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DOWN-UP ALGEBRAS AND CHROMATIC SYMMETRIC FUNCTIONS

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Abstract. We establish Guay-Paquet's unpublished linear relation between certain chromatic symmetric functions by relating his algebra on paths to the q-Klyachko algebra. The coefficients in this relation are q-hit polynomials, and they come up naturally in our setup as connected remixed Eulerian numbers, in contrast to the computational approach of Colmenarejo, Morales and Panova. As Guay-Paquet's algebra is a down-up algebra, we are able to harness algebraic results in the context of the latter and establish expansions of a combinatorial flavor. In particular we resolve a conjecture of Colmenarejo, Morales and Panova on chromatic symmetric functions. This concerns the abelian case of the Stanley–Stembridge conjecture, which we briefly survey.

1. Introduction

Let $G = (V, E)$ be a finite undirected graph with $V = [n] \coloneqq \{1, \ldots, n\}$. The chromatic quasisymmetric function X_G introduced by Shareshian and Wachs [23] is a generalization of Stanley's chromatic symmetric function [25], which in turn is a generalization of Birkhoff's chromatic polynomial. Given the remarkable circle of ideas relating these functions to the cohomology of Hessenberg varieties [23, Section 10] and the Stanley–Stembridge conjecture [27], these functions have garnered substantial attention in the last decade; see for instance [1, 2, 3, 4, 6, 7, 11, 13, 14].

The Stanley–Stembridge conjecture states that X_G is *e-positive* when G is the incomparability graph of a naturally-labeled unit interval order. Such graphs G can be indexed by Dyck paths D and we refer to them as Dyck graphs, writing X_D in place of X_G when there is no scope for confusion. While the aforementioned conjecture is still wide open, there are known partial cases, most notably the *abelian case* [1, 7, 13].

Numerous lines of attack to this conjecture involve the *modular law* [10, 21]. This is a simple linear relation between certain X_G which itself has been a subject of much investigation; see [22] for a deep geometric perspective. Motivated by this law, Guay-Paquet [12] in unpublished work introduced the algebra $\mathcal{P}$ as the *noncommutative* algebra over $\mathbb{C}(q)$ generated by n and e subject to following the *modular* relations:

$$(1+q)\mathsf{ene} = q\mathsf{een} + \mathsf{nee} \tag{1}$$

$$(1+q)\mathsf{nen} = q\mathsf{enn} + \mathsf{nne}. \tag{2}$$

As we will see below, this algebra is in fact known as a down-up algebra. Working in $\mathcal{P}$, Guay-Paquet [12, Theorem 1] established a particularly elegant result which we now state. For undefined jargon in this context, we refer the reader to Sections 3 and 4.

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Theorem 1.1 (Guay-Paquet). *Let $D = UVW$ be a Dyck path where V is an abelian subpath with m north steps (denoted by n) and n east steps (denoted by e), with $m \geq n$. In particular, V may be identified with a partition λ in an $m \times n$ box. Then*

$$X_{UVW} = \sum_{0 \leq k \leq n} \frac{H_k^{m,n}(\lambda)}{(m)_q(m-1)_q \cdots (m-n+1)_q} X_{U\mathsf{e}^k\mathsf{n}^m\mathsf{e}^{n-k}W}.$$

*Here $H_k^{m,n}(\lambda)$ denotes the q-*hit number *of Garsia and Remmel* [9]*, and $(j)_q := (1-q^j)/(1-q)$ is the q-analog of j for $j \geq 0$.*

Informally put, abelian subpaths of Dyck paths may be replaced by special rectangular paths along with coefficients given by q-hit numbers. Hence it suffices to study X_G of the sort that arise on the right-hand side, thereby restricting attention to a much smaller family of Dyck graphs.

Recently, Colmenarejo, Morales and Panova [7] gave an independent proof of Theorem 1.1 relying on intricate rook-theoretic identities. Yet another proof was given independently by Lee and Soh [18].

1.1. **Discussion of results.** Our primary aim is to give a short algebraic proof of Theorem 1.1 by relating the algebra $\mathcal{P}$ to the *q-Klyachko algebra* $\mathcal{K}$. This *commutative* algebra is generated by $(u_i)_{i\in\mathbb{Z}}$ subject to the quadratic relations in (5). Its name reflects the fact that these relations are a deformation of Klyachko's presentation [16] of the S_n-invariant part of the cohomology ring of the permutahedral variety. As the authors demonstrated in [19], $\mathcal{K}$ has intimate links with various subareas within algebraic combinatorics. $\mathcal{K}$ possesses a basis $\mathcal{B}$ of square-free monomials. We will show that the statement in Theorem 1.1 is equivalent to the $\mathcal{B}$-expansion of a monomial $u_1^{c_1}\cdots u_k^{c_k}$ where $c_i > 0$ for $i = 1, \ldots, k$. The resulting coefficients are connected remixed Eulerian numbers [19, 20].

The previous links unearth other interesting properties of $\mathcal{P}$. We briefly describe them postponing explicit statements. It is the case that $\mathcal{P}$ is a down-up algebra introduced by Benkart and Roby [5] (see also [17, Definition 4.14] which implies that $\mathcal{P}$ is the $n = 2$ case of the *quantum pseudoplactic algebra*). As such it possesses a so-called PBW basis of staircase monomials $\mathscr{S}$, which was independently noticed by Guay-Paquet [12].

It is then natural to inquire about the expansion of any monomial w in the basis $\mathscr{S}$. Colmenarejo, Morales and Panova conjectured [7, Conjecture 6.6] that the resulting coefficients, up to an explicit sign, are Laurent polynomials with nonnegative integer coefficients. We resolve this conjecture by giving a simple combinatorial rule in Section 3.2, see Theorem 3.4 and the remarks following it.

Yet another basis comes up as follows: one can identify the diagonal subalgebra $\mathcal{P}^{\text{diag}}$ spanned by words with as many e's as n's, with the polynomial subalgebra in $\mathcal{K}$ generated by u_0 and u_1, these two generators corresponding to the words en and ne in $\mathcal{P}^{\text{diag}}$. The monomials in en and ne thus form a linear basis of $\mathcal{P}^{\text{diag}}$, which can be extended to a third basis for the space $\mathcal{P}_{i,j}$. We refer to this as the *zigzag* basis; see Section 3.3 for the precise description. We give an explicit description for the expansion of any word in the alphabet $\{\mathsf{n}, \mathsf{e}\}$ in this basis; see Theorem 3.13 and the remark following it.

In Section 4.2, we recall how the modular law implies that relations in $\mathcal{P}$ translate to relations amongst chromatic symmetric functions. This leads immediately to the proof of Theorem 1.1, which is directly related to the abelian case of the Stanley–Stembridge conjecture. In Section 4.3 we revisit that case, and attempt an understanding of how the two new formulae — those of Abreu and Nigro, and of Harada and Precup — can be related bijectively to the original work of Stanley [25].

