U.PORTO FACULDADE DE CIÊNCIAS UNIVERSIDADE DO PORTO

DEPARTAMENTO DE MATEMÁTICA

CENTRO DE MATEMÁTICA UNIVERSIDADE DO PORTO

Torsionfree representations and operations on multisets

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joint with V. Futorny (SUSTech & USP) and E. Mendonça (Lyon & USP)

Séminaire Lotharingien de Combinatoire 91

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in memoriam

1947–2022

Felix Klein

In his Erlangen Program, F. Klein, establishes the correspondence

$$\text{groups} \quad \leftrightarrow \quad \text{geometries}$$

E.g. euclidean geometry is defined by the euclidean group of isometries. The geometric objects are the invariants.

$$\operatorname{Isom}\mathbb{R}^n \simeq \left(\begin{array}{ccc|c} & & & x_1 \\ & \mathrm{O}_n(\mathbb{R}) & & \vdots \\ & & & x_n \\ \hline 0 & \cdots & 0 & 1 \end{array}\right) \simeq \mathrm{O}_n(\mathbb{R}) \ltimes \mathbb{R}^n.$$

Emmy Noether

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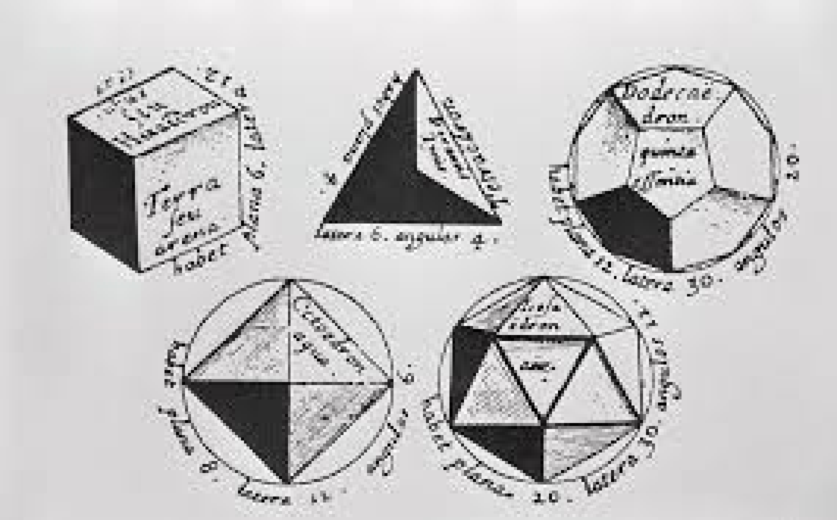
A continuous symmetry of a Hamiltonian system implies that some physical quantity is preserved.

E.g.: preservation of energy, momentum, angular momentum are a consequence of the invariance of physical laws under time translation, space translation, rotational symmetry.

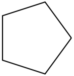
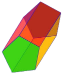
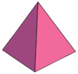
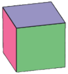
Platonic solids: geometry and its symmetry groups



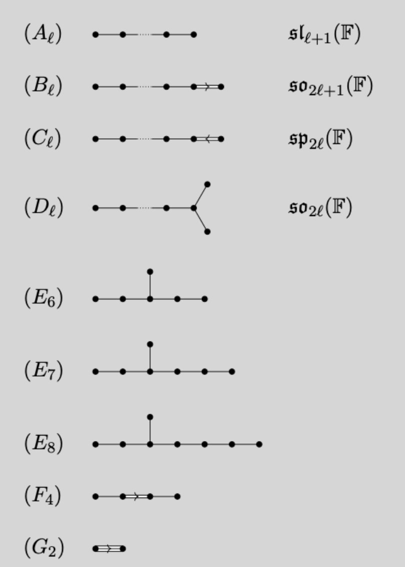
The sphere

$$\mathrm{SO}(3), \quad \mathrm{SL}(2), \quad \mathrm{SU}(2)$$

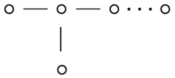
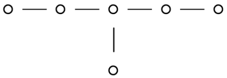
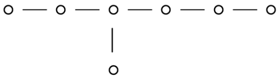
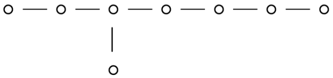
Finite subgroups of SO(3)

