Commutative nilpotent transformation semigroups

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Semigroups

- A **semigroup** is a non-empty set equipped with an associative binary operation.
- $\mathcal{T}_n$ is the semigroup of transformations over $\{1, \ldots, n\}$.

Example

$$\mathcal{T}_2 = \left\{ \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \right\}.$$

Null transformation semigroups

- **Null semigroup:** $S^2 = \{0\}$.
- $\xi(n) = \max\{t^{n-t} : t \in \{1, \ldots, n\}\}$, $n \in \mathbb{N}$.
- $\alpha(n) = \max\{t \in \{1, \ldots, n\} : t^{n-t} = \xi(n)\}$, $n \in \mathbb{N}$.

Theorem (Cameron, East, FitzGerald, Mitchell, Pebody, Quinn-Gregson, 2023)

The maximum size of a null subsemigroup of $\mathcal{T}_n$ is $\xi(n)$.

Example

$\xi(6) = 3^{6-3} = 27$ and $\alpha(6) = 3$

$$\left\{ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 1 & 1 & x & y & z \end{pmatrix} : x, y, z \in \{1, 2, 3\} \right\}.$$

Commutative nilpotent transformation semigroups

- **Nilpotent semigroup:** $\exists m \in \mathbb{N} \quad S^m = \{0\}$.

Theorem (Biggs, Rankin, Reis, 1976)

Let S be a nilpotent subsemigroup of $\mathcal{T}_n$. Then $|S| \leqslant (n-1)!$.

Theorem (Cain, Malheiro, P., 2023)

Let S be a commutative nilpotent subsemigroup of $\mathcal{T}_n$. Then

- $|S| \leqslant \xi(n)$.
- *If $|S| = \xi(n)$ then S is a null semigroup.*

S-partition

Commutative nilpotent semigroup with $S^3 = \{0\}$:

$$\begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6\\ 5 & 5 & 5 & 5 & 5 & 5\end{pmatrix} \qquad \begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6\\ 5 & 1 & 2 & 1 & 5 & 1\end{pmatrix}$$

$$\begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6\\ 5 & 5 & 1 & 5 & 5 & 5\end{pmatrix} \qquad \begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6\\ 5 & 1 & 4 & 1 & 5 & 1\end{pmatrix}$$

$$\begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6\\ 5 & 1 & 6 & 1 & 5 & 1\end{pmatrix}$$

S-partition

Commutative nilpotent semigroup with $S^3 = \{0\}$:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & \mathbf{5} & 6 \\ 5 & 5 & 5 & 5 & \mathbf{5} & 5 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 & 4 & \mathbf{5} & 6 \\ 5 & 1 & 2 & 1 & \mathbf{5} & 1 \end{pmatrix} \qquad A_0 = \{5\}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & \mathbf{5} & 6 \\ 5 & 5 & 1 & 5 & \mathbf{5} & 5 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 & 4 & \mathbf{5} & 6 \\ 5 & 1 & 4 & 1 & \mathbf{5} & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & \mathbf{5} & 6 \\ 5 & 1 & 6 & 1 & \mathbf{5} & 1 \end{pmatrix}$$

S-partition

Commutative nilpotent semigroup with $S^3 = \{0\}$:

$$\begin{pmatrix} \mathbf{1} & 2 & 3 & 4 & 5 & 6 \\ \mathbf{5} & 5 & 5 & 5 & 5 & 5 \end{pmatrix} \qquad \begin{pmatrix} \mathbf{1} & 2 & 3 & 4 & 5 & 6 \\ \mathbf{5} & 1 & 2 & 1 & 5 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{1} & 2 & 3 & 4 & 5 & 6 \\ \mathbf{5} & 5 & 1 & 5 & 5 & 5 \end{pmatrix} \qquad \begin{pmatrix} \mathbf{1} & 2 & 3 & 4 & 5 & 6 \\ \mathbf{5} & 1 & 4 & 1 & 5 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{1} & 2 & 3 & 4 & 5 & 6 \\ \mathbf{5} & 1 & 6 & 1 & 5 & 1 \end{pmatrix}$$

