The Amdeberhan-Konvalinka Conjecture and Symmetric Functions

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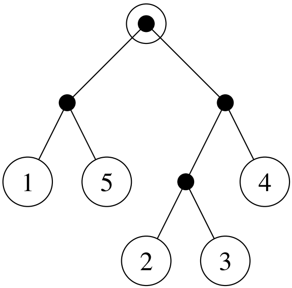
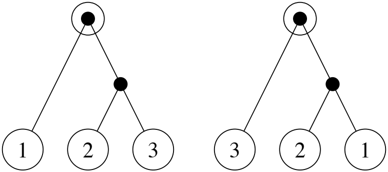
Tanglegrams

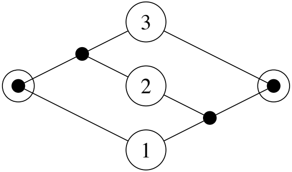
A binary tree (for this talk) is an unordered (rooted) binary tree with labeled leaves and unlabeled internal vertices:



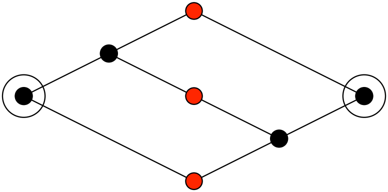
An ordered pair of trees sharing the same set of leaves is called a tanglegram. (The term comes from biology.)



which we can also draw as



Sara Billey, Matjaž Konvalinka, and Frederick A. Matsen IV wanted to count unlabeled tanglegrams



which may defined formally as orbits of tanglegrams under the action of the symmetric group permutating the labels on the leaves.

Burnside's Lemma

To count orbits, we use Burnside's Lemma: If a group G acts on a set S then the number of orbits is

$$\frac{1}{|G|}\sum_{g\in G}\mathrm{fix}(g),$$

where $\mathrm{fix}(g)$ is the number of elements of S fixed by G.

It's not hard to show that $\mathrm{fix}(g)$ depends only on the conjugacy class of g.

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In the case of the symmetric group $\mathfrak{S}_n$, the conjugacy classes correspond to cycle types, which are indexed by partitions of n. If $\lambda = (1^{m_1} 2^{m_2} \cdots)$ is a partition of n then the number of elements of $\mathfrak{S}_n$ of cycle type λ is $n!/z_\lambda$, where $z_\lambda = 1^{m_1} m_1!\, 2^{m_2} m_2! \cdots$.

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If we define $\mathrm{fix}(\lambda)$ be $\mathrm{fix}(g)$ for any $g \in \mathfrak{S}_n$ of cycle type λ, then we may write Burnside's sum for $\mathfrak{S}_n$ as

$$\frac{1}{n!} \sum_{\lambda \vdash n} \mathrm{fix}(\lambda) \frac{n!}{z_\lambda} = \sum_{\lambda \vdash n} \frac{\mathrm{fix}(\lambda)}{z_\lambda}$$

Counting unlabeled binary trees

Now let r_λ be the number of binary trees fixed by a permutation of cycle type λ. Then the number of unlabeled binary trees on n vertices is

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These numbers are sometimes called Wedderburn-Etherington numbers, A001190 in the OEIS (Online Encyclopedia of Integer Sequences).

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So the number of unlabeled tanglegrams with n leaves is

$$\sum_{\lambda \vdash n} \frac{r_\lambda^2}{z_\lambda}.$$

Tangled chains

Billey, Konvalinka, and Matsen define a tangled chain of length k to be a k-tuple of binary trees sharing the same set of leaves.

By the same reasoning, the number of unlabeled tangled chains of length k with n leaves is

$$\sum_{\lambda \vdash n} \frac{r_\lambda^k}{z_\lambda}.$$

A formula for r_λ

Billey, Konvalinka, and Matsen found a remarkable formula for r_λ:

r_λ is zero if λ is not a binary partition (a partition in which every part is a power of 2), and if λ is a binary partition, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 1$, then

$$r_\lambda = \prod_{i=2}^{k} \big(2(\lambda_i + \cdots + \lambda_k) - 1\big).$$

For example, $r_{(4,2,1)} = \big(2 \cdot (2+1) - 1\big)(2 \cdot 1 - 1) = 5 \cdot 1 = 5$.