type	group	group of rotations of
A_n	$\mathbb{Z}_{n+1}$	$n+1$-sided regular polygon 
D_n	$D_{2(n+2)}$	prism 
E_6	$T \simeq A_4$	tetahedron 
E_7	$O \simeq S_4$	cube & octahedron  
E_8	$I \simeq A_5$	dodecahedron & icosahedron  

Dynkin diagrams and the *ADE*-phenomenon

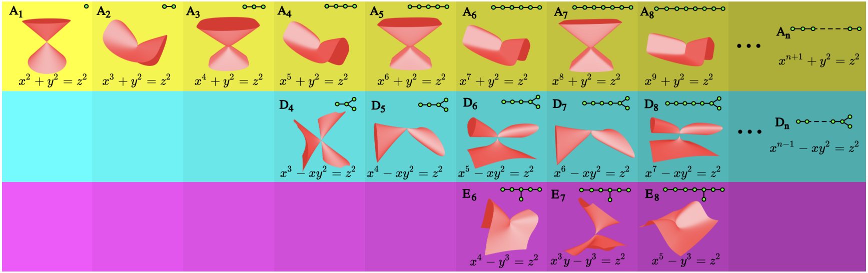


Kleinian/Du Val singularities

Name	Equation	Group	Resolution graph
A_n	$x^2 + y^2 + z^{n+1}$	cyclic $\mathbb{Z}/(n+1)$	
D_n	$x^2 + y^2 z + z^{n-1}$	binary dihedral $\mathrm{BD}_{4(n-2)}$	
E_6	$x^2 + y^3 + z^4$	binary tetrahedral	
E_7	$x^2 + y^3 + yz^3$	binary octahedral	
E_8	$x^2 + y^3 + z^5$	binary icosahedral	

Source: Miles Reid, The Du Val singularities A_n, D_n, E_6, E_7, E_8

Groups and invariant rings



- Let the generator of $G = \mathbb{Z}_2$ act on $\mathbb{A}^2$ by $u \mapsto -u$, $v \mapsto -v$.
- The invariant ring $\mathbb{C}[u, v]^G$ is generated by the quadratic monomials u^2, v^2, uv.
- $\mathbb{C}[u, v]^G$ can be identified with $\mathbb{C}[x, y, z]/\langle xz = y^2 \rangle$.

For each finite subgroup G of $\mathrm{SL}(2,\mathbb{C})$ (or, equivalently, $\mathrm{SU}(2)$), we get the corresponding Kleinian singularity $\mathbb{C}^2/G$ and invariant coordinate ring $\mathbb{C}[u,v]^G$.

The McKay correspondence and representation theory

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The classical McKay observation gives one-to-one correspondences:

$$\{\text{irreducible representations of } G \subseteq \mathrm{SL}(2,\mathbb{C})\} \leftrightarrow \quad \text{basis of } \mathrm{H}^*(Y,\mathbb{Z})$$

$$\{\text{conjugacy classes of } G \subseteq \mathrm{SL}(2,\mathbb{C})\} \leftrightarrow \quad \text{basis of } \mathrm{H}_*(Y,\mathbb{Z})$$

where $f : Y \longrightarrow X$ is a crepant resolution.

`Miles Reid, La correspondance de McKay, Astérisque(2002).`

Back to the noncommutative world

Generalized Heisenberg algebras

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Definition.
The generalized Heisenberg algebra $\mathcal{H}(f)$ is the unital associative algebra with generators x, y, h and relations

$$hx = xf(h), \quad yh = f(h)y, \quad yx - xy = f(h) - h,$$

where $f \in \mathbb{F}[h]$.