$A_0 = \{5\}$

$A_1 = \{1\}$

S-partition

Commutative nilpotent semigroup with $S^3 = \{0\}$:

$$\begin{pmatrix} 1 & \mathbf{2} & 3 & \mathbf{4} & 5 & \mathbf{6} \\ 5 & \mathbf{5} & 5 & \mathbf{5} & 5 & \mathbf{5} \end{pmatrix} \qquad \begin{pmatrix} 1 & \mathbf{2} & 3 & \mathbf{4} & 5 & \mathbf{6} \\ 5 & \mathbf{1} & 2 & \mathbf{1} & 5 & \mathbf{1} \end{pmatrix}$$

$$\begin{pmatrix} 1 & \mathbf{2} & 3 & \mathbf{4} & 5 & \mathbf{6} \\ 5 & \mathbf{5} & 1 & \mathbf{5} & 5 & \mathbf{5} \end{pmatrix} \qquad \begin{pmatrix} 1 & \mathbf{2} & 3 & \mathbf{4} & 5 & \mathbf{6} \\ 5 & \mathbf{1} & 4 & \mathbf{1} & 5 & \mathbf{1} \end{pmatrix}$$

$$\begin{pmatrix} 1 & \mathbf{2} & 3 & \mathbf{4} & 5 & \mathbf{6} \\ 5 & \mathbf{1} & 6 & \mathbf{1} & 5 & \mathbf{1} \end{pmatrix}$$

$$A_0 = \{5\}$$

$$A_1 = \{1\}$$

$$A_2 = \{2, 4, 6\}$$

S-partition

Commutative nilpotent semigroup with $S^3 = \{0\}$:

$$\begin{pmatrix} 1 & 2 & \mathbf{3} & 4 & 5 & 6 \\ 5 & 5 & \mathbf{5} & 5 & 5 & 5 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & \mathbf{3} & 4 & 5 & 6 \\ 5 & 1 & \mathbf{2} & 1 & 5 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & \mathbf{3} & 4 & 5 & 6 \\ 5 & 5 & \mathbf{1} & 5 & 5 & 5 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & \mathbf{3} & 4 & 5 & 6 \\ 5 & 1 & \mathbf{4} & 1 & 5 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & \mathbf{3} & 4 & 5 & 6 \\ 5 & 1 & \mathbf{6} & 1 & 5 & 1 \end{pmatrix}$$

$A_0 = \{5\}$

$A_1 = \{1\}$

$A_2 = \{2, 4, 6\}$

$A_3 = \{3\}$

Property regarding S-partition

Commutative nilpotent semigroup with $S^3 = \{0\}$:

$$\begin{pmatrix} 1 & \mathbf{2} & 3 & \mathbf{4} & 5 & \mathbf{6} \\ 5 & \mathbf{5} & 5 & \mathbf{5} & 5 & \mathbf{5} \end{pmatrix} \quad \begin{pmatrix} 1 & \mathbf{2} & 3 & \mathbf{4} & 5 & \mathbf{6} \\ 5 & \mathbf{1} & 2 & \mathbf{1} & 5 & \mathbf{1} \end{pmatrix}$$

$$\begin{pmatrix} 1 & \mathbf{2} & 3 & \mathbf{4} & 5 & \mathbf{6} \\ 5 & \mathbf{5} & 1 & \mathbf{5} & 5 & \mathbf{5} \end{pmatrix} \quad \begin{pmatrix} 1 & \mathbf{2} & 3 & \mathbf{4} & 5 & \mathbf{6} \\ 5 & \mathbf{1} & 4 & \mathbf{1} & 5 & \mathbf{1} \end{pmatrix}$$

$$\begin{pmatrix} 1 & \mathbf{2} & 3 & \mathbf{4} & 5 & \mathbf{6} \\ 5 & \mathbf{1} & 6 & \mathbf{1} & 5 & \mathbf{1} \end{pmatrix}$$