2. Graded down-up algebra

2.1. **Some generalities.** A *path* P is any word $w \coloneqq w(P)$ in the alphabet $\{\mathsf{n}, \mathsf{e}\}$. Pictorially we depict it by reading w left to right and translating every instance of n (respectively e) as a unit north step (respectively east step) beginning at the origin. We denote the number of n's (respectively e's) by $|w|_{\mathsf{n}}$ (respectively $|w|_{\mathsf{e}}$). We let $\lambda \coloneqq \lambda(P)$ be the partition (in English notation) naturally determined by P in the top left corner of the $|w|_{\mathsf{n}} \times |w|_{\mathsf{e}}$ box. Alternatively, given any $\lambda \subset m \times n$, we may reverse this association to get a path $P \coloneqq P(\lambda)$ starting from $(0, 0)$ to (n, m), which in turn determines a word $w(\lambda)$ in $\{\mathsf{n}, \mathsf{e}\}$. Thus we have the following objects naturally in bijection:

$$\{\lambda \subseteq m \times n\} \leftrightarrow \{\text{paths } P \text{ from } (0, 0) \text{ to } (n, m)\} \leftrightarrow \{w \in \{\mathsf{n}, \mathsf{e}\}^{m+n} \text{ with } |w|_{\mathsf{n}} = m\}.$$

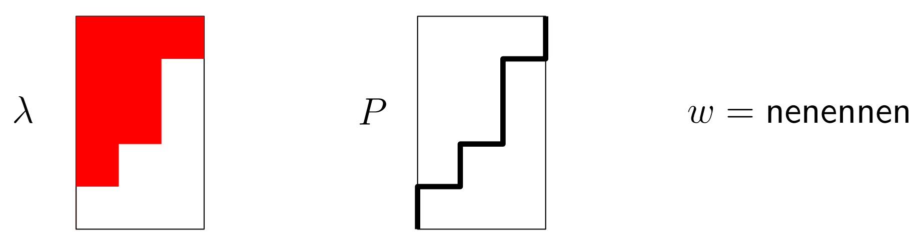

Figure 1. A partition, its associated path, and its associated word

Thus we can, and will, interchangeably use P, w, or λ if it is clear from context.

2.2. **Basic properties of $\mathcal{P}$.** Recall that $\mathcal{P}$ is the $\mathbb{C}(q)$-algebra generated by n and e subject to the modular relations (1) and (2). It turns out that these modular relations imply that $\mathcal{P}$ is an instance of a well-studied class of algebras called *down-up algebras*. These were introduced by Benkart and Roby [5, Section 2] inspired by Stanley's work on differential posets [24]. In the notation of *loc. cit.*, $\mathcal{P}$ is the down-up algebra $A(1 + q, -q, 0)$. At $q = 1$, this recovers the Weyl algebra. While the algebraic properties of down-up algebras have been thoroughly studied, that it encodes the modular law has hitherto not been noted, to the best of our knowledge.

By the *PBW theorem* for down-up algebras [5, Theorem 3.1], the set

$$\mathscr{S} = \{\mathsf{e}^a(\mathsf{ne})^b\mathsf{n}^c \mid a, b, c \in \mathbb{Z}_{\geq 0}\}$$

is a basis for $\mathcal{P}$. We refer to its elements as *staircase* monomials, and to $\mathscr{S}$ as the *staircase basis*. In Section 3.2, we explain how to expand an arbitrary element of $\mathcal{P}$ in this basis.

Observe that the modular relations preserve both the number of n's and e's, so we can endow $\mathcal{P}$ with an algebra bigrading:

$$\mathcal{P} = \bigoplus_{m,n \in \mathbb{Z}_{\geq 0}} \mathcal{P}_{m,n}, \tag{3}$$

where $\mathcal{P}_{m,n}$ is spanned by words w satisfying $|w|_{\mathsf{n}} = m$ and $|w|_{\mathsf{e}} = n$. In particular

$$\mathcal{P}^{\mathrm{diag}} := \bigoplus_{m\in\mathbb{Z}_{\geq 0}} \mathcal{P}_{m,m} \tag{4}$$

is a subalgebra of $\mathcal{P}$.

Involution η. Benkart and Roby [5, p. 329] consider the map η swapping n and e, and extend it to an algebra *anti*automorphism of the free associative algebra generated by n and e. Since the modular relations are preserved under this antiautomorphism, we get an involution η on $\mathcal{P}$. In terms of words, $\eta(w)$ corresponds to reversing w and switching n's for e's. Equivalently, it corresponds to the transposition of a partition $\lambda \subseteq m\times n$ to obtain $\eta(\lambda) = \lambda^t \subseteq n \times m$. Since η exchanges $\mathcal{P}_{m,n}$ and $\mathcal{P}_{m,n}$, we may work under the assumption that $m \leq n$ (or $n \leq m$) whenever convenient.

2.3. **The q-Klyachko algebra.** We give a brisk introduction to the q-Klyachko algebra covering the bare essentials and refer the reader to [19] for more details. The *q-Klyachko algebra* $\mathcal{K}$ is the commutative, graded $\mathbb{C}(q)$-algebra with generators $(u_i)_{i\in\mathbb{Z}}$ and quadratic relations

$$(1+q)u_i^2 = qu_iu_{i-1} + u_iu_{i+1} \tag{5}$$

for all $i \in \mathbb{Z}$. As the authors demonstrated in [19], $\mathcal{K}$ has intimate links with various subareas within algebraic combinatorics. The link to the algebra $\mathcal{P}$ unearthed in this article adds to these various connections.

If $c = (c_i)_{i\in\mathbb{Z}}$ is a sequence of nonnegative integers with finite support,[1] let $u^c := \prod_{i\in\mathbb{Z}} u_i^{c_i}$. In the particular case where the entries of c are 0s or 1s, we may identify c with its support $I \subset \mathbb{Z}$, and then let $u_I := \prod_{i\in I} u_i$. We let $\mathcal{B}$ denote the entire collection of such squarefree monomials u_I. By [19, Proposition 3.9], $\mathcal{B}$ is a basis for $\mathcal{K}$. We may thus decompose uniquely

$$u^c = \sum_I p_c(I) u_I,$$

where the sum is over subsets I of size $|c|$. Let $m = |c| := \sum_i c_i$. By homogeneity $p_c(I) = 0$ unless $|I| = m$. We define

$$A_c(q) = (m)_q! \times p_c(\{1,\ldots,m\}).$$

where $(m)_q! := \prod_{1\leq i\leq m}(i)_q$. Note that $A_c(q)$ is zero if the support of c is not contained in $\{1,\ldots,m\}$; so we can consider $c = (c_1,\ldots,c_m)$, and in this case $A_c(q)$ is a nonzero polynomial with nonnegative integer coefficients. These polynomials were introduced by the authors [19, Section 4.3] under the name *remixed Eulerian numbers.* We now explain how to connect them to q-hit numbers in a special case.

