Colored multiset Eulerian polynomials

Danai Deligeorgaki
(KTH Royal Institute of Technology)

Based on ongoing work with Bin Han & Liam Solus.

Eulerian polynomials

$[n] := \{1, 2, \dots, n\}$

$n=1$	1
$n=2$	$1+x$
$n=3$	$1+4x+x^2$
$n=4$	$1+11x+11x^2+x^3$
$n=5$	$1+26x+66x^2+26x^3+x^4$
$\vdots$	$\vdots$

palindromic, unimodal, ...

- $$A_{[n]}(x) = \sum_{\pi \in S_n} x^{\mathrm{des}(\pi)}$$

$\mathrm{des}(\pi)$ is a permutation statistic

- integer points in polytopes
- volumes of slices of cubes
- faces of simplicial complexes

...

Several identities for $A_{[n]}$.

Carlitz identity: $$\frac{A_{[n]}(x)}{(1-x)^{n+1}} = \sum_{t \geq 0} (t+1)^n x^t$$

Colored Multiset Permutations

Let $M=\{\underbrace{1,1,\dots,1}_{m_1>0},\ \underbrace{2,2,\dots,2}_{m_2>0},\ \dots,\ \underbrace{n,n,\dots,n}_{m_n>0}\}$ be a <u>multiset</u>.

$m := m_1+m_2+\dots+m_n$

$S_M := \{\text{permutations } \underbrace{\pi=\pi_1\cdots\pi_m}_{\text{one-line notation}} \text{ of } M\}$

<u>example</u>

$M=\{1,1,2\}$:

$S_M=\{112,121,211\}$

Definition

Let $r=(r_1,\dots,r_n)\in\mathbb{Z}^n_{>0}$.

- A colored permutation $\pi^{\overset{\text{vector}}{\boxed{c}}} = \pi_1^{c_1}\pi_2^{c_2}\cdots\pi_m^{c_m}\ \overset{(n+1)^1}{\overset{\|}{\pi_{m+1}^{c_{m+1}}}}$ of M is a permutation $\pi\in S_M$ with $c_i\in[r_k]$ if $\pi_i=k$, for $k\in[n]$.

Denote the set of π^c's by S_M^r.

Colored Multiset Permutations

Definition

Let $r = (r_1, \dots, r_n) \in \mathbb{Z}^n_{>0}$.

- A **colored permutation** $\pi^{\boxed{c}} = \pi_1^{c_1}\pi_2^{c_2}\cdots\pi_m^{c_m}\,\overset{(n+1)^1 \,=\, c_{m+1}}{\pi_{m+1}}$ of M is a permutation $\pi \in S_M$ with $c_i \in [r_k]$ if $\pi_i = k$, for $k \in [n]$.

vector

Denote the set of π^c's by S_M^r.

signed permutations

$r_1 = r_2 = 2$

$$M = \{1,1,2,2\}$$

$n = 2, \; m = 4$

$1^1 1^1 2^1 2^1 3^1$	$1^1 1^2 2^1 2^1 3^1$	$1^2 1^1 2^1 2^1 3^1$	$1^2 1^2 2^1 2^1 3^1$	
$1^1 1^1 2^1 2^2 3^1$	$1^1 1^2 2^1 2^2 3^1$	$1^2 1^1 2^1 2^2 3^1$	$1^2 1^2 2^1 2^2 3^1$	$\dots$
$1^1 1^1 2^2 2^1 3^1$	$1^1 1^2 2^2 2^1 3^1$	$1^2 1^1 2^2 2^1 3^1$	$1^2 1^2 2^2 2^1 3^1$	
$1^1 1^1 2^2 2^2 3^1$	$1^1 1^2 2^2 2^2 3^1$	$1^2 1^1 2^2 2^2 3^1$	$1^2 1^2 2^2 2^2 3^1$	$2^4 \cdot 6$ in total

Descents of $\pi^c \in S_{M^r}$

A **descent** of $\pi^c = \pi_1^{c_1} \cdots \pi_m^{c_m}(n+1)^1$ is an $i \in [m]$: $\pi_i^{c_i} > \pi_{i+1}^{c_{i+1}}$

order: $1^1 < 2^1 < \cdots < n^1 < (n+1)^1 < 1^2 < 2^2 < \cdots < n^2 < 1^3 < \cdots < n^{\max\{r_i\}}$

(Steingrímsson)

$(1^1 < 2^1 < 3^1 < 1^2 < 2^2)$

E.g. $M = \{1,1,2,2\}$, $r_1 = r_2 = 2$

$\underline{1^2} 1^1 2^1 \underline{2^2} 3^1$ $\quad$ $1^1 1^1 2^1 2^1 3^1$ $\quad$ . . .

$\underline{1^2}\,\underline{2^1} 1^1 2^1 3^1$ $\quad$ $\underline{2^2} \underline{1^2} \underline{2^1} 1^1 3^1$

DES(π^c) : set of descents of π^c

des(π^c) : number of descents of π^c

colored multiset Eulerian polynomial: $A_{M^r}(x) := \sum_{\pi^c \in S_{M^r}} x^{\operatorname{des}(\pi^c)}$.