$A_0 = \{5\}$

$A_1 = \{1\}$

$A_2 = \{2, 4, 6\}$

$A_3 = \{3\}$

Main Lemma

- $x \in A_i$
- $\alpha_1, \ldots, \alpha_m$ *agree on* $A_{<i}$

$\Longrightarrow$ $(x\alpha_k)\beta$ *are all equal.*

Order the elements of $\{1, \ldots, n\}$

- We order the elements of $\{1, \ldots, n\}$ in a way such that the elements of A_i appear before the elements of A_{i+1}, $i = 0, \ldots, k-1$.

$$A_0 = \{5\}, \quad A_1 = \{1\}, \quad A_2 = \{2, 4, 6\}, \quad A_3 = \{3\}$$

$$5, 1, 2, 4, 6, 3$$

Rewriting the transformations

- We rewrite the transformations of the semigroup in a convenient way.

$$5, 1, 2, 4, 6, 3$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 5 & 5 & 5 & 5 & 5 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 1 & 2 & 1 & 5 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 5 & 1 & 5 & 5 & 5 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 1 & 4 & 1 & 5 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 1 & 6 & 1 & 5 & 1 \end{pmatrix}$$

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$$5, 1, 2, 4, 6, 3$$

$$\begin{pmatrix} 5 & 1 & 2 & 4 & 6 & 3 \\ 5 & 5 & 5 & 5 & 5 & 5 \end{pmatrix} \qquad \begin{pmatrix} 5 & 1 & 2 & 4 & 6 & 3 \\ 5 & 5 & 1 & 1 & 1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 5 & 1 & 2 & 4 & 6 & 3 \\ 5 & 5 & 5 & 5 & 5 & 1 \end{pmatrix} \qquad \begin{pmatrix} 5 & 1 & 2 & 4 & 6 & 3 \\ 5 & 5 & 1 & 1 & 1 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 5 & 1 & 2 & 4 & 6 & 3 \\ 5 & 5 & 1 & 1 & 1 & 6 \end{pmatrix}$$

Words from transformations

- We obtain a set of words over $\{1, \ldots, n\}$ from the semigroup.

$$5, 1, 2, 4, 6, 3$$

$$\begin{pmatrix} 5 & 1 & 2 & 4 & 6 & 3 \\ 5 & 5 & 5 & 5 & 5 & 5 \end{pmatrix} \quad \begin{pmatrix} 5 & 1 & 2 & 4 & 6 & 3 \\ 5 & 5 & 1 & 1 & 1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 5 & 1 & 2 & 4 & 6 & 3 \\ 5 & 5 & 5 & 5 & 5 & 1 \end{pmatrix} \quad \begin{pmatrix} 5 & 1 & 2 & 4 & 6 & 3 \\ 5 & 5 & 1 & 1 & 1 & 4 \end{pmatrix} \quad \longrightarrow$$

$$\begin{pmatrix} 5 & 1 & 2 & 4 & 6 & 3 \\ 5 & 5 & 1 & 1 & 1 & 6 \end{pmatrix}$$

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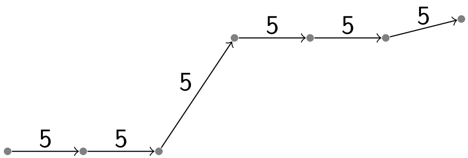
Tree from words



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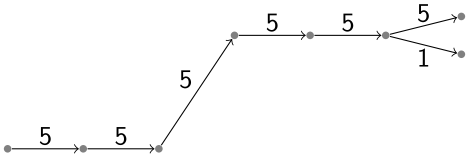
Tree from words



