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Bijective and Automated Approaches to Abel Sums

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Dedicated to Dominique Foata (b. Oct. 12, 1934), on his forthcoming 90th birthday

Abstract. In this tribute to our guru, Dominique Foata, we return to one of our (and Foata's) first loves, and revisit Abel sums and their identities, from two different viewpoints.

Preface

In the very first issue of *Crelle's journal* (the first mathematical periodical solely dedicated to mathematics), Niels Henrik Abel published a two-page paper [A], stating and proving his eponymous identity. This led to an intensive study of general *Abel Sums* by many people (see [R] [C] and their numerous references), and to beautiful bijective approaches pioneered by Dominique Foata and Aimé Fuchs [Fo] [FoFu] that led to Françon's elegant proof [Fr] (see [C], p. 129). This tribute consists of two independent parts. The first part is *bijective*, while the second part is *automated*, elaborating and extending John Majewicz's 1997 Ph.D. thesis [M1] [M2], and more importantly, fully implementing it (and its extension) in a *Maple* package
`https://sites.math.rutgers.edu/~zeilberg/tokhniot/AbelCeline.txt` .

(Majewicz's original *Maple* code, unfortunately, was lost).

Part I: Bijective proofs (à la Foata) of an Abel-type identity and a generalization

In [Ka] (see also [Ka']), the first author proved (as a special case of more general results) the following Abel-type identity. Let n, p by non-negative integers, then

$$\sum_{k=0}^{n} \binom{n}{k} k^k (n-k)^{n-k+p} = \sum_{k=0}^{n} \binom{n}{k} n^k (n-k)! S(p+n-k, n-k) \quad . \tag{1}$$

Here, $S(n, k)$ are the Stirling numbers of the second kind, and $k!S(n, k)$ is the number of maps from a set of size k onto a set of size n. (This property can serve as the definition of these numbers.)

A special case of this formula is attributed to Cauchy (Equation (24) in Chapter 1 of [R]):

$$\sum_{k=0}^{n} \binom{n}{k} k^k (n-k)^{n-k} = \sum_{k=0}^{n} \binom{n}{k} n^k (n-k)! \quad . \tag{2}$$

Here we present a combinatorial proof, in the style of Foata [Fo] [FoFu] for Formulas (2) and (1). We use the notation $[n] = \{1, 2, \ldots, n\}$, and to make the argument clearer we will present both proofs.

The Proofs

Proof of Equation (2)**:** The left-hand side counts triples f, A, B where f is a function from $[n]$ to $[n]$ and the following conditions hold:

$$A \cup B = [n],\ \ A \cap B = \emptyset,\ \ f(A) \subset A \text{ and } f(B) \subset B. \tag{3}$$

Indeed, if $|A| = k$, there are $\binom{n}{k}$ ways to choose A (B is now determined, $B = [n]\backslash A$), k^k ways to choose the restriction of f to A subject to the condition $f(A) \subset A$ and $(n-k)^{n-k}$ ways to choose the restriction of f to B subject to the condition $f(B) \subset B$.

The right-hand side counts triples f, C, D where f is a function from $[n]$ to $[n]$ and the following conditions hold:

$$C \cup D = [n],\ \ C \cap D = \emptyset,\ \ f(D) = D. \tag{4}$$

(Here, $f(D) = D$ means that f is a bijection from D to D; note that we relaxed the condition for C compared to A and strengthened the condition for D compared to B.) Indeed, if $|C| = k$, there are $\binom{n}{k}$ ways to choose C, n^k ways to choose the restriction of f to C (no conditions here) and $(n-k)!$ ways to choose the restriction of f to D subject to the condition $f(D) = D$.

The crucial observation is:

• For *every* function f from $[n]$ to $[n]$ the number of pairs (A, B) that satisfy Equation (3) equals the number of pairs (C, D) that satisfy Equation (4).

Indeed if (A, B) is a pair that satisfies Equation (3), we can take

$$D = f(f(\cdots(f(B))\cdots) \text{ and } C = [n]\backslash D. \tag{5}$$

In other words

$$D = \{d \in [n] \text{ such that for every } k \geq 0 \text{ there exists } b \in B \text{ with } d = f^k(b)\}.$$

Taking the inverse operation $B = f^{-1}(f^{-1}(\cdots(f^{-1}(D))\cdots)$ (and $A = [n]\backslash B$) brings you from a pair (C, D) to a pair (A, B). $\square$

Proof of Equation (1)**:**

Let $X = [n] = \{1, 2, \ldots .n\}$, and $Y = [n+p]$ Consider all triples (f, A, B) where, this time, f maps $[n+p]$ to $[n]$, and A, B satisfy

$$A \cup B = [n], A \cap B = \emptyset, f(A) \subset A \text{ and } f(B \cup [n+1, n+p]) \subset B.$$

We also consider triples (f, C, D) where, f maps $[n+p]$ to $[n]$, and C, D satisfy

$$C \cup D = [n], C \cap D = \emptyset, f(D \cup [n+1, n+p]) = D. \tag{7}$$

Also in this case a stronger statement holds: For every function $f : [n+p] \to [n]$ there is a bijection between between pairs (A, B) satisfying Equation (6) and pairs (C, D) satisfying Equation (7) and the bijection is given again by (5). $\square$

This bijection is implemented in the *Maple* package `AbelBijection.txt`, available from

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/AbelBijection.txt` .