Examples:

1)

$$M=\{1,1,2\}$$
$$r=(1,1)$$

S_M^r :

$1^1 1^1 2^1 3^1$	$1^1 \underline{2}^1 1^1 3^1$	$\underline{2}^1 1^1 1^1 3^1$

2)

$$M=\{1,1,2,2\}$$
$$r=(2,1)$$

S_M^r :

$1^1 1^1 2^1 2^1 3^1$	$1^1 \underline{1}^2 2^1 2^1 3^1$	$\underline{1}^2 1^1 2^1 2^1 3^1$	$1^2 \underline{1}^2 2^1 2^1 3^1$
$2^1 \underline{2}^1 1^1 1^1 3^1$	$2^1 \underline{2}^1 1^1 \underline{1}^2 3^1$	$2^1 2^1 \underline{1}^2 1^1 3^1$	$2^1 2^1 1^2 \underline{1}^2 3^1$
$1^1 2^1 \underline{2}^1 1^1 3^1$	$\underline{1}^2 2^1 \underline{2}^1 1^1 3^1$	$1^1 2^1 2^1 \underline{1}^2 3^1$	$\underline{1}^2 2^1 2^1 \underline{1}^2 3^1$
$1^1 \underline{2}^1 1^1 2^1 3^1$	$1^1 2^1 \underline{1}^2 2^1 3^1$	$\underline{1}^2 \underline{2}^1 1^1 2^1 3^1$	$\underline{1}^2 2^1 \underline{1}^2 2^1 3^1$
$\underline{2}^1 1^1 \underline{2}^1 1^1 3^1$	$2^1 \underline{1}^2 \underline{2}^1 1^1 3^1$	$\underline{2}^1 1^1 2^1 \underline{1}^2 3^1$	$2^1 \underline{1}^2 2^1 \underline{1}^2 3^1$
$\underline{2}^1 1^1 1^1 2^1 3^1$	$2^1 \underline{1}^2 1^1 2^1 3^1$	$\underline{2}^1 1^1 \underline{1}^2 2^1 3^1$	$2^1 1^2 \underline{1}^2 2^1 3^1$

Examples:

1)

$$\boxed{\begin{array}{c} M = \{1,1,2\} \\ r = (1,1) \end{array}}$$

S_M^r : $1^1 1^1 2^1 3^1$ $\quad 1^1 \underline{2}^1 1^1 3^1$ $\quad \underline{2}^1 1^1 1^1 3^1$

$A_{M^r}(x) = 1 + 2x$

2)

$$\boxed{\begin{array}{c} M = \{1,1,2,2\} \\ r = (2,1) \end{array}}$$

S_M^r :

$1^1 1^1 2^1 2^1 3^1$	$1^1 \underline{1}^2 2^1 2^1 3^1$	$\underline{1}^2 1^1 2^1 2^1 3^1$	$1^2 \underline{1}^2 2^1 2^1 3^1$
$2^1 \underline{2}^1 1^1 1^1 3^1$	$2^1 \underline{2}^1 1^1 \underline{1}^2 3^1$	$2^1 2^1 \underline{1}^2 1^1 3^1$	$2^1 2^1 1^2 \underline{1}^2 3^1$
$1^1 2^1 \underline{2}^1 1^1 3^1$	$\underline{1}^2 2^1 \underline{2}^1 1^1 3^1$	$1^1 2^1 2^1 \underline{1}^2 3^1$	$\underline{1}^2 2^1 2^1 \underline{1}^2 3^1$
$1^1 \underline{2}^1 1^1 2^1 3^1$	$1^1 2^1 \underline{1}^2 2^1 3^1$	$\underline{1}^2 \underline{2}^1 1^1 2^1 3^1$	$\underline{1}^2 2^1 \underline{1}^2 2^1 3^1$
$\underline{2}^1 1^1 \underline{2}^1 1^1 3^1$	$2^1 \underline{1}^2 \underline{2}^1 1^1 3^1$	$\underline{2}^1 1^1 2^1 \underline{1}^2 3^1$	$2^1 \underline{1}^2 2^1 \underline{1}^2 3^1$
$\underline{2}^1 1^1 1^1 2^1 3^1$	$2^1 \underline{1}^2 1^1 2^1 3^1$	$\underline{2}^1 1^1 \underline{1}^2 2^1 3^1$	$2^1 1^2 \underline{1}^2 2^1 3^1$

$A_{M^r}(x) = 1 + 13x + 10x^2$

Aim: Understand inequalities that hold amongst ⑦

the coefficients of $A_{\mu}^{r}(x) = p_0 + p_1 x + \dots + p_d x^{d}$ $d = \deg(p)$

or its sym. decomposition $A_{\mu}^{r}(x) = \underset{\deg(a)=d}{a(x)} + x\,\underset{\deg(b)<d}{b(x)}$ a, b palindromic

- palindromic $((p_0, p_1, \dots, p_d)$ is symmetric $)$
- unimodal $(p_0 \le p_1 \le \dots \le p_k \ge \dots \ge p_d$ for $0 \le k \le d)$
- log-concave $(p_k^2 \ge p_{k-1} p_{k+1}$ for $1 \le k \le d-1)$
- (b_i) γ-positive $\left(= \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \gamma_i x^i (1+x)^{d-2i}, \quad \gamma_i \ge 0 \right)$
- real-rooted (all roots lie in $\mathbb{R}$)
- alternatingly increasing $(0 \le p_0 \le p_d \le p_1 \le \dots \le p_{\lfloor \frac{d+1}{2} \rfloor})$