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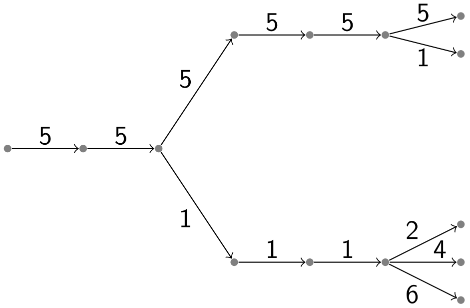
Tree from words



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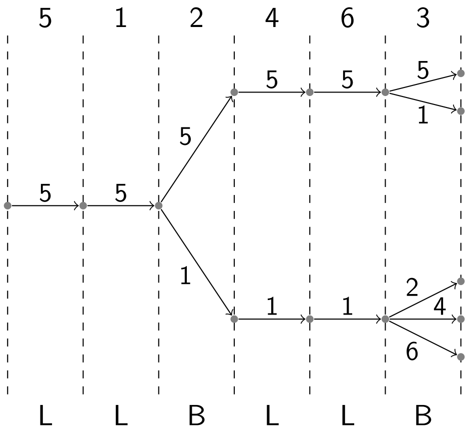
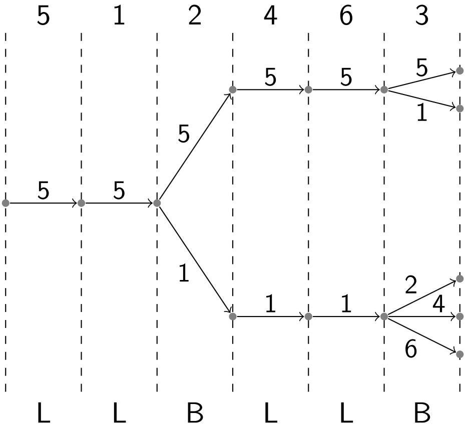
Tree



Property of the tree



Main Lemma

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- $\alpha_1, \ldots, \alpha_m$ agree on $A_{<i}$

$\implies (x\alpha_k)\beta$ are all equal.

$A_0 = \{5\}$

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$A_2 = \{2, 4, 6\}$

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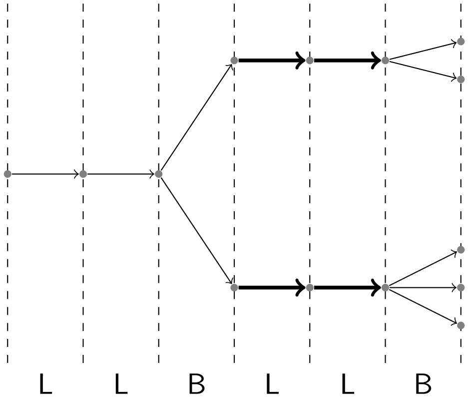
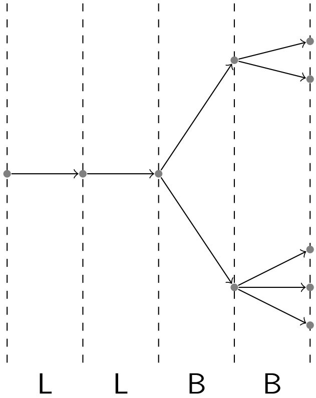
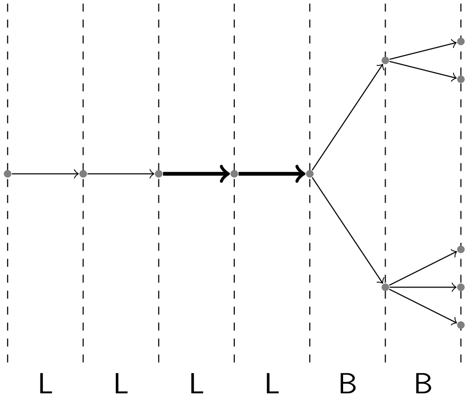
Modifying the tree