Quantum generalized Heisenberg algebras

In recent work with Razzavinia, we generalized all of the previous classes of algebras as follows.

Definition. *(qGHAs)*
Let $\mathbb{F}$ be an arbitrary field and fix $q \in \mathbb{F}$ and $f, g \in \mathbb{F}[h]$. The quantum generalized Heisenberg algebra is the algebra $\mathcal{H}_q(f, g)$ generated by x, y, h with relations

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Example

Thus, generalized Heisenberg algebras are precisely the qGHA with $q = 1$ and $g = f(h) - h$, i.e. $\mathcal{H}(f) = \mathcal{H}_1(f, f - h)$.

Smith algebras

S.P. Smith introduced a class of algebras *similar* to $U(\mathfrak{sl}_2)$, defined by the relations

$$[H, A] = A, \quad [H, B] = -B, \quad [A, B] = g(H),$$

where g is a polynomial.

Smith's algebras $\mathcal{S}(g)$ are the qGHAs $\mathcal{H}_1(h-1, g)$.

Smith's algebras are related to:

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Smith's algebras are related to:

- invariant rings of differential operators
- the Zhu algebra of a vertex operator algebra
- noncommutative Kleinian singularities!

S. Rueda introduced a more general class of algebras, defined by the relations

$$[H, A] = A, \quad [H, B] = -B, \quad AB - qBA = g(H),$$

where g is a polynomial and $q \in \mathbb{F}^*$.

Rueda's algebras are the qGHAs $\mathcal{H}_q(h-1, g)$.

The Weyl algebra

The Weyl algebra

$$\mathbb{A}_1(\mathbb{F}) = \langle\ t, \partial \mid [\partial, t] = 1\ \rangle$$

is the algebra of differential operators on $\mathbb{F}[t]$ with polynomial coefficients, in case $\mathrm{char}(\mathbb{F}) = 0$.

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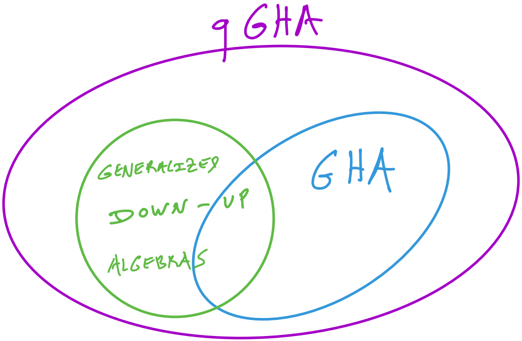
$\mathbb{A}_1(\mathbb{F})$ can be realized as $\mathcal{H}_1(h, 1)/\langle h \rangle$.


q GHA
GENERALIZED
DOWN-UP
ALGEBRAS
GHA

Representations of quantum generalized Heisenberg algebras

Representations of quantum generalized Heisenberg algebras classify a very large class of creation and annihilation operators under very general assumptions.

Finite-dimensional representations

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Theorem.

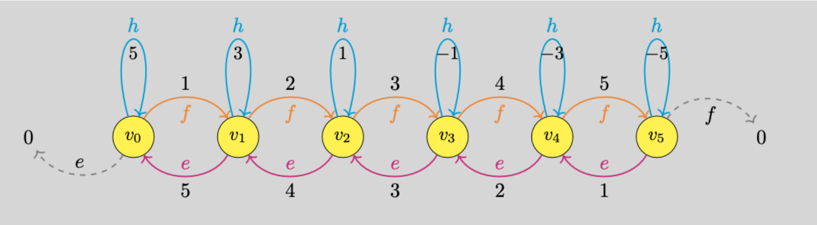
Any simple n-dimensional $\mathcal{H}_q(f,g)$-module is isomorphic to exactly one of the following ($\gamma \in \mathbb{F}^*$ and $\lambda, \mu : \mathbb{Z} \to \mathbb{F}$):