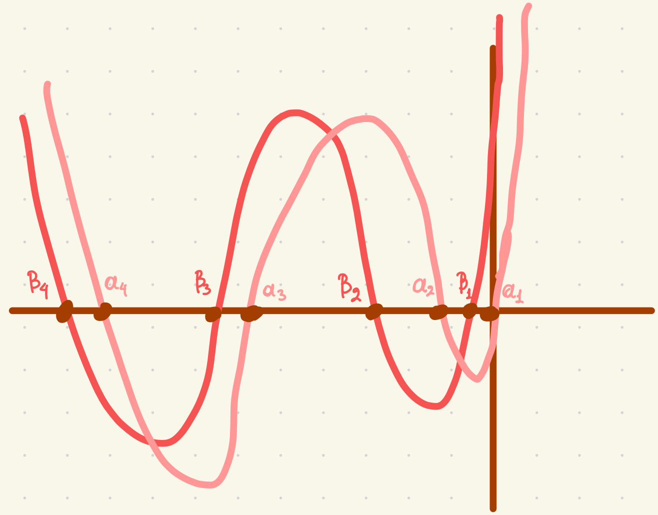
Given real-rooted polynomials we can ask how their roots relate along the real line.

- p, q with only real zeros

$p: \alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_d, \quad q: \beta_1 \geq \beta_2 \geq \dots \geq \beta_m$

- q **interlaces** p $(q < p)$ if

$$\alpha_1 \geq \beta_1 \geq \alpha_2 \geq \beta_2 \geq \dots$$



Examples:

$p(x) = 1 + 4x + x^2$

$q(x) = 1 + x$

$q < p$

$$\alpha_1 \geq \beta_1 \geq \alpha_2$$

$\alpha_1 = -2-\sqrt{3}$, $\beta_1 = -1$, $\alpha_2 = -2+\sqrt{3}$

$p(x) = 1 + x + x^2$ --- $\Delta < 0$

$q(x) = 1 + x$

p has non-real zeros

Aim: Understand inequalities that hold amongst the coefficients of $A\mu^r(x) = p_0 + p_1 x + \dots + p_d x^d$, $d = \deg(p)$

or its sym. decomposition $A\mu^r(x) = \underset{\deg(a)=d}{a(x)} + x\,\underset{\deg(b)<d}{b(x)}$ a, b palindromic

$b < a$

a, b have non-negative coefficients $\Rightarrow$ $A\mu^r, a, b$:

- ∘ **palindromic** $((p_0, p_1, \dots, p_d)$ is symmetric$)$
- ◇ **unimodal** $(p_0 \le p_1 \le \dots \le p_k \ge \dots \ge p_d$ for $0 \le k \le d)$
- □ **log-concave** $(p_k^2 \ge p_{k-1} p_{k+1}$ for $1 \le k \le d-1)$
- ✢ **bi-γ-positive** $\left(\dots = \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \gamma_i x^i (1+x)^{d-2i},\ \gamma_i \ge 0\right)$
- ▫ **real-rooted** (all roots lie in $\mathbb{R}$)
- ✿ **alternatingly increasing** $\left(0 \le p_0 \le p_d \le p_1 \le \dots \le p_{\lfloor \frac{d+1}{2} \rfloor}\right)$

Examples

$$A_{M^r}(x) = \overset{\deg(a)=d}{a(x)} + x\,\overset{\deg(b)<d}{b(x)}$$

$a(x)$, $b(x)$ palindromic

1) $\boxed{\begin{matrix} M=\{1,1,2,2\} \\ r=(2,1) \end{matrix}}$

$$A_{M^r}(x) = 1+13x+10x^2$$
$$= \underbrace{1+4x+x^2}_{a(x)} + x(\underbrace{9+9x}_{b(x)})$$

2) $\boxed{\begin{matrix} M=\{1,1,1,1,1,2,2\} \\ r=(1,1) \end{matrix}}$

$$A_{M^r}(x) = 1+10x+10x^2$$
$$= \underbrace{1+x+x^2}_{a(x)} + x(\underbrace{9+9x}_{b(x)})$$

$b < a$

- $\underset{-2-\sqrt{3}}{a_1} \geq \underset{-1}{\beta_1} \geq \underset{-2+\sqrt{3}}{a_2}$

- $a(x)$ has non-real roots
- $A_{M^r}(x)$ altern. increasing

Goal 1

Generalize the Carlitz identity $\dfrac{A_{[n]}(x)}{(1-x)^{m+1}} = \sum_{t\geq 0} (t+1)^n x^t$

for $M = \{\underbrace{1,\ldots,1}_{m_1}, \ldots, \underbrace{n,\ldots,n}_{m_n}\}$, $m = m_1 + \ldots + m_n$ and $\mathbf{r} = (r_1, \ldots, r_n)$.