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The total number of of binary trees with n leaves is

$$r_{(1^n)} = 1 \cdot 3 \cdots (2n-3).$$

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But the formula is still somewhat mysterious.

The Amdeberhan-Konvalinka conjecture

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Let q be a prime. We say that a partition λ is q-ary if every part of λ is a power of q. Define $r_{\lambda,q}$ by

$$r_{\lambda,q} = \begin{cases} 0, & \text{if } \lambda \text{ is not } q\text{-ary} \\ \prod_{j=2}^{l(\lambda)} (q\lambda_j + q\lambda_{j+1} + \cdots + q\lambda_{l(\lambda)} - 1) & \text{if } \lambda \text{ is } q\text{-ary} \end{cases}$$

(Here $l(\lambda)$ is the number of parts of λ.)

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The Amdeberhan-Konvalinka Conjecture: For every positive integer k,

$$\sum_{\lambda \vdash n} \frac{r_{\lambda,q}^k}{z_\lambda}$$

is an integer.

Symmetric functions

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There are several important bases for Λ^n, indexed by partitions of n, but we only need three of them for now.

Bases for symmetric functions

First, the monomial symmetric functions: If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ then m_λ is the sum of all distinct monomials of the form

$$x_{i_1}^{\lambda_1} \cdots x_{i_k}^{\lambda_k}.$$

Next, the power sum symmetric functions are defined by

$$p_n = \sum_{i=1}^{\infty} x_i^n$$

and $p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_k}$.

Finally, the complete symmetric functions

$$h_n = \sum_{i_1 \leq \cdots \leq i_n} x_{i_1} \cdots x_{i_n}.$$

and $h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_k}$.

Integral symmetric functions

A symmetric function is called integral if its coefficients are integers. (This is equivalent to its coefficients being integers in the monomial basis, or any of the other common bases except for the power sum basis.)

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For example $\frac{1}{2}p_1^2 + \frac{1}{2}p_2$ is integral because it is equal to

$$\sum_{i \leq j} x_i x_j = h_2 = m_{(2)} + m_{(1,1)}.$$

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If f is an integral symmetric function expressed in terms of the p_λ, then setting each p_j to 1 gives an integer, since setting each p_j to 1 is equivalent to setting $x_1 = 1$, $x_i = 0$ for $i > 1$.

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The cycle index of any species (and more generally, the characteristic of any representation of $\mathfrak{S}_n$) is integral.

The Kronecker product

We recall the operation of Kronecker product on symmetric functions, defined by

$$p_\lambda * p_\mu = z_\lambda \delta_{\lambda,\mu}\, p_\lambda,$$

or equivalently,

$$\frac{p_\lambda}{z_\lambda} * \frac{p_\mu}{z_\mu} = \delta_{\lambda,\mu} \frac{p_\lambda}{z_\lambda},$$

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and linearity. (It corresponds to tensor products of $\mathfrak{S}_n$ representations.)

Theorem. If f and g are integral symmetric functions then so is $f * g$.

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Proof sketch. By linearity, it is enough to prove this when $f = h_\lambda$ and $g = h_\mu$. But $h_\lambda * h_\mu$ is the cycle index for the species $E_\lambda * E_\mu$, where if $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ then $E_\lambda = E_{\lambda_1} E_{\lambda_2} \cdots E_{\lambda_k}$.

Amdeberhan-Konvalinka symmetric functions

We can generalize the Amdeberhan-Konvalinka conjecture to prime powers. Let m be a power of the prime q and define the symmetric function

$$u_m(n,\alpha) = \sum_{\lambda \vdash_q n} \frac{p_\lambda}{z_\lambda} \alpha \prod_{j=2}^{l(\lambda)} (m\lambda_j + m\lambda_{j+1} + \cdots + m\lambda_{l(\lambda)} + \alpha),$$

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The Amdeberhan-Konvalinka conjecture follows from this theorem, since the Amdeberhan-Konvalinka number $\sum_{\lambda \vdash n} r_{\lambda,q}^k / z_\lambda$ is obtained by setting each p_λ to 1 in the kth Kronecker power $\left(-u_q(n,-1)\right)^{*k}$.