(a) $\mathrm{A}_{q,f,g}(\lambda,\mu)/\mathbb{F}[t^{\pm 1}](t^n - \gamma)$ [x acts invertibly]

(b) $\mathrm{B}_{q,f,g}(\lambda,\mu)/\mathbb{F}[t^{\pm 1}](t^n - \gamma)$ (duals to the above) [x does not act invertibly but y acts invertibly]

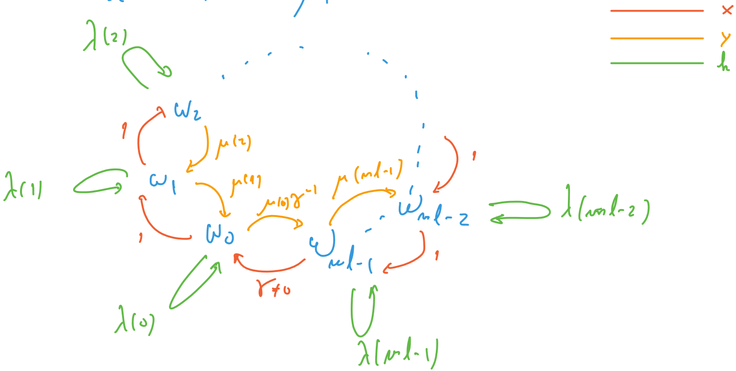
(c) $\mathrm{C}_{q,f,g}(\alpha)/\mathbb{F}[t]t^n$ [neither x not y act invertibly]

Representations of $\mathfrak{sl}_2$ in characteristic 0



A

$n = l m \qquad l = |\lambda|, \ m = |\mu|$



E.g.: $n = 2 \cdot 3 = 6$ ($m = 2$, $l = 3$)

$$f(h) = 1 + h - 3\binom{h}{2} = 1 + h - \frac{3}{2}h(h-1)$$

$$g(h) = 1$$

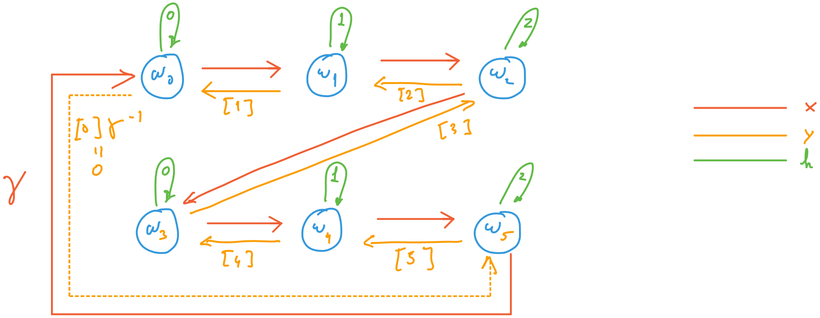
$\gamma \in \mathbb{F}^*$

$$yx - qxy = 1$$

EITHER $q = 1$ WITH $\mathrm{char}\,\mathbb{F} = 2$ OR q PRIMITIVE 6-TH ROOT OF 1

$\lambda(i) = j \in \{0, 1, 2\}$ s.t. $i \equiv j \pmod 3$ $l = 3$

$$\mu(i) = [i] = \frac{q^i - 1}{q - 1} \quad (\text{OR } i, \text{ IF } q = 1)$$


x
y
h

Back to Kleinian singularities of type A

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- For the above choice, there is a filtration by finite-dimensional subspaces of $\mathbb{A}_1(\mathbb{F})^{\mathbb{Z}_n}$ such that the graded ring has the form

$$\mathbb{F}[X, Y, Z]/\langle XZ - Y^n \rangle.$$

Free actions on the Cartan subalgebra

What follows is joint work with V. Futorny & E. Mendonça.

- In Block's classification of irreducible of simple $\mathfrak{sl}_2$-modules one finds, along with the weight modules, also Whittaker modules and modules which are torsionfree over the Cartan subalgebra $\mathbb{F}h$.