2.3.1. *q-hit numbers.* Consider a partition λ inside an $m \times n$ square. We give a combinatorial description for the q-hit numbers $H_k^{m,n}(\lambda)$ where $\lambda \subseteq m\times n$; these numbers were first introduced by Garsia and Remmel [9], the combinatorial interpretation given here is due to Dworkin [8].

Let $R(m,n,\lambda,k)$ denote the set of maximal nonattacking rook placements on an $m\times n$ board such that there are exactly k rooks inside λ. Given $p \in R(m,n,\lambda,k)$ we let $\mathrm{stat}(p)$ denote the number of *unattacked* cells in the $m \times n$ board. Unattacked cells are certain cells that do not contain rooks and are defined as follows. A cell in λ is *unattacked* if it does not lie below a rook,

[1]The support of c is the set of indices i such that $c_i > 0$.

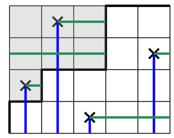

FIGURE 2. Unattacked cells in a maximal nonattacking rook placement on a 4×5 board

or to the right of a rook, or to the left of a rook outside λ. A cell outside λ is unattacked if it does not lie below a rook or to the right of a rook outside λ. This given, we have

$$H_k^{m,n}(\lambda) \coloneqq \sum_{p \in R(m,n,\lambda,k)} q^{\mathrm{stat}(p)}. \tag{6}$$

If $m = n$, we write $H_k^m(\lambda)$. It is straightforward to check that, assuming $m \geq n$, we have

$$(m-n)_q! \times H_k^{m,n}(\lambda) = H_k^m(\lambda). \tag{7}$$

See Figure 2 for a maximal nonattacking rook placement p where $m = 4$, $n = 5$ and $\lambda = (3,3,1,0)$. The six unattacked cells tell us that $q^{\mathrm{stat}(p)} = q^6$, which is the contribution of p to $H_2^{4,5}(\lambda)$.

Assume $m = n$ for the rest of this section, and fix $\lambda \subset m \times m$. Define the *area sequence* $\mathrm{a}(\lambda) \coloneqq (a_1, \ldots, a_m)$ by setting

$$a_i = i - \lambda_{m+1-i}.$$

As i runs from 1 to m, the a_i run from $1 - \lambda_m \leq 1$ to $m - \lambda_1 \geq 0$ with $a_{i+1} - a_i \in \{1, 0, -1, \ldots\}$. It follows that the set of entries underlying $\mathrm{a}(\lambda)$ is an interval containing 0 or 1. Note further that the multisets underlying $\mathrm{a}(\lambda)$ and $\mathrm{a}(\lambda^t)$ are equal — as may be seen by a standard pairing of north and east steps at the same height for instance.

Now consider the monomial $u(\lambda)$ in $\mathcal{K}$ defined as follows:

$$u(\lambda) \coloneqq \prod_{1 \leq i \leq m} u_{a_i}. \tag{8}$$

Example 2.1. Consider $\lambda = (5,5,3,3,3,0) \subset 6 \times 6$ as shown in Figure 3. We have $\mathrm{a}(\lambda) = (1,-1,0,1,0,1)$ and $\mathrm{a}(\lambda^t) = (1,0,1,-1,0,1)$. Additionally, $u(\lambda) = u_{-1}u_0^2u_1^3$.

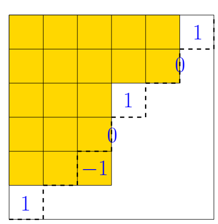

FIGURE 3. $\lambda = (5,5,3,3,3,0)$ inside a 6×6 board with $\mathrm{a}(\lambda) = (1,-1,0,1,0,1)$

By the properties of $\mathrm{a}(\lambda)$ given above, the expansion of $u(\lambda)$ in the basis $\mathcal{B}$ is as follows:

$$u(\lambda) = \sum_{k=0}^{m} c_k u_{[1-k, m-k]}. \tag{9}$$

As established in [20, § 4.2], the remixed Eulerian numbers occurring in this expansion are q-hit numbers:

$$u(\lambda) = \sum_{k=0}^{m} \frac{H_k^m(\lambda)}{(m)_q!} u_{[1-k,m-k]}. \tag{10}$$

3. Basis expansions in the algebra $\mathcal{P}$

We consider expansions of elements of $\mathcal{P}$ in three different bases. The first one is the rectangular basis considered by Guay-Paquet [12] for which our main result is Theorem 3.1. Its proof makes use of the q-Klyachko algebra introduced by the authors [19, 20]. In Section 4 we will obtain Theorem 1.1 as a corollary.

We give two other expansions: first, in the staircase basis $\mathscr{S}$, thus proving a conjecture of Colmenarejo, Morales and Panova [7]. Second, we give an explicit expansion in what we call the zigzag basis.

3.1. **Expansion in the rectangular basis.**

Given nonnegative integers $m \geq n$, define the set of *rectangular monomials* as follows:

$$\mathscr{R}_{m,n} = \{\mathsf{e}^k\mathsf{n}^m\mathsf{e}^{n-k} \mid 0 \leq k \leq n\}. \tag{11}$$

For $m < n$, we obtain $\mathscr{R}_{m,n}$ using η.

Our aim in this subsection is to expand any monomial w in $\mathcal{P}$ in terms of monomials in $\mathscr{R}_{m,n}$. The next result is essentially in [12] though not stated as such. The reader should compare this statement to Theorem 1.1: as we will see in Section 4.2, it will in fact imply it.

Theorem 3.1. *Fix nonnegative integers $m \geq n$. Let $\lambda \subset m \times n$, and consider the corresponding path $w := w(\lambda)$. Then in $\mathcal{P}_{m,n}$ we have*

$$w(\lambda) = \sum_{i=0}^{n} \frac{H_i^{m,n}(\lambda)}{(m)_q(m-1)_q\cdots(m-n+1)_q}\mathsf{e}^i\mathsf{n}^m\mathsf{e}^{n-i}. \tag{12}$$

Proof. We first consider the case $m = n$. Consider the map ψ_m

$$\begin{aligned}\psi_m : \mathcal{P}_{m,m} &\to \mathcal{K}\\ w(\lambda) &\mapsto u(\lambda),\end{aligned} \tag{13}$$

extended by linearity. We first claim that ψ_m is well defined, postponing the proof to Lemma 3.3.

Consider the $m+1$ *rectangular* monomials $\square_{i,m} = \mathsf{e}^i\mathsf{n}^m\mathsf{e}^{m-i}$. A direct computation shows $\psi_m(\square_{i,m}) = u_{[1-i,m-i]}$. It follows that the elements $\square_{i,m}$ are independent in $\mathcal{P}$ as their images are independent in $\mathcal{K}$. Now note that $\mathcal{P}_{m,m}$ has dimension $m+1$ [5, Theorem 3.1].

