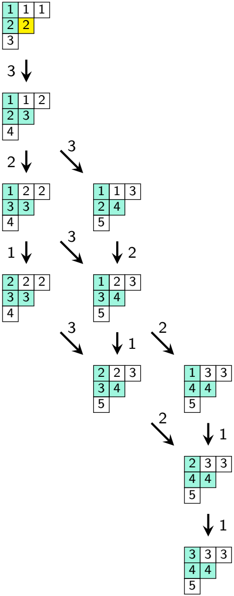

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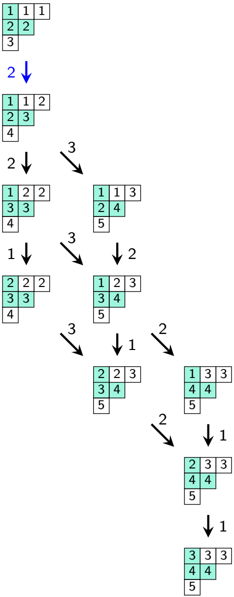

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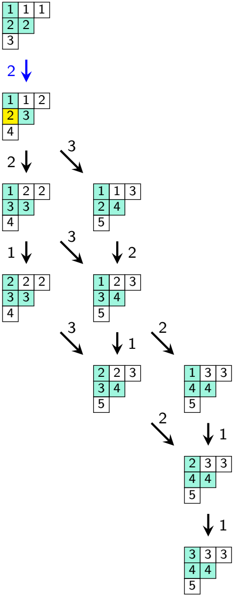

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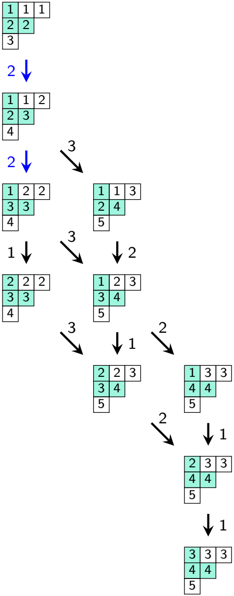

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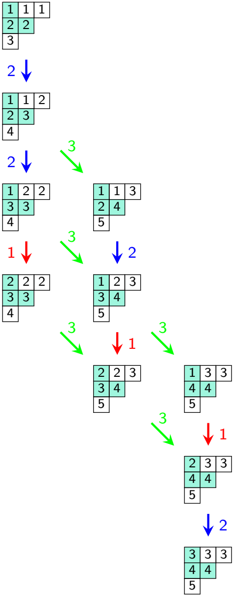

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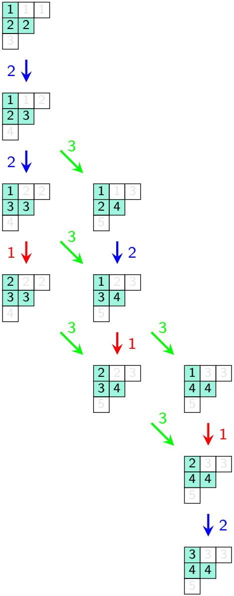

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Theorem (Cain, Malheiro, Rodrigues, R '23)

Let α *and* β *be composition of the same natural number. Then,* $\Gamma(\mathrm{hypo}, \alpha)$ *and* $\Gamma(\mathrm{hypo}, \beta)$ *are isomorphic as* **unlabelled** *directed graphs, under the map* $T \longmapsto \mathfrak{r}(\mathfrak{a}(T), \beta)$.

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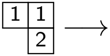

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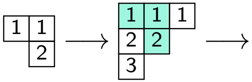

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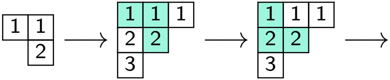

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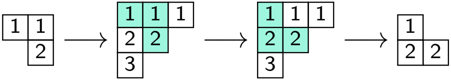
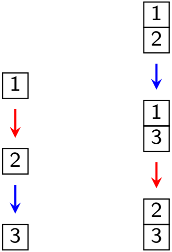
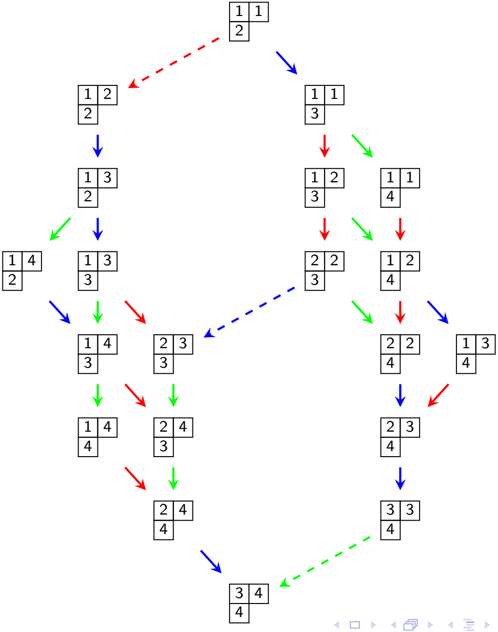