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First suppose that g can be expressed as a sum of monic terms, that is monomials $x_1^{a_1} x_2^{a_2} \dots$ with coefficient 1.
In this case, if $g = t_1 + t_2 + \cdots$, where the t_i are monic terms, then

$$f[g] = f(t_1, t_2, \dots).$$

For example

$$f[e_2] = f(x_1x_2, x_1x_3, x_2x_3, \dots)$$
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For the general case, we can give a different characterization of plethysm when f and g are expressed in terms of power sums. First, $p_j[g]$ is the result of replacing each p_i in g with p_{ij}. Then $f[g]$ is obtained by replacing each p_j in f with $p_j[g]$.

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So for fixed g, the map $f \mapsto f[g]$ is a homomorphism.

Integrality of plethysm

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This is clear when the first definition of plethysm applies (g is a sum of monic terms), and it's not too hard to prove in the general case.

If f is a symmetric function of the form

$$p_1 + \text{higher order terms}$$

then f has a unique plethystic inverse of the same form, which we write as $f^{[-1]}$. It satisfies

$$f \circ f^{[-1]} = f^{[-1]} \circ f = p_1.$$

If f is integral then so is $f^{[-1]}$.

Back to the Billey-Konvalinka-Matsen formula

We now explain how to obtain the Billey-Konvalinka-Matsen formula by repeated application of the binomial theorem. We will then use essentially the same method for the general case.

We start with the cycle index for binary trees,

$$Z_R = \sum_{\lambda} r_\lambda \frac{p_\lambda}{z_\lambda}.$$

As we saw using combinatorial species, Z_R satisfies the plethystic equation

$$Z_R = p_1 + h_2[Z_R],$$

where $h_2 = \sum_{i \leq j} x_i x_j = \frac{1}{2} p_1^2 + \frac{1}{2} p_2$.

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where $h_2 = \sum_{i \le j} x_i x_j = \frac{1}{2}p_1^2 + \frac{1}{2}p_2$. This is the symmetric function refinement of the exponential generating function equation

$$B(x) = x + B(x)^2/2.$$

for binary trees.

We may rewrite the equation for Z_R as

$$(p_1 - h_2) \circ Z_R = p_1$$

so Z_R is the plethystic inverse of

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So if we set $C = 1 - Z_R$ then

$$p_1 = -\frac{1}{2}C^2 + \frac{1}{2}p_2[C].$$

We rearrange this into $C^2 = p_2[C] - 2p_1$ and take square roots to get

$$C = (p_2[C] - 2p_1)^{1/2}.$$

We can use this formula to get an explicit formula for the expansion of C, or more generally, of $C^{-\alpha}$ in power sums. (Note that C has constant term 1.)

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Applying the binomial theorem gives

$$C^{-\alpha} = (p_2[C]-2p_1)^{-\alpha/2} = \sum_{m_1=0}^{\infty} (-2)^{m_1} \binom{-\alpha/2}{m_1} p_1^{m_1} p_2[C]^{-\alpha/2-m_1}$$

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$$\begin{aligned} C^{-\alpha} &= \sum_{m_1=0}^{\infty} (-2)^{m_1} \binom{-\alpha/2}{m_1} p_1^{m_1} (p_4[C] - 2p_2)^{-\alpha/4-m_1/2} \\ &= \sum_{m_1,m_2=0}^{\infty} (-2)^{m_1+m_2} \binom{-\alpha/2}{m_1} \binom{-\alpha/4 - m_1/2}{m_2} \\ &\qquad \times p_1^{m_1} p_2^{m_2} p_4[C]^{-\alpha/4-m_1/2-m_2}. \end{aligned}$$

Continuing in this way, we get the expansion of $C^{-\alpha}$ into powers of $p_1, p_2, p_4, p_8, \ldots$

Continuing in this way, we get the expansion of $C^{-\alpha}$ into powers of $p_1, p_2, p_4, p_8, \ldots$

We can rearrange the product of binomial coefficients to get

$$C^{-\alpha} = F_2(\alpha) = 1 + \sum_{n=1}^{\infty} \sum_{\lambda \vdash_2 n} \frac{p_\lambda}{z_\lambda} \alpha \prod_{j=2}^{l(\lambda)} (2\lambda_j + 2\lambda_{j+1} + \cdots + 2\lambda_{l(\lambda)} + \alpha),$$

and in particular,

$$Z_R = 1 - C = \sum_{n=1}^{\infty} \sum_{\lambda \vdash_2 n} \frac{p_\lambda}{z_\lambda} \prod_{j=2}^{l(\lambda)} (2\lambda_j + 2\lambda_{j+1} + \cdots + 2\lambda_{l(\lambda)} - 1),$$