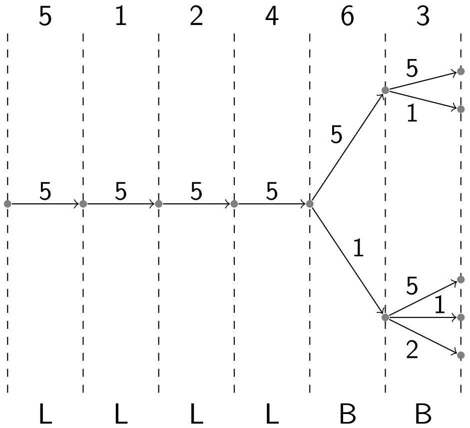
Modifying the tree



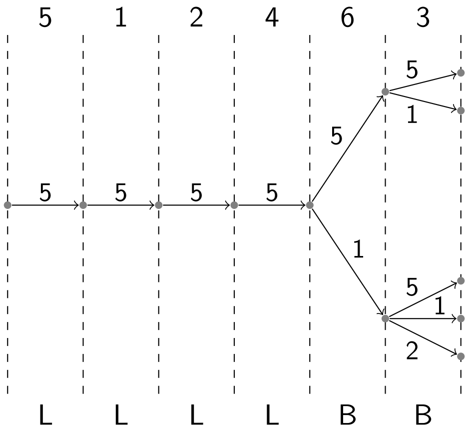
Modifying the tree



Modifying the tree



Words from tree



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Transformations from words

- We obtain a null subsemigroup of $\mathcal{T}_n$ from the set of words. The null semigroup has the same size as the initial commutative nilpotent semigroup. So $|S| \leqslant \xi(n)$.

$$5, 1, 2, 4, 6, 3$$

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$$\longrightarrow$$

$$\begin{pmatrix} 5 & 1 & 2 & 4 & 6 & 3 \\ 5 & 5 & 5 & 5 & 5 & 5 \end{pmatrix} \quad \begin{pmatrix} 5 & 1 & 2 & 4 & 6 & 3 \\ 5 & 5 & 5 & 5 & 1 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 5 & 1 & 2 & 4 & 6 & 3 \\ 5 & 5 & 5 & 5 & 5 & 1 \end{pmatrix} \quad \begin{pmatrix} 5 & 1 & 2 & 4 & 6 & 3 \\ 5 & 5 & 5 & 5 & 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 5 & 1 & 2 & 4 & 6 & 3 \\ 5 & 5 & 5 & 5 & 1 & 2 \end{pmatrix}$$

Commutative nilpotent transformation semigroups

Theorem (Cain, Malheiro, P., 2023)

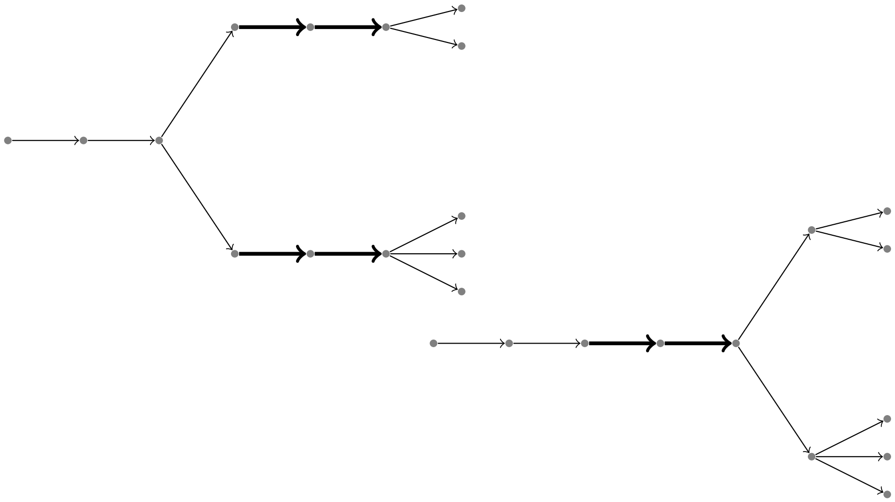
Let S be a commutative nilpotent subsemigroup of $\mathcal{T}_n$. Then