Color / Set: $\frac{A_{M^{\mathbf{r}}}(x)}{(1-x)^{m+1}} =$	$\mathbf{r} = (1,\ldots,1)$	$\mathbf{r} = (2,\ldots,2)$	$\mathbf{r} = (3,\ldots,3)$	
$\mathbf{m} = (1,\ldots,1)$	$\sum_{t\geq 0} (t+1)^n x^t$	$\sum_{t\geq 0} (2t+1)^n x^t$	$\sum_{t\geq 0} (3t+1)^n x^t$	$\cdots$ (Steingrimsson) 1994
$\mathbf{m} = (2,\ldots,2)$	$\sum_{t\geq 0} \left((t+1)(\frac{t}{2}+1)\right)^n x^t$			
$\mathbf{m} = (3,\ldots,3)$	$\sum_{t\geq 0} \left((t+1)(\frac{t}{2}+1)(\frac{t}{3}+1)\right)^n x^t$			
	$\vdots$	$\vdots$		
	(MacMahon) 1915	(Lin) 2013		

Theorem: For a multiset $M=\{\underbrace{1,\dots,1}_{m_1},\dots,\underbrace{n,\dots,n}_{m_n}\}$, $m=|M|$, $\mathbf{r}=(r_1,\dots,r_n)$:

$$\frac{A_M^{\mathbf{r}}(x)}{(1-x)^{m+1}} = \sum_{t\geq 0}\prod_{k=1}^{n}\binom{r_k t+m_k}{m_k}x^t$$

(The proof uses Gessel-Stanley's "barred permutations")

Set \ Color	$\frac{A_M^{\mathbf{r}}(x)}{(1-x)^{m+1}}=$ $\mathbf{r}=(1,\dots,1)$	$\mathbf{r}=(2,\dots,2)$	$\mathbf{r}=(3,\dots,3)$	$\dots$
$\mathbf{m}=(1,\dots,1)$	$\sum_{t\geq 0}(t+1)^n x^t$	$\sum_{t\geq 0}(2t+1)^n x^t$	$\sum_{t\geq 0}(3t+1)^n x^t$	
$\mathbf{m}=(2,\dots,2)$	$\sum_{t\geq 0}\left((t+1)(\frac{t}{2}+1)\right)^n x^t$			
$\mathbf{m}=(3,\dots,3)$	$\sum_{t\geq 0}\left((t+1)(\frac{t}{2}+1)(\frac{t}{3}+1)\right)^n x^t$			
$\vdots$				

$$\sum_{t\geq 0}\left((r\,t+1)(r\,\tfrac{t}{2}+1)\cdots(r\,\tfrac{t}{m_1}+1)\right)^n x^n$$

(The case $m_1=\dots=m_n$ and $r_1=\dots=r_n$.)

More generally, for $M=\{\underbrace{1,\dots,1}_{m_1},\dots,\underbrace{n,\dots,n}_{m_n}\}$, $|M|=m$, $\mathbf{r}=(r_1,\dots,r_n)$,

$$\boxed{\sum_{\pi^c\in S_{M^r}} \frac{z_{\pi_1}^{c_1-1}\cdots z_{\pi_m}^{c_m-1}\prod_{\substack{i=2,\dots,m+1\\ i-1\in DES(\pi^c)}} z_{\pi_i}^{r_{\pi_i}}\cdots z_{\pi_m}^{r_{\pi_m}} z_{m+1}}{\prod_{i=1}^{m+1}\left(1-z_{\pi_i}^{r_{\pi_i}}\cdots z_{\pi_m}^{r_{\pi_m}} z_{m+1}\right)} = \sum_{t\geq 0} z_{m+1}^t \prod_{k=1}^{n}\begin{bmatrix} r_k t+m_k \\ m_k\end{bmatrix}_{z_k}}$$

More generally, for $M=\{\underbrace{1,\dots,1}_{m_1},\dots,\underbrace{n,\dots,n}_{m_n}\}$, $|M|=m$, $\mathbf{r}=(r_1,\dots,r_n)$,

$$\boxed{\sum_{\pi^c\in S_{M^r}} \frac{z_{\pi_1}^{c_1-1}\cdots z_{\pi_m}^{c_m-1}\prod_{\substack{i=2,\dots,m+1\\ i-1\in \mathrm{Des}(\pi^c)}} z_{\pi_i}^{r_{\pi_i}}\cdots z_{\pi_m}^{r_{\pi_m}}x}{\prod_{i=1}^{m+1}\left(1-z_{\pi_m}^{r_{\pi_i}}\cdots z_{\pi_m}^{r_{\pi_m}}x\right)} = \sum_{t\geq 0} x^t \prod_{k=1}^{n} \begin{bmatrix} r_k t+m_k \\ m_k\end{bmatrix}_{z_k}}$$

Specializations

$r=2$: Chow-Gessel, 2007

$z_1=\dots=z_n=q$, $r_1=\dots=r_n=r$:

$$\frac{\sum_{\pi^c\in S_{M^r}} x^{\mathrm{des}(\pi^c)} q^{\mathrm{fdmaj}(\pi^c)}}{\prod_{i=0}^{m}\left(1-q^{r i}x\right)} = \sum_{t\geq 0} x^t\prod_{k=1}^{n}\begin{bmatrix} rt+m_k\\ m_k\end{bmatrix}_q$$

$z_1=\dots=z_n=1$:

$$\frac{A_{M^r}(x)}{(1-x)^{m+1}} = \sum_{t\geq 0}\prod_{k=1}^{n}\binom{r_k t+m_k}{m_k}x^t$$

also specializes to the identities of MacMahon, Lin, Steingrímsson, q-binomial theorem...