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- In Block's classification of irreducible of simple $\mathfrak{sl}_2$-modules one finds, along with the weight modules, also Whittaker modules and modules which are torsionfree over the Cartan subalgebra $\mathbb{F}h$.
- The latter are opposite to weight modules in the sense that the action is as far from semisimple as possible.

Set-up

We are interested in actions of $\mathcal{S}(g)$ on $\mathbb{F}[h]^n$ where h acts freely.

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for some $u \in \mathbb{F}[h]$ (for simplicity, assume that u is monic).
- $z_u = xy - u(h)$ is the Casimir (generates the center)
- Use a correspondence between finite multisets in $\mathbb{F}$ and monic polynomials in $\mathbb{F}[h]$:

$$X \quad \mapsto \quad \text{poly}_X = \prod_{\lambda \in X} (h - \lambda)$$
$$\mathrm{R}_f \quad \leftarrow\!\shortmid \quad f$$

- $\underline{X}$ is the underlying set obtained from X.

Rank n case

Find $Q, P \in \mathrm{M}_n(\mathbb{F}[h])$ such that

$$Q(h-1)P(h) - P(h+1)Q(h) = g(h)I,$$

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Using actions by differential operators we have constructed families of simple $\mathbb{F}[h]$-torsionfree modules for n arbitrary.

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  - $C \in \mathbb{F}$ is the central charge;
  - $\lambda \in \mathbb{F}^\times$;
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The category of these modules is not semisimple, but...

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- finite length;
- simple socle $\mathsf{F}_\lambda A_C(X^\star)$;
- composition factors in category $\mathcal{O}$ except for $\mathsf{F}_\lambda A_C(X^\star)$;
- explicit computation of composition length.

Rank 1 case: details

Let $\mathfrak{U}_1$ denote the category of $\mathbb{K}[h]$-free $\mathcal{S}_u$-modules.

- Fix $C \in \mathbb{F}$
- $\mathrm{R}_{u(h)+C} = X \coprod Y$ multiset partition
- $q(h) = \mathrm{poly}_X$
- $p(h) = \mathrm{poly}_{Y+1}$ (so $p(h+1) = \mathrm{poly}_Y$)

$A_C(X) = \mathbb{F}[h]$ (regular $\mathbb{F}[h]$-module) with action

$$xf(h) = f(h+1)q(h) \quad \text{and} \quad yf(h) = f(h-1)p(h),$$

for all $f(h) \in \mathbb{F}[h]$.

Theorem.
The following is a skeleton of the category $\mathfrak{U}_1$

$$\left\{ \mathsf{F}_\lambda A_C(X) \mid C \in \mathbb{F},\ \lambda \in \mathbb{F}^\times \text{ and } X \subseteq \mathsf{R}_{u+C} \text{ (submultiset)} \right\}.$$

(F_λ twists the action by $x \mapsto \lambda x$, $y \mapsto \lambda^{-1} y$.)

Classification

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(F_λ twists the action by $x \mapsto \lambda x$, $y \mapsto \lambda^{-1} y$.)

Theorem.
$A_C(X)$ is simple if and only if $(X - Y) \cap \mathbb{Z}_{\geq 1} = \emptyset$.

Algorithm to determine all composition series for $A_C(X)$.

For $\alpha, \beta \in \mathbb{F}$, set

$$\alpha \preccurlyeq \beta \iff \beta - \alpha \in \mathbb{N} \subseteq \mathbb{F}.$$

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- If $(X - Y) \cap \mathbb{Z}_{\geq 1} = \emptyset$ then $A_C(X)$ is simple so $A_C(X) \supsetneq (0)$ is a composition series.

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$$X \star \beta = \{\hat{\beta}\} \cup X \setminus \{\beta\}$$

where $\hat{\beta} \in Y_{\precnsim \beta}$ is such that $\kappa_\beta = \beta - \hat{\beta} \geq 1$ is minimum.