See the front of this article
`https://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/abelKZ.html` for a sample input file and its corresponding output file.

Remark: If we give each element $k \in [n]$ a weight w_k and a function f from $[n+p]$ to $[n]$ the weight $w(f) := \prod_{k=1}^{n} w_{f(k)}$ then our bijective proof gives a Hurwitz-type generalization of Abel's formula, see Hurwitz [H] and Excercise 30 in Section 2.3.4.4 of Knuth [Kn]. (For Formula (1) we obtain a Hurwitz-type generalization of the Stirling numbers of the second kind.)

Part II: Automating Abel sums

Abel's original identity had many proofs, but the one by Shalosh B. Ekhad and John Majewicz [EM] (written 30 years ago, and dedicated to Dominique Foata on his 60^{th} birthday) has the distinction that it was *computer-generated*, yet with a bit of patience, *humanly readable* and verifiable. In order to motivate the sequel, let us reproduce it in its entirety.

Abel's identity: For any non-negative integer n,

$$\sum_{k=0}^{n} \binom{n}{k} (r+k)^{k-1}(s-k)^{n-k} = \frac{(r+s)^n}{r} \quad . \tag{8}$$

Proof: Let $F_{n,k}(r, s)$ and $a_n(r, s)$ denote, respectively, the summand and sum on the LHS of (8), and let $G_{n,k} := (s-n)\binom{n-1}{k-1}(k+r)^{k-1}(s-k)^{n-k-1}$. Since

$$F_{n,k}(r,s)-sF_{n-1,k}(r,s)-(n+r)F_{n-1,k}(r+1,s-1)+(n-1)(r+s)F_{n-2,k}(r+1,s-1) = G_{n,k}-G_{n,k+1} \,,$$

(check!), we have by summing from $k = 0$ to $k = n$, due to the telescoping on the right:

$$a_n(r,s) - sa_{n-1}(r,s) - (n+r)a_{n-1}(r+1,s-1) + (n-1)(r+s)a_{n-2}(r+1,s-1) = 0. \tag{9}$$

Since $(r+s)^n \cdot r^{-1}$ also satisfies this recurrence (check!), with the same initial conditions $a_0(r,s) = r^{-1}$ and $a_1(r,s) = (r+s) \cdot r^{-1}$, Equation (8) follows. $\square$

This proof was derived using John Majewicz's brilliant adaptation of *Sister Celine's technique* [Fa1] [Fa2] [Z] (see also [PWZ], Chapter 4). Recall that Sister Celine was interested in finding *pure recurrence relations* of the form

$$c_0(n)a_n + c_1(n)a_{n+1} + \cdots + c_L(n)a_{n+L} = 0 \quad , \tag{10}$$

where $c_0(n), \ldots, c_L(n)$ are polynomials in n for sequences a_n, that are defined by expressions of the form

$$a_n := \sum_{k=-\infty}^{\infty} F_{n,k} \quad ,$$

where $F_{n,k}$ is proper hypergeometric (see [PWZ] for the definition, in particular, $F_{n+1,k}/F_{n,k}$ and $F_{n,k+1}/F_{n,k}$ are both rational functions of n and k). The way she did it was to search for a recurrence of the form:

$$\sum_{i=0}^{L} \sum_{j=0}^{M} b_{ij}(n) F_{n+i,k+j} \ = \ 0 \quad , \tag{11}$$

for *some* positive integers L and M. (In her case by hand, but nowadays it has all been fully automated.)

Dividing by $F_{n,k}$, and clearing denominators, looking at the numerator, and then setting all the coefficients of powers of k to 0, we obtain a system of *linear equations* (with coefficients that are polynomials in n) for the *undetermined* $b_{ij}(n)$.