Carlitz 1975 :

$$\frac{\displaystyle\sum_{\pi\in S_M} x^{\mathrm{des}(\pi)}\, q^{\mathrm{maj}(\pi)}}{\displaystyle\prod_{i=0}^{m}(1-q^i x)} = \sum_{t\geq 0} x^t\,[t+1]_q^n$$

Lin 2015 — multiset multivariate $\mathbf{r}=(2,\dots,2)$

different proofs

Bagno-Biagioli 2007 — color vectors $\mathbf{r}=(r,\dots,r)$

Beck-Braun 2013 — multivariate version

Theorem

$$\frac{\displaystyle\sum_{\pi^c\in S_M^r} \prod_{i=1}^{m}(x z_{\pi_1}\cdots z_{\pi_i})^{a_i(\pi^c)} \prod_{\substack{a_i(\pi^c)=0\\ i\in \mathrm{DES}(\pi^c)}}(x z_{\pi_1}\cdots z_{\pi_i})^{r}}{\displaystyle(1-x)\prod_{i=1}^{m}\left(1-(x z_{\pi_1}\cdots z_{\pi_i})^{r}\right)} = \sum_{t\geq 0} x^t \prod_{k=1}^{n}\begin{bmatrix} t+m_k\\ m_k\end{bmatrix}_{z_k}$$

¡The right handside does not depend on r!

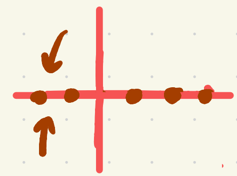
Questions?

$$A_{M^r}(x) := \sum_{\pi^c \in S_{M^r}} x^{\mathrm{des}(\pi^c)} = \underbrace{a(x)}_{\deg(a)=d} + x\underbrace{b(x)}_{\deg(b)<d}$$

palindromic ?

unimodal ?

real-rooted ?



...

Tools:

$$\frac{A_{M^r}(x)}{(1-x)^{m+1}} = \frac{\sum_{\pi^c \in S_{M^r}} x^{\mathrm{des}(\pi^c)}}{(1-x)^{m+1}} = \sum_{t \geq 0} \prod_{k=1}^{n} \binom{r_k t + m_k}{m_k} x^t$$

Set \ Color	$\mathbf{r}=(1,\dots,1)$	$\mathbf{r}=(2,\dots,2)$	$\mathbf{r}=(3,\dots,3)$	
$\frac{A_M^{\mathbf{r}}(x)}{(1-x)^{m+1}}=$				
$\mathbf{m}=(1,\dots,1)$	$\sum_{t\geq 0}(t+1)^n x^t$	$\sum_{t\geq 0}(2t+1)^n x^t$	$\sum_{t\geq 0}(3t+1)^n x^t$	$\dots$
$\mathbf{m}=(2,\dots,2)$	$\sum_{t\geq 0}\left((t+1)(\frac{t}{2}+1)\right)^n x^t$	$\sum_{t\geq 0}\left((2t+1)(t+1)\right)^n x^t$	$\sum_{t\geq 0}\left((3t+1)(\frac{3t}{2}+1)\right)^n x^t$	
$\mathbf{m}=(3,\dots,3)$	$\sum_{t\geq 0}\left((t+1)(\frac{t}{2}+1)(\frac{t}{3}+1)\right)^n x^t$	$\sum_{t\geq 0}\left((2t+1)(t+1)(\frac{2t}{3}+1)\right)^n x^t$	$\sum_{t\geq 0}\left((3t+1)(\frac{3t}{2}+1)(t+1)\right)^n x^t$	
	$\vdots$		$\ddots$	

For $\mathbf{r}=(1,\dots,1)$, $A_M^{\mathbf{r}}$ is

(Carlitz, Hoggatt 1978) palindromic for $\mathbf{m}=(k,\dots,k)$

(Simion, 1984) real-rooted

(Lin, Xu, Zhao, 2022) bi-γ-positive for $\mathbf{m}=(k,\dots,k,1)$

Recall: For $\mathbf{m}=(5,2)$, i.e., $M=\{1,1,1,1,1,2,2\}$:

$$A_M^{\mathbf{r}}(x)=1+10x+10x^2=\underbrace{1+x+x^2}_{a}+x\cdot\underbrace{(9+9x)}_{b}$$

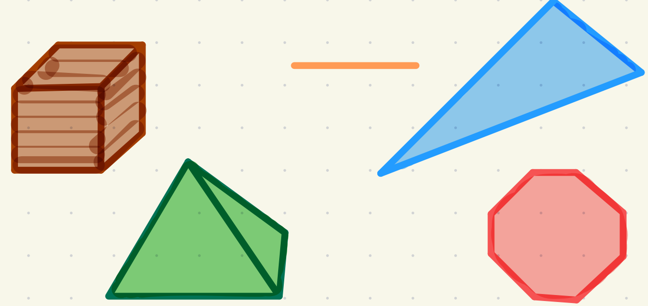
For $\mathbf{m}=(1,\dots,1)$, $\mathbf{r}=(c,\dots,c)$, $A_M^{\mathbf{r}}$ satisfies

(Steingrimsson, 1994) real-rootedness

(Brändén, Solus, 2021) $b<a$

using polytopes!

Polytopes

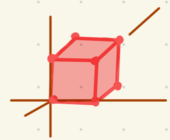


in $\mathbb{R}^d$

- A **polytope** is:

  - The convex hull $\operatorname{conv}\{v_1, \dots, v_n\} := \left\{ \sum_{i=1}^{n} \lambda_i v_i \;\middle|\; 0 \leq \lambda_i, \sum_{i=1}^{n} \lambda_i = 1 \right\}$

    of finitely many points $v_1, \dots, v_n \in \mathbb{R}^d$.