Lyndon symmetric functions

To generalize this, we introduce the "Lyndon symmetric functions"

$$L_m = \frac{1}{m} \sum_{d|m} \mu(d) p_d^{m/d}$$

Then L_m counts "primitive necklaces" or "Lyndon words" and in particular L_m is integral. In particular, if m is a power of a prime q then

$$L_m = \frac{1}{m}(p_1^m - p_q^{m/q}).$$

Lemma. For all $m > 1$,

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Therefore $-L_m[1-p_1]$ has a plethystic inverse.

Proof. We have

$$\begin{aligned} -L_m[1-p_1] &= -\frac{1}{m}\sum_{d|m}\mu(d)p_d^{m/d}[1-p_1] \\ &= -\frac{1}{m}\sum_{d|m}\mu(d)(1-p_d)^{m/d}. \end{aligned}$$

The constant term is $-\frac{1}{m}\sum_{d|m}\mu(d) = 0$. The p_1 term comes from $d = 1$:

$$-\frac{1}{m}(1-p_1)^m = -\frac{1}{m}(1-mp_1+\cdots) = -\frac{1}{m} + p_1 + \cdots .$$

Since L_m is integral, so is $L_m[1 - p_1]$. Therefore $-L_m[1 - p_1]$ has an integral plethystic inverse, B_m, so $-L_m[1 - B_m] = p_1$.

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Let $C_m = 1 - B_m$ so that $-L_m[C_m] = p_1$. Thus C_m is a right plethystic inverse of $-L_m$.

Since L_m is integral, so is $L_m[1 - p_1]$. Therefore $-L_m[1 - p_1]$ has an integral plethystic inverse, B_m, so $-L_m[1 - B_m] = p_1$.

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If m is a power of the prime q, then

$$L_m = \frac{1}{m}(p_1^m - p_q^{m/q}).$$

so

$$C_m^m - p_q[C_m]^{m/q} = -mp_1,$$

so

$$C_m = (p_q[C_m]^{m/q} - mp_1)^{1/m}$$

As before, we can expand by the binomial theorem and iterate to get the explicit formula for C_m^α.

Recap

(1) Let m be a an integer greater than 1, and let B_m be the plethystic inverse of $-L_m[1 - p_1]$, where

$$L_m = \frac{1}{m} \sum_{d|m} \mu(d) p_d^{m/d}.$$

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Then B_m is integral.

(2) Now let m be a power of the prime q, and let $C_m = 1 - B_m$. Then for all α,

$$\begin{aligned} C_m^{-\alpha} &= 1 + \sum_{n=1}^{\infty} \sum_{\lambda \vdash_q n} \frac{p_\lambda}{z_\lambda} \alpha \prod_{j=2}^{l(\lambda)} (m\lambda_j + m\lambda_{j+1} + \cdots + m\lambda_{l(\lambda)} + \alpha) \\ &= \sum_{n=0}^{\infty} \sum_{\lambda \vdash_q n} \frac{p_\lambda}{z_\lambda} \prod_{j=1}^{l(\lambda)} (m\lambda_{j+1} + m\lambda_{j+2} + \cdots + m\lambda_{l(\lambda)} + \alpha). \end{aligned}$$

Questions

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2. Does B_m have a combinatorial interpretation?
3. Is B_m Schur positive? If so, does it have a representation-theoretic interpretation?

We can ask similar questions about C_m^{-1} and $1 - C_m^k$ for $k = 1, 2, \ldots, m-1$, which are all integral (and positive when m is a prime power).

How can we find a combinatorial interpretation for a symmetric function?

Given a symmetric function $F = F(x_1, x_2, \dots)$, there are two one-variable generating functions associated with it that may be helpful in understanding what it counts.

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It can be obtained by setting $p_1 = z$, $p_2 = p_3 = \cdots = 0$ in F.

If F is the symmetric function generating for some type of "partially labeled objects" (for example, a cycle index via Pólya's theorem) then $\widetilde{F}$ counts unlabeled objects (all labels are the same) and $\mathscr{F}$ counts totally labeled objects (all labels are different).