Connection to quasi-crystal graphs

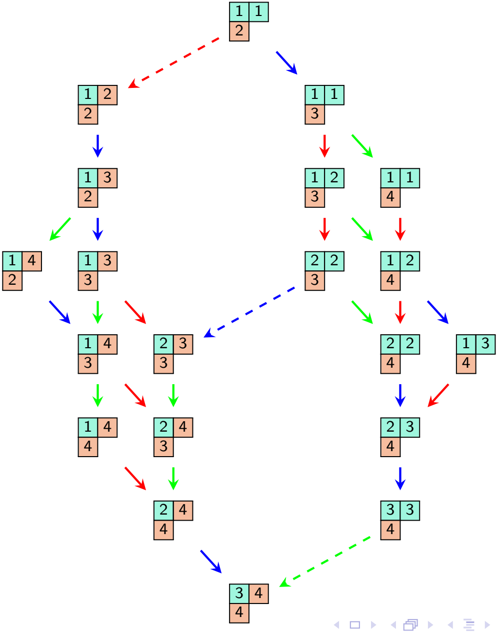
- These are a consequence of a result by Maas-Gariépy, but the notion of quasi-arrays gives us an explicit isomorphism.
- The converse holds for infinite rank but not for finite. The following quasi-crystal components are isomorphic (as unlabelled graphs) but (1) and $(1, 1)$ do not have the same number of parts.



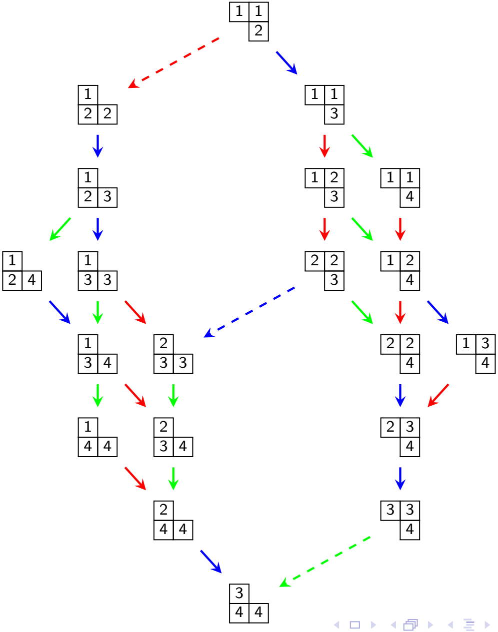
Connection to quasi-crystal graphs



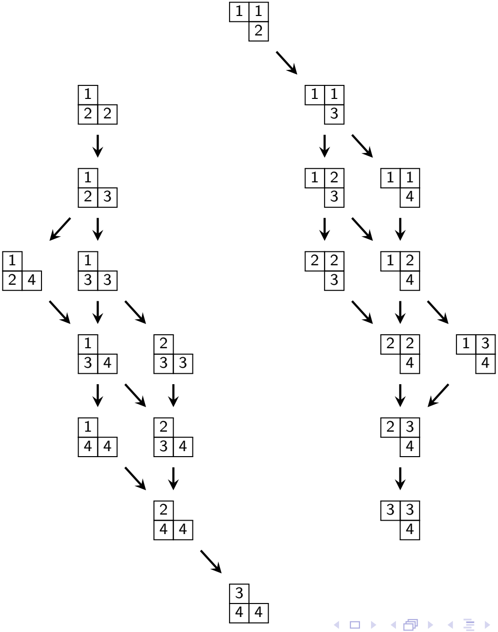
Connection to quasi-crystal graphs



Connection to quasi-crystal graphs



Connection to quasi-crystal graphs

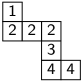


Schur functions and fundamental quasi-symmetric functions

Theorem (Cain, Malheiro, Rodrigues, R ’23)

Let α be a composition and λ the partition obtained by reordering α. Then, any component of $\Gamma(\mathrm{plac})$ *comprising words whose associated Young tableaux have shape λ contains a component of* $\Gamma(\mathrm{hypo})$ *comprising words whose associated quasi-ribbon tableaux have shape α.*

- As a consequence, the fundamental quasi-symmetric function F_α appears in the decomposition of s_λ.
- Idea: perform “slide left, slide up” on a quasi-ribbon tableau of shape α and obtain a Young tableau of shape λ:



$\alpha = (1, 3, 1, 2)$

Schur functions and fundamental quasi-symmetric functions

Theorem (Cain, Malheiro, Rodrigues, R '23)