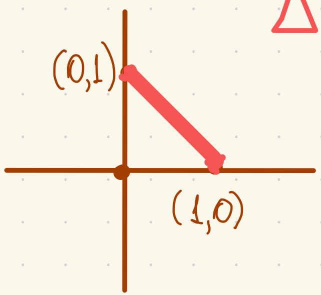
    Example: $\operatorname{conv}\{(x,y,z) \mid x,y,z \in \{0,1\}\} =$  in $\mathbb{R}^3$

  - Bounded intersection of $k < \infty$ closed half-spaces $\{\overset{\in\mathbb{R}^d}{x} \mid \overset{\in\mathbb{R}^{k\times d}}{A} x \leq \overset{\in\mathbb{R}^k}{b}\}$ in $\mathbb{R}^d$.

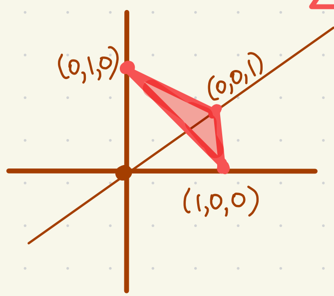
    Example: $$\left\{ (x,y,z) \;\middle|\; \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \leq \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} = \{(x,y,z) \mid 0 \leq x,y,z \leq 1\}$$

Standard Simplex:

$$\Delta_{d-1} = \operatorname{conv}\{ e_i \mid i \in [d] \} \subseteq \mathbb{R}^d, \quad e_i := (0,\dots,0,\underset{\substack{\downarrow \\ i\text{-th position}}}{1},0,\dots,0)$$

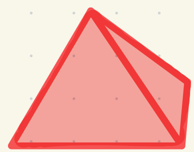


Δ_1



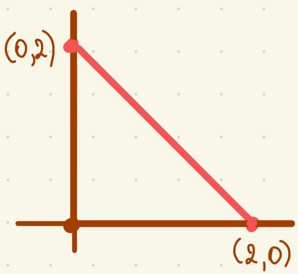
Δ_2

$\mathbb{R}^4$: Δ_3



...

$\boxed{\text{Dilated}}$ $\gg$ $k\Delta_{d-1} = \operatorname{conv}\{ k e_i \mid i \in [d] \} \subseteq \mathbb{R}^d. \quad (k \geq 1)$



$2\Delta_1$

...

Ehrhart Polynomials

Let $P = \mathrm{conv}\{v_1, \dots, v_n\}$ for $v_1, \dots, v_n \in \mathbb{Z}^d$.

(Ehrhart, '62)

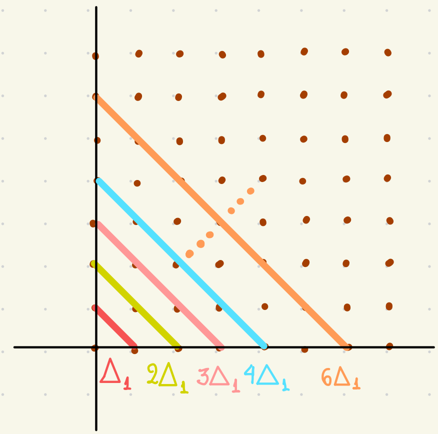
$\mathrm{erh}_P(k) = |kP \cap \mathbb{Z}^d|$ is a polynomial.

E.g.

$\Delta_1 = \mathrm{conv}\{(1,0),(0,1)\} \rhd\rhd \; \mathrm{ehr}_{\Delta_1}(k) = k+1$

$r\Delta_1 = \mathrm{conv}\{(r,0),(0,r)\} \rhd\rhd \; \mathrm{ehr}_{r\Delta_1}(k) = rk+1$

$\mathbb{Z}^2$

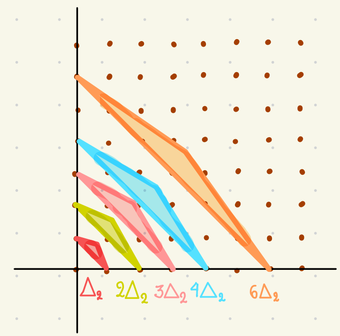


More generally,

$\Delta_d = \mathrm{conv}\{e_1, e_2, \dots, e_{d+1}\} \rhd\rhd \; \mathrm{ehr}_{\Delta_d}(k) = \binom{k+d}{d}$

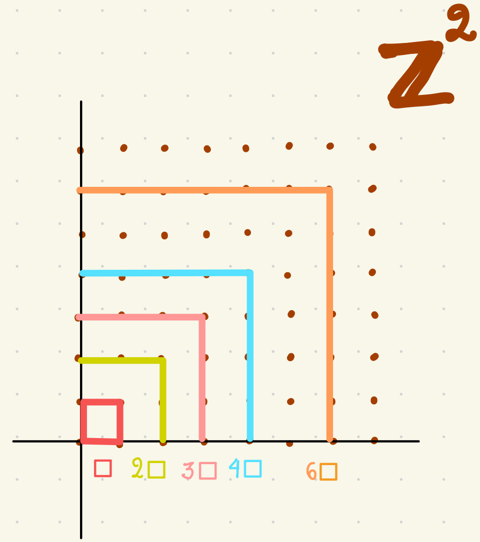
$r\Delta_d = \mathrm{conv}\{re_1, re_2, \dots, re_{d+1}\} \rhd\rhd \; \mathrm{ehr}_{r\Delta_d}(k) = \binom{rk+d}{d}$

$\mathbb{Z}^3$



(see "Computing the Continuous Discretely" book by Matthias Beck, Sinai Robins)

Ehrhart Polynomials



E.g.