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For $m = 2$, the equation for $B = B_2$ may be written

$$B = p_1 + \frac{1}{2}(B^2 + p_2[B]) = p_1 + h_2[B]$$

so $\widetilde{B} = \widetilde{B}_2$ satisfies

$$\widetilde{B}(z) = z + \frac{1}{2}\big(\widetilde{B}(z)^2 + \widetilde{B}(z^2)\big) = z + h_2[\widetilde{B}(z)]$$

This means that $\widetilde{B}(z)$ counts unlabeled binary trees, as we would expect. (Wedderburn-Etherington numbers.)

n	0	1	2	3	4	5	6	7	8	9
$\widetilde{B}_n$	0	1	1	1	2	3	6	11	23	46

For $m = 3$, the equation for $B = B_3$ may be written

$$B = p_1 + B^2 - \frac{1}{3}\big(B^3 - p_3[B]\big),$$

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The coefficients of $\widetilde{B}(z)$ are A352702 in the OEIS (also A107092) but the OEIS has no useful information about these numbers.

n	0	1	2	3	4	5	6	7	8	9
$[z^n]\,\widetilde{B}_3(z)$	0	1	1	2	4	9	22	55	142	376

Exponential generating functions

For the exponential generating functions, things are much nicer. Writing $\mathscr{B}$ for $\mathscr{B}_m$ and $\mathscr{C}$ for $\mathscr{C}_m$, we have (for all m, not just prime powers)

$$\mathscr{C} = (1 - mz)^{1/m}$$

$$\begin{aligned}\mathscr{B} = 1 - (1 - mz)^{1/m} &= \sum_{n=1}^{\infty} (-1)^{n-1} \binom{1/m}{n} (mz)^n \\ &= \sum_{n=1}^{\infty} (m-1)(2m-1)\cdots((n-1)m-1)\frac{z^n}{n!},\end{aligned}$$

and more generally,

$$\begin{aligned}1 - \mathscr{C}^k &= 1 - (1 - mz)^{k/m} \\ &= \sum_{n=1}^{\infty} k(m-k)(2m-k)\cdots((n-1)m-k)\frac{z^n}{n!}.\end{aligned}$$

These exponential generating functions have several combinatorial interpretations, which are most easily approached through differential equations. We introduce another auxiliary symmetric function $T = C^{-1} = (1 - B)^{-1}$, with corresponding exponential generating function

$$\begin{aligned}\mathscr{T} &= \frac{1}{\mathscr{C}} = \frac{1}{1 - \mathscr{B}} \\ &= \frac{1}{(1 - mz)^{1/m}} \\ &= 1 + \sum_{n=1}^{\infty} (m+1)(2m+1)\cdots\bigl((n-1)m+1\bigr)\frac{z^n}{n!}.\end{aligned}$$

To find combinatorial interpretations for these exponential generating functions we take derivatives.

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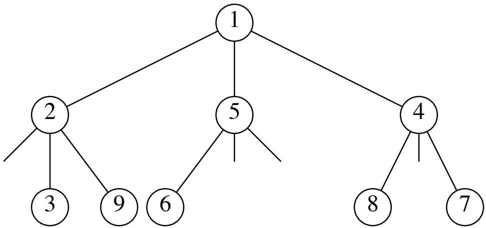
We find that

$$\mathscr{T}'(z) = \frac{1}{(1 - mz)^{(m+1)/m}} = \mathscr{T}(z)^{m+1}$$

The theory of exponential generating functions gives us a combinatorial interpretation from this differential equation: $\mathscr{T}(z)$ counts $(m+1)$-ary increasing trees.

For example, with $m = 2$, a ternary increasing tree is



- Every vertex has 3 children, but some of them may be empty. (The empty tree is a ternary increasing tree.)
- Every vertex is less than all of its descendants.

To find the combinatorial interpretation for $1 - \mathscr{C}^k = 1 - (1 - mz)^{k/m}$, and in particular for $\mathscr{B}(z)$, which is the case $k = 1$, we take its derivative:

$$\frac{d}{dz}\left(1 - (1 - mz)^{k/m}\right) = \frac{k}{(1 - mz)^{(m-k)/m}} = k\mathscr{T}^{m-k}$$

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So we know what the coefficient of $x_1 \cdots x_n$ in B counts. But what about the other coefficients?