Thus the $\square_{i,m}$ for $0 \leq i \leq m$ give a basis[2] of $\mathcal{P}_{m,m}$, and if we write

$$w = \sum_{0\leq i\leq m} c_{\lambda,i}\,\square_{i,m}$$

[2]The $m+1$ staircase monomials $\delta_i := \mathsf{e}^{m-i}(\mathsf{ne})^i\mathsf{n}^{m-i}$ for $0 \leq i \leq m$ are easily seen to be spanning, so in fact both families are bases.

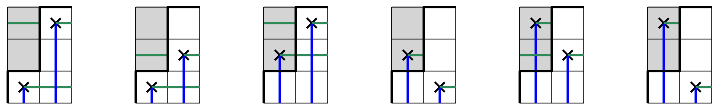

FIGURE 4. Nonattacking rook placements on 3×2 board

then by applying ψ_m we obtain

$$\psi_m(w) = u(\lambda) = \sum_{0 \le i \le m} c_{\lambda,i}\, u_{[1-i,m-i]}.$$

Comparison with (9), (10) implies the claim of the theorem for $m = n$.

Assume now that $m > n$, and consider $w' = w\mathsf{e}^{m-n}$. The corresponding partition λ' coincides with λ but sits inside an $m \times m$ square. We can thus compute the area sequence $a(\lambda')$, and notice that $1, \dots, m-n$ all occur in the sequence.[3] It follows that we can *a priori* restrict (9) to a smaller set of $n+1$ intervals (those containing $\{1, \dots, m-n\}$) and obtain

$$u(\lambda') = \sum_{i=0}^{n} \frac{H_i^m(\lambda)}{(m)_q!}\, u_{[1-i,m-i]} = \sum_{i=0}^{n} \frac{H_i^{m,n}(\lambda)}{(m)_q(m-1)_q \cdots (m-n+1)_q}\, u_{[1-i,m-i]}. \tag{14}$$

Then the rest of the proof follows the square case. The $n+1$ staircase monomials $\delta_{i,m,n} = \mathsf{e}^{n-i}(\mathsf{ne})^i\mathsf{n}^{m-i}$ are a basis of $\mathcal{P}_{m,n}$, so the $n+1$ rectangular monomials $\square_{i,m,n} = \mathsf{e}^i\mathsf{n}^m\mathsf{e}^{n-i}$ also form one since their images $\psi_m(\square_{i,m,n}\mathsf{e}^{m-n}) = u_{[1-i,m-i]}$ are independent in $\mathcal{K}$. We infer that (14) is the image by ψ_m of the rectangular basis expansion of $w(\lambda')$, and thus the coefficient of $\square_{k,m,n}$ in the expansion of $w(\lambda') = w(\lambda)\mathsf{e}^{m-n}$ is given by $\frac{H_k^{m,n}(\lambda)}{(m)_q(m-1)_q\cdots(m-n+1)_q}$. This is the content of (12) after multiplication by e^{m-n}. $\square$

Example 3.2. Let $w = \mathsf{nennee} \in \mathcal{P}_{3,2}$ and let $\lambda \coloneqq \lambda(w) = (1,1,0)$. Consider the six non-attacking rook placements on the 3×2 board in Figure 4. The leftmost two rook placements contribute to $H_0^{3,2}(\lambda)$ and the remaining to $H_1^{3,2}(\lambda)$. We thus get

$$H_0^{3,2}(\lambda) = q + q^2$$
$$H_1^{3,2}(\lambda) = 1 + q + q^2 + q^3.$$

Theorem 3.1 then says

$$\mathsf{nenne} = \frac{q+q^2}{(3)_q(2)_q}\,\mathsf{nnnee} + \frac{1+q+q^2+q^3}{(3)_q(2)_q}\,\mathsf{ennee}.$$

We now prove the lemma used in the proof of Theorem 3.1.

Lemma 3.3. *The map ψ_m is well-defined.*

Proof. We must verify that ψ_m is invariant under the modular relations (1), (2).

Let us start with relation (1). We need to show that, for any words w_1, w_2 such that $W \coloneqq w_1\mathsf{ene}w_2$ is in $\mathcal{P}_{m,m}$, the map ψ_m takes the same values on $(1+q)w_1\mathsf{ene}w_2$ and on $q\,w_1\mathsf{een}w_2 + w_1\mathsf{nee}w_2$. Let $(a_k)_k$ be the area sequence of $\lambda(W)$, and define i so that a_i corresponds to the

[3] this can be seen on the area sequence of the conjugate partition, which starts with $1, \dots, m-n$.

highlighted n in W (that is, w_1 has $i-1$ occurrences of n). The term $(1+q)W$ is sent via ψ_m to $(1+q)u_{a_i}\prod_{j\neq i}u_{a_j}$, while the other term is sent to $(q\,u_{a_i-1}+u_{a_i+1})\prod_{j\neq i}u_{a_j}$. To conclude with the q-Klyachko relation $(1+q)u_{a_i}^2=q\,u_{a_i}u_{a_i-1}+u_{a_i}u_{a_i+1}$, it suffices to show that $a_j=a_i$ for a $j\neq i$.

The factor ene in W tells us that $a_{i+1}\leq a_i\leq a_{i-1}$. If any of these inequalities is an equality we are done. Otherwise, if $a_i\leq 0$, then $a_{i+1}<a_i$, $a_m\geq 0$ and the fact that $a_{k+1}-a_k\leq 1$ for any k together imply that there exists $j>i+1$ such that $a_j=a_i$ as desired. If $a_i\geq 1$, consider $a'=(1-a_m,\ldots,1-a_1)$ so that $a'_{m+1-i}=1-a_i\leq 0$ and apply the same reasoning to show that there exists $j<i-1$ such that $a_j=a_i$.

The case of the relation (2), namely $(1+q)\mathsf{nen}=q\,\mathsf{enn}+\mathsf{nne}$ can be dealt with similarly. It is simpler, since the occurrence of nen implies that we have the needed $u_{a_i}^2$ in the image already. ☐

We will return to the consequences of Theorem 3.1 to chromatic symmetric functions in Section 4.

3.2. **The staircase basis.** Fix positive integers m and n. Define $\mathscr{S}_{m,n}\coloneqq\mathscr{S}\cap\mathcal{P}_{m,n}$. We know that $\mathscr{S}_{m,n}$ is a basis for $\mathcal{P}_{m,n}$. In this section we give an expansion for any monomial $w\in\mathcal{P}_{m,n}$ in this basis. Like before, we let $\delta_k\coloneqq\mathsf{e}^a(\mathsf{ne})^k\mathsf{n}^b$ where a,b are such that $\delta_k\in\mathscr{S}_{m,n}$.