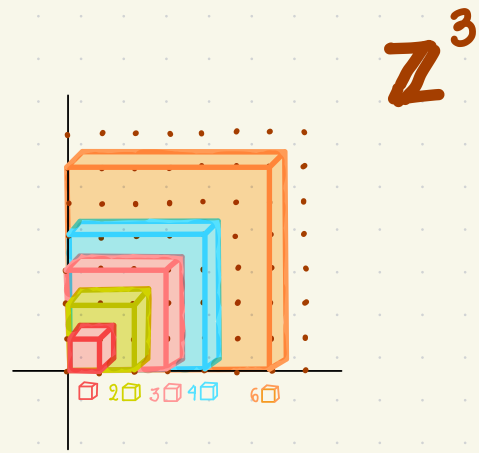
$[0,1]^2 = \text{conv}\{(x,y) \in \{0,1\}^2\}$ ⪢ $\text{ehr}_{\square}(k) = (k+1)^2$

$[0,r]^2 = \text{conv}\{(x,y) \in \{0,r\}^2\}$ ⪢ $\text{ehr}_{r\square}(k) = (rk+1)^2$

More generally,

$[0,1]^d$ ⪢ $\text{ehr}_{[0,1]^d}(k) = (k+1)^d$

$[0,r]^d$ ⪢ $\text{ehr}_{[0,r]^d}(k) = (rk+1)^d$



General rule?

Let $P = r_1 \Delta_{m_1} \times r_2 \Delta_{m_2} \times \cdots \times r_n \Delta_{m_n}$ $(r_i, m_i \in \mathbb{N}^+)$.

product polytope

Then $\mathrm{ehr}_P(k) = \mathrm{ehr}_{r_1\Delta_1}(k) \times \cdots \times \mathrm{ehr}_{r_n\Delta_n}(k)$

$$= \binom{r_1 k + m_1}{m_1} \times \cdots \times \binom{r_n k + m_n}{m_n}.$$

Theorem (D-Han-Solus, 2023⁺)

For $M = \{\underbrace{1,\ldots,1}_{m_1}, \ldots, \underbrace{n,\ldots,n}_{m_n}\}$, $m = |M|$, $r = (r_1, \ldots, r_n)$:

$$\frac{A_{M^r}(x)}{(1-x)^{m+1}} = \sum_{t \geq 0} \prod_{j=1}^{n} \binom{rt + m_j}{m_j} x^t = \sum_{t \geq 0} \mathrm{ehr}_{r_1\Delta_{m_1} \times \cdots \times r_n \Delta_{m_n}}(k)\, x^t$$

Results:

$$\frac{A_{M^r}(x)}{(1-x)^{m+1}} = \sum_{t\geq 0} \mathrm{ehr}_{r_1\Delta_{m_1}\times\cdots\times r_n\Delta_{m_n}}(k)\,x^t$$

$\left\{\begin{array}{l}\text{restrict to values}\\ r_j \geq m_j+1 \quad \forall j\in[n]\end{array}\right\}$ $\Downarrow$ $\left\{\begin{array}{l}r_1\Delta_{m_1}\times\cdots\times r_n\Delta_{m_n} \text{ has}\\ \text{an interior point}\end{array}\right\}$

Consequences

$b(x) \prec a(x)$ $\Longrightarrow$ real-rooted $\Longrightarrow$ unimodal . . .

palindromic $\Longleftrightarrow$ $r_j = m_j+1$ for $j\in[n]$

$r_j \geq m_j+1 \quad \forall j\in[n]$

Set \ Color; $\frac{A_{M^{\mathbf{r}}}(x)}{(1-x)^{m+1}} =$	$\mathbf{r}=(1,\dots,1)$	$\mathbf{r}=(2,\dots,2)$	$\mathbf{r}=(3,\dots,3)$	$\mathbf{r}=(4,\dots,4)$	$\dots$
$\mathbf{m}=(1,\dots,1)$	$\sum_{t\geq 0}(t+1)^n x^t$	$\sum_{t\geq 0}(2t+1)^n x^t$	$\sum_{t\geq 0}(3t+1)^n x^t$	$\sum_{t\geq 0}(4t+1)^n x^t$	
$\mathbf{m}=(2,\dots,2)$	$\sum_{t\geq 0}\left((t+1)(\frac{t}{2}+1)\right)^n x^t$	$\sum_{t\geq 0}\left((2t+1)(t+1)\right)^n x^t$	$\sum_{t\geq 0}\left((3t+1)(\frac{3t}{2}+1)\right)^n x^t$	$\sum_{t\geq 0}\left((4t+1)(2t+1)\right)^n x^t$	
$\mathbf{m}=(3,\dots,3)$	$\sum_{t\geq 0}\left((t+1)(\frac{t}{2}+1)(\frac{t}{3}+1)\right)^n x^t$	$\sum_{t\geq 0}\left((2t+1)(t+1)(\frac{2t}{3}+1)\right)^n x^t$	$\sum_{t\geq 0}\left((3t+1)(\frac{3t}{2}+1)(t+1)\right)^n x^t$	$\sum_{t\geq 0}\left((4t+1)(2t+1)(\frac{4t}{3}+1)\right)^n x^t$	
$\vdots$			$\ddots$		

$A_{M^{\mathbf{r}}}$ palindromic

$b < a$ for $r_j \geq m_j + 1$: The proof extends work of Brändén and Solus (2021).