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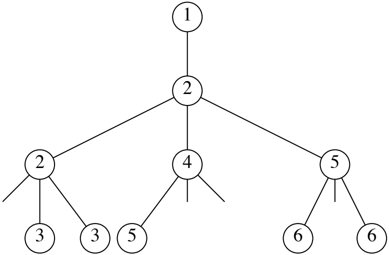
We might hope that they would count B-trees with repeated labels allowed:



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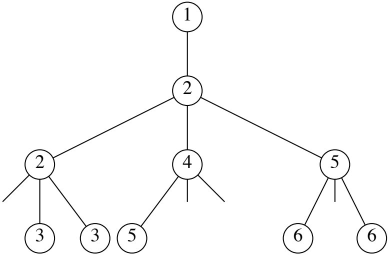

But of course this doesn't work.

Some ideas

The differential equation approach might be generalizable to symmetric functions.

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$$\frac{\partial B}{\partial p_1} = \frac{1}{(1-B)^{m-1}}.$$

Unfortunately, this only determines the terms in B that contain p_1, and this isn't enough.

There are more general differential operators on symmetric functions with nice combinatorial properties, $\partial_n = h_n^\perp$. For example,

$$\partial_2 = \frac{1}{2}\frac{\partial^2}{\partial p_1^2} + \frac{\partial}{\partial p_2},$$

and we have

$$\partial_2 B_m = \frac{\frac{1}{2}(m-1)}{(1-B_m)^{2m-1}}$$

for m odd.

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for m odd. But I couldn't get very far with this idea.

Another idea is to look at B-trees with repeated labels allowed but with certain inequalities that must be satisfied.

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An idea for proving Schur positivity is to find a symmetric group representation on the vector space spanned by B-trees. We can start by having the group act by permuting the labels. But this doesn't preserve the increasing condition, so some kind of "straightening" will be necessary.

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More models

There are some additional combinatorial models for the coefficients of $\mathscr{B}$ (and $\mathscr{T}$) that give alternatives to the increasing $(m+1)$-ary trees.

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First we have m-Stirling permutations. These are permutations of the multiset $\{1^m, 2^m, \ldots, n^m\}$ with the property that between two occurrences of i only numbers larger than i appear, for example, with $m = 2$, 1221344553. There is a simple bijection Φ from m-Stirling permutations to $(m+1)$-ary increasing trees.

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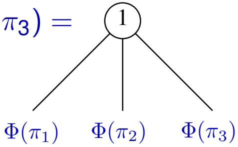
It may be defined recursively (for $m = 2$) by

$\Phi(\pi_1 1 \pi_2 1 \pi_3) =$



and similarly for all m.

$(m-1)$-colored increasing ordered trees

Another interpretation is suggested by the differential equation

$$\mathscr{B}' = \frac{1}{(1-\mathscr{B})^{m-1}}.$$

(Recall that $\mathscr{B}(z) = 1 - (1 - mz)^{1/m}$.)

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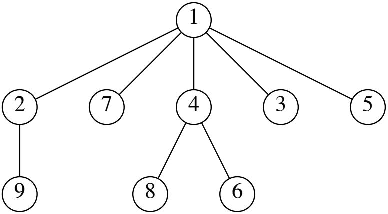
(Recall that $\mathscr{B}(z) = 1-(1-mz)^{1/m}$.)

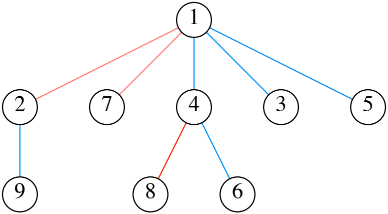
For $m=2$, this is $\mathscr{B}' = 1/(1-\mathscr{B})$ so $\mathscr{B}$ counts increasing ordered trees:



More generally, the differential equation

$$\mathscr{B}' = \frac{1}{(1-\mathscr{B})^{m-1}}.$$

means that $\mathscr{B}$ counts ordered trees in which the edges are colored in colors 1, 2, ..., $m-1$ with the property that among the children of any vertex, those of color 1 come first, then those of color 2, and so on. (But not every color must appear.)



Thank you!