We begin by stating our claim.

Theorem 3.4. *Let $w\in\mathcal{P}_{m,n}$ be a monomial, and $m_w\geq 0$ be the largest integer such that the path $P(\delta_{m_w})$ lies weakly below the path $P(w)$. In $\mathcal{P}$, consider the basis expansion*

$$w=\sum_{k\geq 0}(-1)^{m_w-k}c_{w,k}(q)\,\delta_k. \tag{15}$$

Then $c_{w,k}\in\mathbb{Z}_{\geq 0}[q]$ and vanishes unless $k\leq m_w$.

While there are in general many ways to employ the modular relations to express an arbitrary monomial w in terms of staircase monomials, we are guided by the aim that e's and $\mathsf{n}'s$ move to the left and right respectively, and in doing so, force a string of ne's in between. At the same time, we want the signs to behave nicely in a predictable manner. We will solely need the two relations

$$\mathsf{n}^i\mathsf{e}=(i)_q\,\mathsf{ne}\,\mathsf{n}^{i-1}-q(i-1)_q\,\mathsf{en}\,\mathsf{n}^{i-1}, \tag{16}$$

$$\mathsf{ne}^i=(i)_q\mathsf{e}^{i-1}\,\mathsf{ne}-q(i-1)_q\,\mathsf{e}^{i-1}\,\mathsf{en}. \tag{17}$$

These relations follow from the modular relations easily. The second one follows from the first by applying the involution η. Additionally, and crucially, observe that the coefficients involved are, up to a sign, polynomials in $\mathbb{Z}_{\geq 0}[q]$.

We state next our crucial definition that governs how the aforementioned relations apply in the course of our procedure.

Definition 3.5. Consider a factor w' in w where $w'=\mathsf{n}^i\mathsf{e}$ or $w'=\mathsf{ne}^i$ with $i\geq 2$ maximal. We say that w' is *critical* if $P(w)$ shares an edge with the path $P(\delta_{m_w})$ at one of the letters in w'.

Note that, by definition of m_w, the letter in the critical factor that corresponds to $P(\delta_{m_w})$ is necessarily the starting n if $w'=\mathsf{n}^i\mathsf{e}$, and the last e if $w'=\mathsf{ne}^i$.

Lemma 3.6. *Fix a word w in $\{\mathsf{n},\mathsf{e}\}$ The following are equivalent.*

(1) *w does not possess a critical factor.*
(2) *w corresponds to a staircase monomial.*

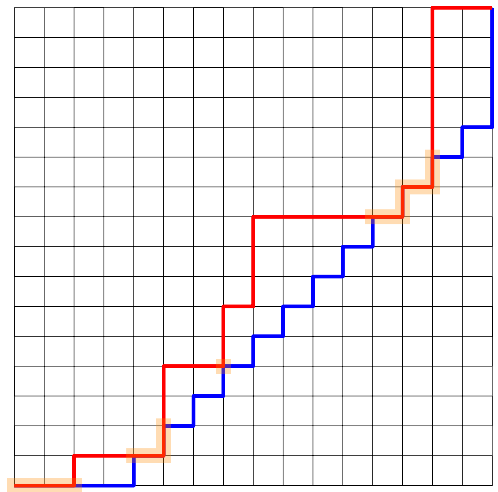

FIGURE 5. A path $P(w)$ (in red) and the associated $P(\delta_{m_w})$ (in blue). The subpaths where the two touch are highlighted.

Proof. It is immediate that monomials in $\mathscr{S}_{m,n}$ do not contain critical factors. Hence assume $w \notin \mathscr{S}_{m,n}$ and consider the path $P(\delta_{m_w})$. It agrees, i.e., shares an edge, with $P(w)$ at various junctures. At one of the two extremes (or both) of any maximal factor of agreement, there must be a critical factor for w. At the right extreme, this will be a factor of the form $\mathsf{n}^i\mathsf{e}$. At the left extreme this will be the transposed version, i.e., ne^i. □

The rewriting procedure: We now describe a rewriting procedure that takes as input any linear combination of words $C = \sum_w f_w w$.

(1) Pick w such that $f_w \neq 0$ and w possesses a critical factor. If no such w exists, the procedure terminates and outputs C.
(2) Pick any critical factor v in w. Modify C by replacing the critical factor v in w according to the relations (16), (17) applied from left to right. Go back to the first step.

In the second step of the procedure, let w_V, w_H be the two words that are obtained from a word w after applying the relations (16), (17). Here w_V comes with a positive weight $(i)_q$, while w_H comes with a negative weight $-q(i-1)_q$.

Figure 6 shows an execution of this algorithm for $w = \mathsf{nneeenne}$, representing naturally the rewriting procedure as a binary tree. We omitted the weights on the edges to keep the picture legible.

Proof of Theorem 3.4. First note that the rewriting procedure will necessarily end, as the shapes corresponding to the words are strictly increasing after each step of the procedure. It follows that the final output will be a linear combination of words with no critical factors, which represents the same element in $\mathcal{P}$ as the starting linear combination since we only apply relations that are valid in $\mathcal{P}$. By Lemma 3.6, this will indeed be the expansion in staircase monomials as desired.

Now a key remark is that, for any word w, and any of its critical factors, we have $m_{w_V} = m_w$ while $m_{w_H} = m_w - 1$, where w_H, w_V are defined above. Since $m_{\delta_k} = k$, any sequence of rewritings

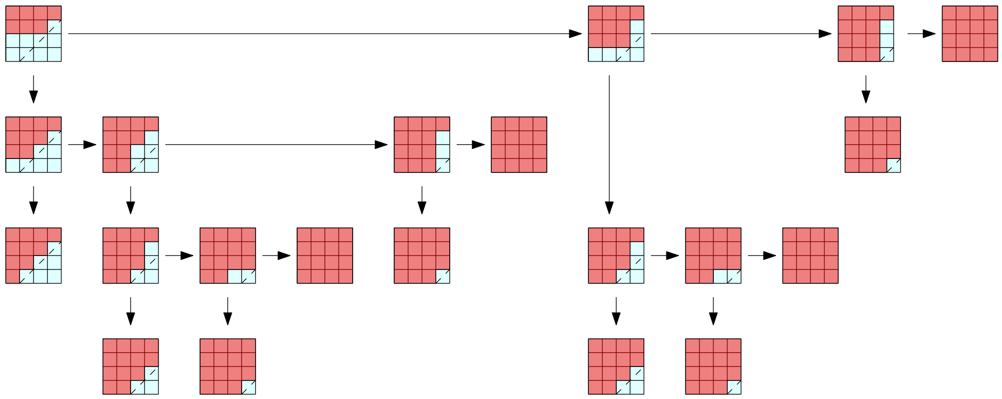

FIGURE 6. Rewriting algorithm applied to $w = \mathsf{nneeenen}$. Horizontal (respectively vertical) arrows represent $w \to w_H$ (respectively $w \to w_V$).

that goes from w to δ_k will then necessarily involve $m_w - k$ sign switches, as w_H comes with a negative weight while w_V has a positive weight.