Some igredients: Let $P := r_1\Delta_{m_1} \times \dots \times r_n \Delta_{m_n}$.

- $A_{M^{\mathbf{r}}}$ palindromic $\Leftrightarrow$ P Gorenstein (De Negri, Hibi, 1997) $\Leftrightarrow$ $r_j = m_j + 1$ for $1 \leq j \leq n$
- For $r_j \geq m_j + 1$, P has degree $m = m_1 + \dots + m_n$ $\Rightarrow$ $a(x), b(x)$ have nonnegative coefficients
- $\mathrm{Ehr}_P(t) = \prod_{j=1}^{n} \binom{r_j t + m_j}{m_j} = \sum_{i=0}^{m} c_i t^i (1+t)^{m-i}$ for $c_i \geq 0$ ("magic positive")
- Through the "subdivision operator" $\mathcal{E} : \binom{x}{k} \to x^k$, it suffices to compare roots of $\mathrm{Ehr}_P(x)$ (for different P).

Bi-γ-positivity

Consider the multisets (and no colors):

$$M_1 = \{\underbrace{1,\ldots,1}_{m_1},\ldots,\underbrace{n,\ldots,n}_{m_1},n+1\} \;\&\; M_2 = \{\underbrace{1,\ldots,1}_{m_1},\ldots,\underbrace{n,\ldots,n}_{m_1}\}$$

Theorem (Lin-Ma-Ma-Zhou, 2021):
The polynomial $A_{M_1}(x)$ is bi-γ-positive with symmetric decomposition

$$A_{M_1}(x) = \underbrace{\alpha(x)}_{\gamma\text{-positive}} + \underbrace{x^{(m_1-1)} A_{M_2}(x)}_{\gamma\text{-positive}}.$$

(D-Han-Solus, 2023+)
γ-coefficients count certain "weakly increasing trees" on $\{1,\ldots,1,\ldots,n\text{-}1,\ldots,n\text{-}1,n,n+1\}$

(Lin-Ma-Ma-Zhou, 2021)
γ-coefficients count certain "weakly increasing trees" on $\{1,\ldots,1,\ldots,n\text{-}1,\ldots,n\text{-}1,n\}$

Question: What about colored multiset Eulerian polynomials?

palindromic

bi-γ-positive

Color / Set	$\mathbf{r}=(1,\dots,1)$	$\mathbf{r}=(2,\dots,2)$	$\mathbf{r}=(3,\dots,3)$	$\mathbf{r}=(4,\dots,4)$	
$\mathbf{m}=(1,\dots,1)$	$A_{\{1,\dots,1\}}(x)$	$A_{\mu^r}(x)$	$A_{\mu^r}(x)$	$A_{\mu^r}(x)$	
$\mathbf{m}=(2,\dots,2)$, $\mathbf{m}'=(2,\dots,2,1)$	$A_{\{2,\dots,2\}}(x)$, $a(x)+x\cdot A_{\{2,\dots,2\}}(x)$	$A_{\mu^r}(x)$	$A_{\mu^r}(x)$	$A_{\mu^r}(x)$	$\dots$
$\mathbf{m}=(3,\dots,3)$, $\mathbf{m}'=(3,\dots,3,1)$	$A_{\{3,\dots,3\}}(x)$, $a(x)+2x\cdot A_{\{3,\dots,3\}}(x)$	$A_{\mu^r}(x)$	$A_{\mu^r}(x)$	$A_{\mu^r}(x)$	

$$A_{\mu^r}(x) = \underbrace{a(x)}_{\gamma\text{-coefficients}} + x\,\underbrace{b(x)}$$ symmetric decomposition

Question: Find a combinatorial interpretation for the γ-coefficients of the polynomials in the symmetric decomposition of A_{μ^r} whenever $r_j \geq m_j+1$, $1 \leq j \leq n$.

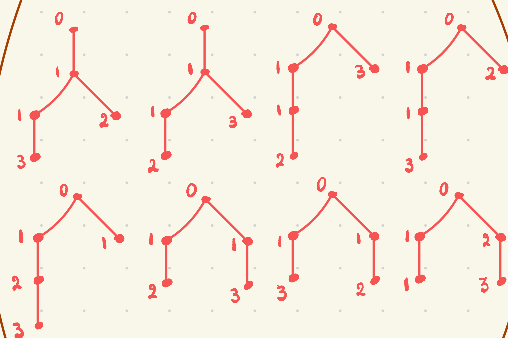
Example: $M_1=\{1,1,2,2,3\}$ $(r_1=r_2=r_3=1)$

$$A_{M_1}(x)=\sum_{\pi\in S_{M_1}} x^{\mathrm{des}(\pi)} = \underline{1+12x+15x^2+2x^3} = \underbrace{1+11x+11x^2+x^3}_{a(x)} + x\underbrace{(1+4x+x^2)}_{A_{M_2}(x),\ M_2=\{1,1,2,2\}}$$

weakly increasing tree interpretation:

$$\sum_{\pi\in S_{M_1}} x^{\mathrm{des}(\pi)} = \#\Big($$

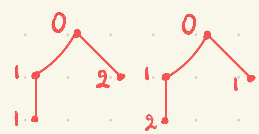
$$\Big)(x+1)^3 + \#\Big($$



$$\Big)x(x+1)$$

$$+\,x\Big(\#\Big($$

$$\Big)(x+1)^2 + \#\Big($$



$$\Big)x\Big)$$

Thank you! :)