Having found the $b_{ij}(n)$, summing Equation (11) from $k = -\infty$ to $k = \infty$, we obtain Equation (10) with

$$c_i(n) = \sum_{j=0}^{M} b_{ij}(n) \quad , \quad 0 \leq i \leq L \quad .$$

In his Ph.D. thesis [M1] [M2] (written under the direction of the second author), John Majewicz adapted Sister Celine's method to *Abel-type* sums, of the form

$$a_n(r, s) = \sum_{k=0}^{n} F_{n,k} \, (r + k)^{k-1+p} (s - k)^{n-k+q} x^k \quad , \tag{12}$$

where $F_{n,k}$ is hypergeometric in n and k. Here p and q are arbitrary integers, and x is any number (or symbol). It is no longer the case that $a_n(r, s)$ satisfies a *pure* recurrence in n, with r and s **fixed**, but it does satisfy a *functional recurrence*, similar (but often much more complicated) to Equation (10). Denoting the summand of (12) by $\overline{F}_{n,k}(r, s)$

$$\overline{F}_{n,k}(r, s) \ := \ F_{n,k}(r + k)^{k-1+p} (s - k)^{n-k+q} x^k \quad ,$$

one looks for polynomials $b_{ij}(n)$ (that also depend on r, s, p, q and x, but are **free** of k), such that

$$\sum_{i=0}^{L} \sum_{j=0}^{M} b_{ij}(n) \overline{F}_{n+i,k+j}(r - j, s + j) \ = \ 0 \quad . \tag{13}$$

Dividing by $\overline{F}_{n,k}(r, s)$ (since $\overline{F}_{n+i,k+j}(r - j, s + j)/\overline{F}_{n,k}(r, s)$ is still a rational function of n and k), clearing denominators, looking at the numerator, and setting all the coefficients of powers of k to 0, we get again a system of linear equations for the undetermined quantities $b_{ij}(n)$. We then ask *Maple* to kindly `solve` them, and if *in luck* we get a non-zero solution. It can be shown ([M1] [M2]) that one can always find orders L and M for which such a system is solvable (for sufficiently large L and M, there are more unknowns than equations). Having found such a recurrence for $\overline{F}_{n,k}$, summing over k we obtain a functional recurrence for $a_n(r,s)$:

$$\sum_{i=0}^{L}\sum_{j=0}^{M} b_{ij}(n)a_{n+i}(r-j,s+j) \,=\, 0 \quad . \tag{14}$$

Implementation

We fully implemented the Celine–Majewicz algorithm in a *Maple* packge `AbelCeline.txt` available from

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/AbelCeline.txt` .

The function call is

`FindOpe(F,n,k,r,s,R,S,N,K,MaxOrd);` ,

where `MaxOrd` is the maximal order of the recurrence you are willing to tolerate.

It inputs a hypergeometric term $F_{n,k}$ and outputs the recurrence in the form of the corresponding operator, where N, R, and S are the forward shift operators in n, r, and s, respectively. To get a computer-generated paper, in *humanese*, the function call is

`Paper(F,n,k,r,s,R,S,N,K,MaxOrd):` .

Sample Output

Typing

`Paper(binomial(n,k)*x**k,n,k,r,s,2):`,

we get in 0.12 seconds the following deep fact.

Let, for any integers p and q and number (or symbol) x

$$a_n(r,s) := \sum_{k=0}^{n} \binom{n}{k} (r+k)^{k-1+p}(s-k)^{n-k+q}\, x^k \quad ,$$

then

$$a_n(r,s) = (nx+rx)\, a_{n-1}(r+1,s-1) + s a_{n-1}(r,s) + (-nrx-nsx+rx+sx)\, a_{n-2}(r+1,s-1) \quad .$$

Typing

`Paper(1/(k!**2*(n-k)!)*x**k,n,k,r,s,3):`

we get that the sequence of polynomials in (r, s) defined by

$$a_n(r,s) := \sum_{k=0}^{n} \frac{1}{k!^2(n-k)!}\,(r+k)^{k-1+p}(s-k)^{n-k+q}\,x^k$$

satisfies the functional recurrence

$$a_n(r,s) = \frac{x\,(n+r)\,a_{n-1}(r+1,s-1)}{n^2} + \frac{\left(2n^2-2ns-2n+s\right)s a_{n-1}(r,s)}{(n-1-s)\,n^2}$$
$$-\frac{\left(n^2+2nr-2rs-s^2-n-r\right)x a_{n-2}(r+1,s-1)}{(n-1-s)\,n^2}$$
$$-\frac{(n-s)\,s^2 a_{n-2}(r,s)}{(n-1-s)\,n^2} + \frac{\left(nr+ns-rs-s^2\right)x a_{n-3}(r+1,s-1)}{(n-1-s)\,n^2} \quad .$$

To see numerous other examples, read the output files in the front of this article:

`https://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/abelKZ.html` .

In particular, to see the functional recurrence satisfied by the 'innocent' sum:

$$a_n(r,s) := \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}(r+k)^{k-1+p}(s-k)^{n-k+q}\,x^k \quad ,$$

see the output file

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/oAbelCeline4.txt` .