It follows that the global sign of the coefficient of δ_k is $(-1)^{m_w-k}$, and thus that $c_{w,k} \in \mathbb{Z}_{\geq 0}[q]$. It is also immediate from the procedure that $c_{w,k} = 0$ if $k > m_w$. □

Example 3.7. Consider $w = \mathsf{nneeenen}$ as in Figure 6. There are exactly two paths from the root to a leaf representing $\delta_2 = \mathsf{eenenenn}$, both of which involve a single horizontal edge. By considering the weights for each path we conclude that the coefficient of δ_2 in w is

$$-q(2)_q(2)_q(1)_q - q(2)_q(1)_q(1)_q(1)_q = -q(1+q)(2+q).$$

Remark 3.8. Theorem 3.4 implies easily [7, Conjecture 6.6].[4] The staircase basis in [7, Section 6] corresponds to staircase paths in the top left corner. To expand in this basis, one needs to use the "reverse" rewriting rules, which are obtained from (16), (17) by reversing the words and changing q to q^{-1}:

$$\mathsf{e}^i\mathsf{n} = q^{1-i}(i)_q \mathsf{en}\, \mathsf{e}^{i-1} - q^{1-i}(i-1)_q\, \mathsf{ne}\, \mathsf{e}^{i-1}, \tag{18}$$

$$\mathsf{en}^i = q^{1-i}(i)_q \mathsf{n}^{i-1}\, \mathsf{en} - q^{1-i}(i-1)_q\, \mathsf{n}^{i-1}\, \mathsf{ne}. \tag{19}$$

This results in polynomials in q^{-1} for the coefficients, instead of the polynomials in q that we obtain in 3.4.

The proof in fact tells us a little bit more — we must have all $c_{w,k} \neq 0$ for $0 \leq k \leq m_w$. Moreover, since the only way to hit the staircase monomial δ_{m_w} is by applying moves $w \to w_V$ at all stages, we get an explicit description for c_{w,m_w} as a product of q-integers. For instance, for w in Figure 5, we have

$$c_{w,m_w} = (3)_q \;\; (2)_q(3)_q(2)_q(3)_q(3)_q(4)_q(5)_q \;\; (6)_q(5)_q.$$

It is easy to give a characterization of this product in terms of w. More generally, it would be interesting to find a combinatorial interpretation for all the coefficients $c_{w,k}$.

[4] Their conjecture is stated in terms of chromatic symmetric functions, but we explain in Section 4.2 why this can be expressed in the algebra $\mathcal{P}$.

3.3. **The zigzag basis.** For the purposes of this section, we set $\mathsf{s} := \mathsf{en}$ and $\mathsf{t} := \mathsf{ne}$.

Proposition 3.9. *$\mathcal{P}^{\mathrm{diag}}$ is isomorphic to the commutative polynomial algebra $\mathbb{C}(q)[\mathsf{s},\mathsf{t}]$.*

Proof. Recall map ψ_m defined in the proof of Theorem 3.1. Define the map ψ as the direct sum of the maps ψ_m, so that ψ has domain the subalgebra $\mathcal{P}^{\mathrm{diag}}$. Then ψ is an algebra homomorphism into $\mathcal{K}$ since $u(\lambda \cdot \mu) = u(\lambda)u(\mu)$ as is directly verified for any two shapes λ, μ inside squares.

We showed in the proof of Theorem 3.1 that ψ sends the basis $\square_{i,m}$, $0 \leq i \leq m$, of $\mathcal{P}^{\mathrm{diag}}$ to the independent vectors $u_{[1-i,m-i]}$ of $\mathcal{K}$. It is therefore injective with image spanned by all elements u_I where I is an interval containing 0 or 1 plus the identity element $u_\emptyset$. Equivalently, this image is the subalgebra of $\mathcal{K}$ generated by u_0 and u_1. It is free on these generators since the degree m subspace has maximal dimension $m+1$. Pulling the result back to $\mathcal{P}^{\mathrm{diag}}$ gives us the result since $\psi(\mathsf{s}) = u_0$ and $\psi(\mathsf{t}) = u_1$. □

Remark 3.10. Benkart and Roby [5, Proposition 3.5] establish that, for a general down-up algebra $A(\alpha,\beta,\gamma)$, the subalgebra A_0 (i.e., the analog of $\mathcal{P}^{\mathrm{diag}}$) is always a commutative subalgebra. The proof in *loc. cit.* is elementary albeit involved.[5] Kirkman, Musson and Passman [15] show that the subalgebra generated by ud and du in a general down-up algebra $A(\alpha,\beta,\gamma)$ over a field K is a polynomial algebra in those two generators provided that $\beta \neq 0$. Recalling that $\mathcal{P}$ is $A(1+q,-q,0)$, this gives another proof of Proposition 3.9.

We are thus naturally led to the question of expanding a monomial $w \in \mathcal{P}_{m,m}$ in terms of s and t. We consider a more general rectangular version. Fix nonnegative integers $m \geq n$. Consider the set of *zigzag* monomials defined as follows:

$$\mathscr{Z}_{m,n} = \{\mathsf{s}^a\mathsf{t}^{n-a}\mathsf{n}^{m-n} \mid 0 \leq a \leq n\} \tag{20}$$

For $m < n$, define $\mathscr{Z}_{m,n}$ by employing η. These zigzag monomials show up in [15, Section 2.1].

Fix a word $w \in \mathcal{P}_{m,n}$ with associated path and partition being P and λ respectively. Define the sequence $b(\lambda) = (b_1, \ldots, b_n)$ as follows:

$$b_i = \begin{cases} m+1-i-\lambda'_i, & \lambda_{m+1-i} < i,\\ i - \lambda_{m+1-i}, & \lambda_{m+1-i} \geq i. \end{cases}$$

Informally, the sequence $b(\lambda)$ measures the distance from the diagonal in the same vein as the area sequence $a(\lambda)$ from before. Figure 7 gives an example where $m = 11$ and $n = 9$. We either take the heights of the green shaded rectangle or the lengths of the red shaded rectangles. These capture the two cases that occur in the definition, and we get $b(\lambda) = (2, 1, -1, 0, 4, 5)$.

For $i \in \mathbb{Z}$ define $\mathbf{v}_i \in \mathcal{P}_{1,1}$ for $i \in \mathbb{Z}$ as follows:

$$\mathbf{v}_i = \begin{cases} (i)_q\,\mathsf{t} - q(i-1)_q\,\mathsf{s}, & i \geq 1,\\ q^i\left((1-i)_q\,\mathsf{s} - (-i)_q\,\mathsf{t}\right), & i \leq 0. \end{cases}$$

This choice will become transparent during the course of the following proof. Note that $\mathbf{v}_0 = \mathsf{s}$ and $\mathbf{v}_1 = \mathsf{t}$.