- $|S| \leqslant \xi(n)$.
- *If* $|S| = \xi(n)$ *then* S *is a null semigroup.*

Largest commutative nilpotent transformation semigroups

- Let S be a commutative nilpotent subsemigroup of $\mathcal{T}_n$ such that $|S| = \xi(n)$. Let N be the null subsemigroup of $\mathcal{T}_n$ obtained from S.
- The trees of S and N have the same number of linear columns.
- All the linear columns of the tree of S are located in the trunk of the tree.
- The structure of the tree of S is equal to the structure of the tree of N.
- S is a null semigroup.

Largest commutative nilpotent transformation semigroups



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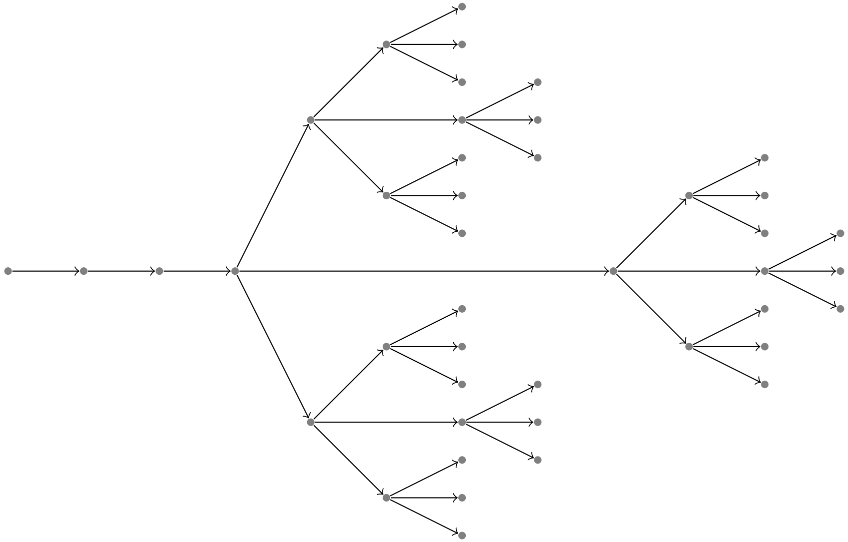
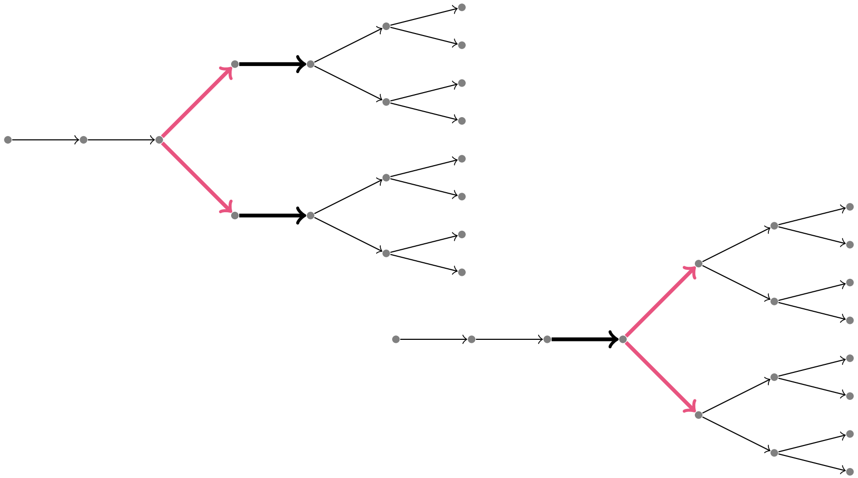
Largest commutative nilpotent transformation semigroups



Largest commutative nilpotent transformation semigroups



Largest commutative nilpotent transformation semigroups

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Largest commutative nilpotent transformation semigroups