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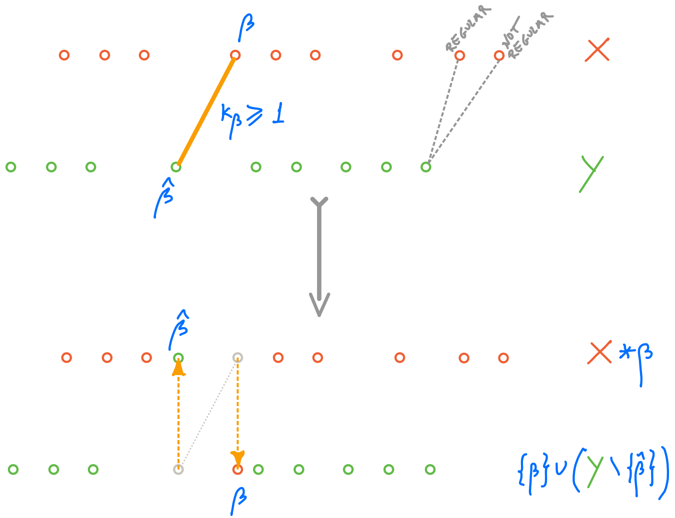
Say that $\beta \in X$ is regular if

$$\hat{\beta} + 1, \hat{\beta} + 2, \cdots, \hat{\beta} + (\kappa_\beta - 1) \quad \text{are not in } X.$$



β
REGULAR
NOT REGULAR
X
$k_\beta \geqslant 1$
Y
$\hat\beta$
$\hat\beta$
X * β
β
{β} ∪ (Y \ {$\hat\beta$})

Theorem.
The composition series for $A_C(X)$ are of the form

$$A_C(X_0) \supsetneq A_C(X_1) \supsetneq A_C(X_2) \supsetneq \cdots \supsetneq A_C(X_m) \supsetneq (0)$$

where, for $0 \leq k \leq m-1$,

- $X_0 = X$ and $X_{k+1} = X_k \star \beta_{k+1}$;
- $\beta_{k+1} \in X_k$ is regular with respect to X_k;
- $\mathrm{R}_{u+C} = X_{k+1} \coprod Y_{k+1}$;
- $(X_m - Y_m) \cap \mathbb{Z}_{\geq 1} = \emptyset$.

Main results

Theorem.

(a) $A_C(X)$ has finite length $m+1$, with

$$m \leq \ell(X) := \sum_{\beta\in X} |Y_{\not\preccurlyeq\beta}| \geq 0.$$

(b) m and $X^\star = X \star \beta_1 \star \cdots \star \beta_m$ do not depend on the choices of the β_k and $A_C(X^\star) = \mathrm{soc}\,(A_C(X))$ is simple (in fact the unique simple submodule of $A_C(X)$).

(c) $A_C(X)/A_C(X\star\beta)$, for $\beta \in X$ regular is simple and finite-dimensional, with

$$\dim A_C(X)/A_C(X\star\beta) = \kappa_\beta.$$

(d) All simple finite-dim'l $\mathcal{S}_u$-modules occur as composition factors of $A_C(X)$, for suitable $C \in \mathbb{F}$ and $X \subseteq \mathrm{R}_{u+C}$.

Theorem.
In the Grothendieck group $K_0(\mathcal{S}_u) = \{[M] \mid M \in \mathcal{S}_u\text{−mod}\}$, we have:

$$[A_C(X)] = [A_C(X^\star)] + \sum_{\beta \in \underline{\mathsf{R}_{u+C}}} \varphi_X(\beta)[L(\beta)],$$

where $\varphi_X(\beta) = \min\left\{|Y_{\npreccurlyeq\beta}|, |X_{\succcurlyeq\beta}|\right\}$.

Main results (cont'd)

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Corollary.
The module $A_C(X)$ has length $1 + \sum_{\beta \in \underline{R_{u+C}}} \varphi_X(\beta)$.