Notice how complicated things get!

Getting differential recurrences

Since r and s, as opposed to n and k, are 'continuous variables', and since the functional recurrences gotten by Majewicz's Abel–Celine technique get so complicated even with very simple summands, it occurred to us to look for *differential* recurrences. For any bi-variate (proper) hypergeometric term $F_{n,k}$, defining (as above)

$$a_n(r,s) := \sum_{k=0}^{n} F_{n,k}(r+k)^{k-1+p}(s-k)^{n-k+q}\,x^k \quad ,$$

where p and q are integers (but may be left symbolic) and x is a number (again, it can be left symbolic), one looks for differential-recurrence equations of the form

$$\sum_{i=0}^{L}\sum_{j=0}^{M} b_{i,j}(n,r,s)\frac{d^i}{dr^i}a_{n+j}(r,s) \;=\; 0 \quad ,$$

and

$$\sum_{i=0}^{L}\sum_{j=0}^{M} c_{i,j}(n,r,s)\frac{d^i}{ds^i}a_{n+j}(r,s) \;=\; 0 \quad ,$$

Together these pairs of differential recurrence equations, combined with initial conditions, uniquely determine $a_n(r,s)$.

In fact, one can consider far more general 'kernels'. We can find a pair of differential recurrence equations for sums of the form

$$a_n(r,s) := \sum_{k=0}^{n} F_{n,k} K(r,s,n,k) \quad ,$$

for any *kernel* $K(r,s,n,k)$ such that $(\frac{\partial K}{\partial r})/K$, and $(\frac{\partial K}{\partial s})/K$ are rational functions of (r,s) (and k).

We proceed analogously. One applies the generic operator to the summand $\overline{F}(n,k)$, expands, clears denominators, looks at the numerator, then equates all the powers of k to 0, getting a system of linear equations that *Maple* can `solve` for you.

To get these pairs of equations, in verbose, human-readable form, type:

```
PaperD(F,n,k,r,s,MaxOrd,KER):
```

where `MaxOrd` is the maximum order you are willing to tolerate, and `KER` is the kernel. For example, typing

```
PaperD(binomial(n,k),n,k,r,s,2,(r+k)**(k-1+p)*(s-k)**(n-k+q)*x**k):    ,
```

we obtain in a fraction of a second the facts that the sequence of polynomials in r and s, $a_n(r,s)$ (for any p and q and x)

$$a_n(r,s) := \sum_{k=0}^{n} \binom{n}{k} (r+k)^{k-1+p}(s-k)^{n-k+q}\, x^k \quad ,$$

satisfy the pair of differential-recurrence equations (the first in r, the second in s)

$$-\left(pn+ns-n+p+s-1\right)a_n(r,s) + \left(nr+ns+r+s\right)\left(\frac{\partial}{\partial r}a_n(r,s)\right)$$
$$+(n+p)a_{n+1}(r,s) - (n+r+1)\left(\frac{\partial}{\partial r}a_{n+1}(r,s)\right) = 0 \quad ,$$

and

$$-\left(n+1\right)\left(q+n-s+1\right)a_n(r,s) + q a_{n+1}(r,s) + \left(n-s+1\right)\left(\frac{\partial}{\partial s}a_{n+1}(r,s)\right) = 0 \quad .$$

Note that setting $x=1$ and $p=0, q=0$ we obtain yet-another (automatic) proof of the original identity (8). In fact it is closer in spirit to Niels Abel's original proof, that also used differentiation (or rather integration).

Remark: The proof technique of this part can prove (2) and can be adapted (with some pre-processing) to prove (1).

To see many other examples, look at the output files

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/oAbelCeline5.txt` ,

and

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/oAbelCeline6.txt` .

To see examples with more complicated kernels, see

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/oAbelCeline7.txt` .

Readers can easily find many other such deep facts by playing with `AbelCeline.txt`. Enjoy!

Comment: The anonymous referees and the non-anonymous editor (Christian Krattenthaler) made the following interesting remarks. We thank them for allowing us to quote them *verbatim.*

The paper has a nice and interesting observation (namely that in some cases it might be more fruitful to consider differential operators instead of difference operators). However, the corresponding theory has already been developed in very general form by Frédéric Chyzak, Manuel Kauers, and Bruno Salvy in the interesting paper [CKS],

In this general form (Ore algebras) the corresponding algorithms have been implemented in Christoph Koutschan's Mathematica *package. It may however be that the present paper's specialized implementation is faster and more efficient than the general-purpose implementation of Koutschan.*

It also seems that one may be able to get differential recurrences, where one takes **both** *differentiations with respect to* r *and with respect to* s*.*

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