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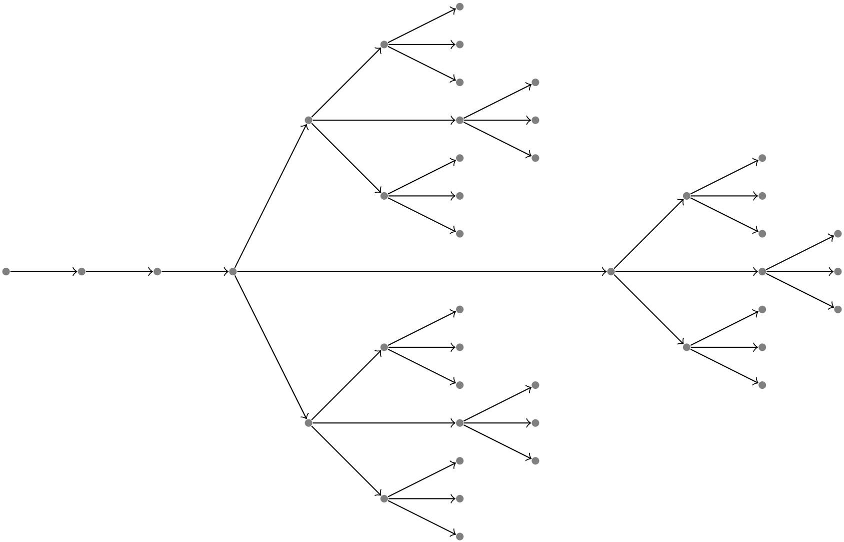
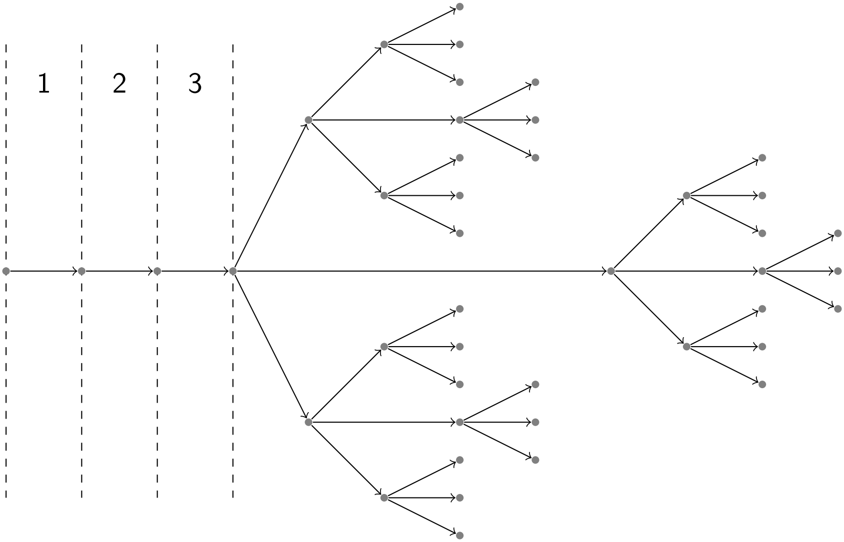
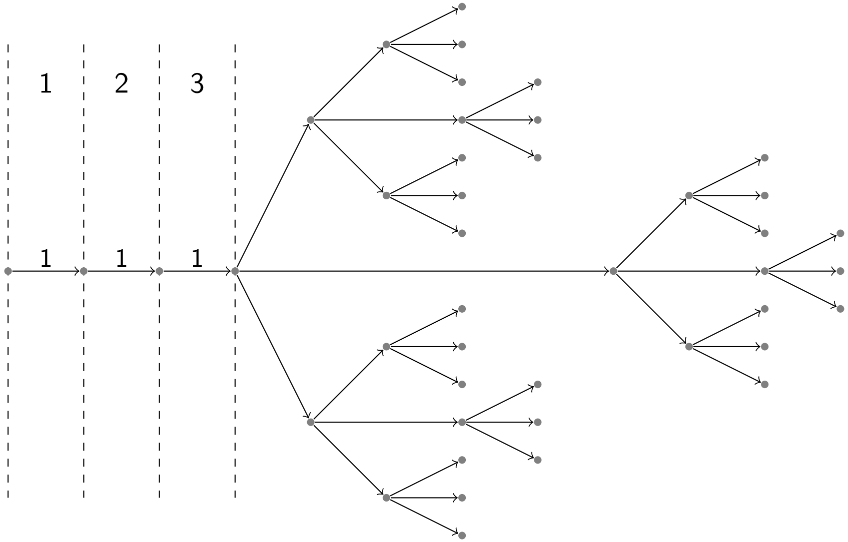
Largest commutative nilpotent transformation semigroups