[5]The reader should note that the grading employed in [5] is not our bigrading, but a weaker one that can be defined for any down-up algebra.

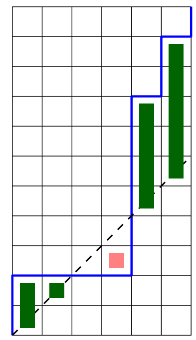

FIGURE 7. Illustration of the sequence $b(\lambda)$.

Proposition 3.11. *Fix a monomial $w \in \mathcal{P}_{m,n}$ where $m \geq n$. We have*

$$w = \mathbf{v}_{b_1} \cdots \mathbf{v}_{b_n} \cdot \mathsf{n}^{m-n}.$$

Proof. If $n = 0$, there is nothing to show as w must necessarily equal n^{m-n}. So we assume $n \geq 1$ and consider two cases.

Suppose $w = \mathsf{n}^i \mathsf{e} w'$ where $i \geq 1$. Then i must necessarily equal $m - \lambda'_1$, which is b_1. We thus have

$$w = \mathsf{n}^{b_1}\mathsf{e}w' = \Big((1-(b_1)_q)\mathsf{en}\cdot\mathsf{n}^{b_1-1} + (b_1)_q\mathsf{ne}\cdot\mathsf{n}^{b_1-1}\Big)\,w' = \mathbf{v}_{b_1}\cdot\mathsf{n}^{b_1-1}w'. \tag{21}$$

Now $\mathsf{n}^{b_1-1}P'$ is a word representing a path in a smaller bounding box, and we can proceed by induction.

On the other hand, if $w = \mathsf{e}^i\mathsf{n}w'$ with $i \geq 1$, we must have $i = \lambda_m = 1 - b_1$. Now mimicking the above argument we get

$$\begin{aligned} w = \mathsf{e}^{1-b_1}\mathsf{n}w' &= \big((1-(1-b_1)_{q^{-1}})\mathsf{ne} + (1-b_1)_{q^{-1}}\mathsf{en}\big)\,\mathsf{e}^{-b_1}w' \\ &= q^{b_1}((1-b_1)_q\mathsf{en} - (-b_1)_q\mathsf{ne})\mathsf{e}^{-b_1}w' = \mathbf{v}_{b_1}\cdot\mathsf{e}^{-b_1}w'. \end{aligned} \tag{22}$$

Again $\mathsf{e}^{-b_1}P'$ is a word representing a path in a smaller bounding box, and we can proceed by induction. □

We note that the $\mathbf{v}_{b_i}$ all commute, so the product can be written in various ways.

Example 3.12. Referring to Figure 7, we have $w = \mathsf{n}^2\mathsf{e}^4\mathsf{n}^6\mathsf{en}^2\mathsf{en}$. Noting that $b(\lambda) = (2, 1, -1, 0, 4, 5)$, we get

$$w = ((2)_q\,\mathsf{s} - q\,\mathsf{t})\cdot\mathsf{s}\cdot q^{-1}((2)_q\,\mathsf{t} - \mathsf{s})\cdot\mathsf{t}\cdot((4)_q\,\mathsf{s} - q(3)_q\,\mathsf{t})\cdot((5)_q\,\mathsf{s} - q(4)_q\,\mathsf{t})\cdot\mathsf{n}^{11-6}.$$

Theorem 3.13. *Consider the basis expansion in $\mathcal{P}$*

$$w = \sum_{0\leq r\leq n} c_{w,r}\,\mathsf{s}^r\mathsf{t}^{n-r}\mathsf{n}^{m-n}.$$

Then $c_{w,r}$ is a globally signed *Laurent polynomial, i.e., its nonzero coefficients are either all positive or all nonnegative.*

Proof. We extract the coefficient of $\mathsf{s}^r\mathsf{t}^{n-r}$ in $\mathbf{v}_{b_1}\cdots\mathbf{v}_{b_n}$. Let $S\subset\binom{[n]}{r}$. Define c_S as

$$c_S := \prod_{\substack{i\in S\\ b_i\geq 1}}(-q(b_i-1)_q)\prod_{\substack{i\in S\\ b_i\leq 0}}\left(q^{b_i}(1-b_i)_q\right)\prod_{\substack{i\notin S\\ b_i\geq 1}}(b_i)_q\prod_{\substack{i\notin S\\ b_i\leq 0}}\left(-q^{b_i}(-b_i)_q\right)$$

Now, c_S is the coefficient that appears as one scans $\mathbf{v}_{b_1}\cdots\mathbf{v}_{b_n}$ left to right and picks up the coefficient of s if $i\in S$, and that of t if $i\notin S$. It follows from Proposition 3.11 that

$$c_{w,r}=\sum_{S\in\binom{[n]}{r}}c_S,$$

For an S to contribute to this expression, we must necessarily have all i for which $b_i=0$ belong to S, and all i for which $b_i=1$ belong to $[n]\setminus S$. If these constraints are not satisfied, then $c_S=0$.

Assuming these constraints are met, we claim that the sign of c_S only depends on $|S|$ and w. Indeed, the exponent of -1 is the number of $i\in S$ with $b_i>1$ plus the number of $i\notin S$ with $b_i<0$. We leave it to the reader to verify that this quantity has the same parity as $|S|$ plus the number of i with $b_i\leq 0$. The claim follows. □

Remark 3.14. If $\lambda\subseteq m\times m$, then it is seen that $b(\lambda)$ is a rearrangement of the area sequence $a(\lambda)$ introduced in Section 2.3.1. So the form of Theorem 3.13 simplifies in the square case.

4. The abelian case of the Stanley–Stembridge conjecture

We relate here the algebra $\mathcal{P}$ to chromatic symmetric functions, following Guay-Paquet [12]. To keep our exposition brief, we refer the reader to [26, Chapter 7] for any undefined notions pertaining to the ring QSym of quasisymmetric functions, and its distinguished subring Sym of symmetric functions. Given a strong composition α, we let M_α and F_α denote the corresponding *monomial* and *fundamental* quasisymmetric functions respectively.

4.1. **Chromatic quasisymmetric functions.** Consider a graph $G=([n],E)$. A *coloring* κ of G is an attribution of a *color* in $\mathbb{Z}_+=\{1,2,\ldots\}$ to each vertex of G; it is *proper* if $\kappa(i)\neq\kappa(j)$ whenever $\{i,j\}\in E$. An *ascent* (respectively *descent*) of a coloring κ is an edge $\{i<j\}\in E$ such that $\kappa(i)<\kappa(j)$ (respectively $\kappa(i)>\kappa(j)$). Denote the number of ascents (respectively descents) by $\operatorname{asc}(\kappa)$ (respectively $\operatorname{des}(\kappa)$).