Let α be a composition and λ the partition obtained by reordering α. Then, any component of $\Gamma(\text{plac})$ *comprising words whose associated Young tableaux have shape λ contains a component of* $\Gamma(\text{hypo})$ *comprising words whose associated quasi-ribbon tableaux have shape α.*

- As a consequence, the fundamental quasi-symmetric function F_α appears in the decomposition of s_λ.
- Idea: perform "slide left, slide up" on a quasi-ribbon tableau of shape α and obtain a Young tableau of shape λ:

1		
2	2	2
3		
4	4	

$\alpha = (1, 3, 1, 2)$

Schur functions and fundamental quasi-symmetric functions

Theorem (Cain, Malheiro, Rodrigues, R '23)

Let α be a composition and λ the partition obtained by reordering α. Then, any component of $\Gamma(\text{plac})$ *comprising words whose associated Young tableaux have shape λ contains a component of* $\Gamma(\text{hypo})$ *comprising words whose associated quasi-ribbon tableaux have shape α.*

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1	2	2
2	4	
3		
4		

$\alpha = (1, 3, 1, 2)$

Schur functions and fundamental quasi-symmetric functions

Theorem (Cain, Malheiro, Rodrigues, R '23)

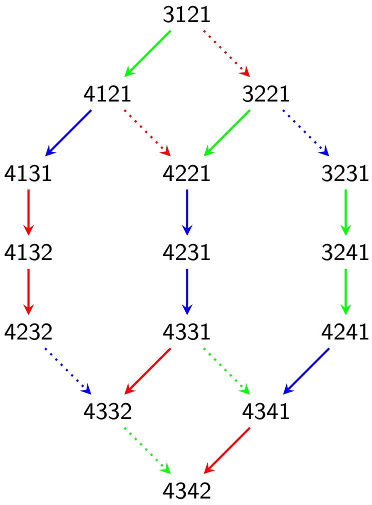
Let α be a composition and λ the partition obtained by reordering α. Then, any component of Γ(plac) *comprising words whose associated Young tableaux have shape λ contains a component of* Γ(hypo) *comprising words whose associated quasi-ribbon tableaux have shape α.*

- As a consequence, the fundamental quasi-symmetric function F_α appears in the decomposition of s_λ.
- Idea: perform "slide left, slide up" on a quasi-ribbon tableau of shape α and obtain a Young tableau of shape λ:

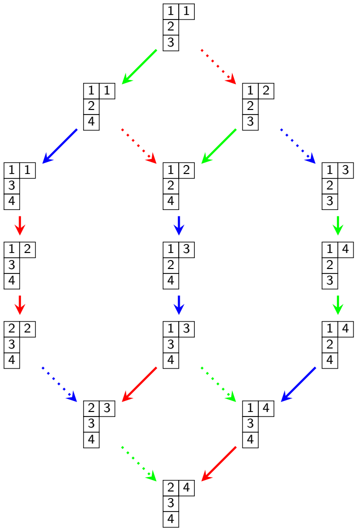
1	2	2
2	4	
3		
4		

$\lambda = (3, 2, 1, 1)$

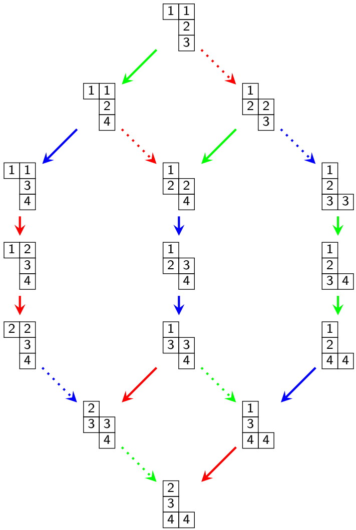
Schur functions and fundamental quasi-symmetric functions



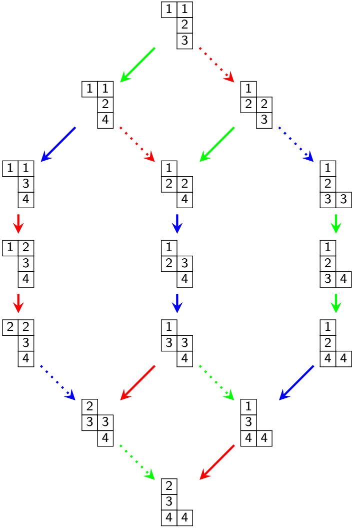
Schur functions and fundamental quasi-symmetric functions



Schur functions and fundamental quasi-symmetric functions



Schur functions and fundamental quasi-symmetric functions



$$s_{211} = F_{211} + F_{121} + F_{112}$$

Thank you!