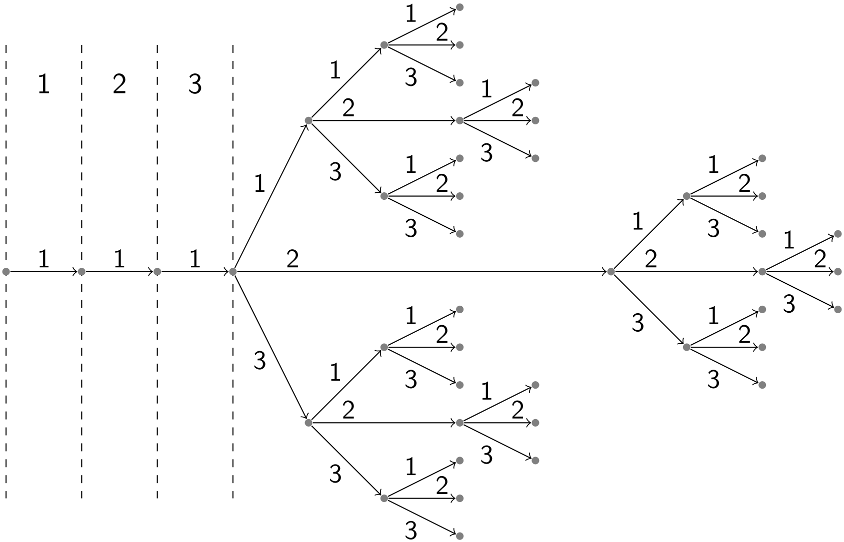
Largest commutative nilpotent transformation semigroups



Largest commutative nilpotent transformation semigroups



Largest commutative nilpotent transformation semigroups



Largest commutative nilpotent transformation semigroups

- Let S be a commutative nilpotent subsemigroup of $\mathcal{T}_n$ such that $|S| = \xi(n)$. Let N be the null subsemigroup of $\mathcal{T}_n$ obtained from S.
- The trees of S and N have the same number of linear columns.
- All the linear columns of the tree of S are located in the trunk of the tree.
- The structure of the tree of S is equal to the structure of the tree of N.
- S is a null semigroup.

Commutative nilpotent transformation semigroups

Theorem

Let S be a commutative nilpotent subsemigroup of $\mathcal{T}_n$. Then

- $|S| \leqslant \xi(n)$.
- *If* $|S| = \xi(n)$ *then* S *is a null semigroup.*

Commutative semigroups with one idempotent

Theorem (Cain, Malheiro, P., 2023)

The maximum size of a commutative subsemigroup of $\mathcal{T}_n$ with one idempotent is

$$\begin{cases} n, & \text{if } n \in \{2, 3\}, \\ \xi(n), & \text{otherwise.} \end{cases}$$

Theorem (Cain, Malheiro, P., 2023)

Let S be a maximum-order commutative subsemigroup of $\mathcal{T}_n$ with one idempotent. Then

- *If $n \in \{2, 3\}$, then S is a group.*
- *If $n = 4$, then S is either a group or a null semigroup.*
- *If $n \in \mathbb{N} \setminus \{2, 3, 4\}$, then S is a null semigroup.*

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