The *chromatic quasisymmetric function* of G [23] is the generating function of proper colorings weighted by ascents:

$$X_G(x,q)=\sum_{\kappa:V\to\mathbb{Z}_+\text{proper}}q^{\operatorname{asc}(\kappa)}x_{\kappa(1)}x_{\kappa(2)}\ldots x_{\kappa(n)}. \tag{23}$$

It is clearly in QSym, homogeneous of degree n. The chromatic symmetric function is $X_G(x,1)$ and was originally defined by Stanley [25].

Letting ρ be the linear involution ρ on QSym defined by sending $M_{\alpha_1,\ldots,\alpha_k}$ to $M_{\alpha_k,\ldots,\alpha_1}$, one has

$$\sum_{\kappa:V\to\mathbb{Z}_+\text{proper}}q^{\operatorname{des}(\kappa)}x_{\kappa(1)}x_{\kappa(2)}\ldots x_{\kappa(n)}=q^{|E|}X_G(x,q^{-1})=\rho(X_G).$$

Since ρ leaves Sym stable, we can use descents or ascents indifferently in the definition of X_G when it happens to be symmetric, which is precisely the case we will be interested in.

As mentioned in the introduction, a particular class of graphs of interest to us are *Dyck graphs*.

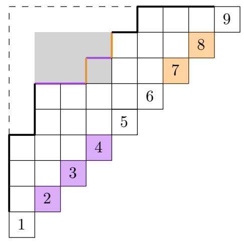

FIGURE 8. A Dyck path with an abelian subpath in bold and the abelian rectangle highlighted.

Definition 4.1. A simple graph $G = ([n], E)$ is a Dyck graph if for any $\{i < j\} \in E$, then $\{i' < j'\} \in E$ for all $i \leq i' < j' \leq j$.

Dyck graphs arise as incomparability graphs of natural unit interval orders; we will have no need for this description. A Dyck path D uniquely determines a Dyck graph; given all the ways to index Dyck paths, we inherit various ways to index Dyck graphs, which we will employ.

Proposition 4.2 ([23])**.** *For G a Dyck graph, $X_G(x,q)$ is a symmetric function.*

4.2. **Guay-Paquet's rectangular formula.** Let G be a Dyck graph on $[n]$ corresponding to Dyck path D. Let $I = \{i-a+1, \ldots, i\}$, $J = \{j, j+1, \ldots, j+b-1\}$ with $i < j$ be subsets of $[n]$ such that $(i-a+1, j-1) \in E$ and $(i+1, j+b-1) \in E$. This forms an "abelian rectangle" $[i-a+1, i] \times [j, j+b-1]$. In terms of paths, this abelian rectangle corresponds to a certain "abelian" subpath of D with a north steps and b east steps. We refer the reader to [13, Section 3] for an explanation for why the term abelian.

Figure 8 depicts a Dyck path D. The labeled squares along the diagonal give the vertex set of the associated Dyck graph. Edges are given by squares below the path and above the diagonal squares. In this example, we have $I = \{2,3,4\}$ and $J = \{7,8\}$, and the resulting abelian rectangle $I \times J$ in gray. The subpath of D in this shaded region gives the abelian subpath.

The modular law [10] says that, if a subpath ene or nen is part of an abelian subpath, then

$$(1+q)X_{U\mathsf{ene}V} = qX_{U\mathsf{een}V} + X_{U\mathsf{nee}V}, \tag{24}$$

$$(1+q)X_{U\mathsf{nen}V} = qX_{U\mathsf{enn}V} + X_{U\mathsf{nne}V}. \tag{25}$$

Fix U and W, and consider the set of all Dyck paths UvW where v is the abelian subpath. Let us assume that v has a north steps and b east steps, so that the abelian rectangle has dimensions $a \times b$. We denote the $\mathbb{C}(q)$-linear span of the chromatic symmetric functions X_{UvW} by $\mathcal{X}_{a,b}$.

Consider the map on the $\mathbb{C}(q)$-linear span of words with a n's and b e's, with image in $\mathcal{X}_{a,b}$, defined by

$$v \mapsto X_{UvW} \tag{26}$$

and extended linearly. Comparing the relations (1), (2) of $\mathcal{P}$ and the modular laws (24), we have in fact a map defined on $\mathcal{P}_{a,b}$. We can thus apply this map to the relation in Theorem 3.1, and this gives precisely Theorem 1.1.

4.3. **The Stanley–Stembridge conjecture.** The Stanley–Stembridge conjecture [25, 27] asserts that, if G is a Dyck graph, then the chromatic symmetric function $X_G|_{q=1}$ has a positive expansion in the e_λ basis. Shareshian and Wachs [23] then refined it by conjecturing that the e-expansion of X_G had coefficients that are polynomials in q with nonnegative coefficients. Writing

$$X_G = \sum_\lambda c_\lambda^G e_\lambda, \tag{27}$$

the conjecture says the following.

Conjecture 4.3. *For any Dyck graph G and any partition λ, the coefficient c_λ^G is in $\mathbb{N}[q]$.*

Recall that an *acyclic orientation* A of a graph G is an orientation of its edges such that the resulting directed graph has no directed cycles. Assuming $V(G) = [n]$, an *ascent* of A is an edge $i \to j$ with $1 \le i < j \le n$. If G is nonempty, then A has at least one *source*, i.e., a vertex with no incoming edge.

The following theorem was proved in [23, Theorem 5.3], the case $q = 1$ being already known to Stanley [25, Theorem 3.3].

Theorem 4.4. *For any Dyck graph G and any $k \ge 1$, the sum of c_λ^G over all partitions with k parts is the number of acyclic orientations of G with k sources, counted with weight $q^{\#\text{ascents of } A}$.*

In particular the sum over all λ of c_λ^G is enumerated by acyclic orientations of G.

4.4. **The abelian case.** One case has been particularly studied and proved in different ways: the so-called *abelian case*. In the language of Section 4.2, this is when the Dyck graph G on n vertices has an associated abelian rectangle of maximal size $a \times b$ with $a + b = n$. In terms of Dyck paths, it means that the number of initial n's plus the number of final e's is $\ge n$; equivalently, the associated shape $\lambda = \lambda(G)$ satisfies $\lambda_1 + \ell(\lambda) \le n$.

In this section we will record, and comment on two known e-expansions of X_G when G is abelian.

The *source sequence* $\mathrm{ss}(A) = (m_1, \dots, m_k)$ of A is defined recursively as follows: if S_1 is the set of sources of A, then $m_1 = |S_1|$ and $(m_2, \dots, m_k)$ is the source sequence of the acyclic orientation obtained by restricting A to $G \setminus S_1$.

Let G be an abelian Dyck graph with $\lambda = \lambda(G)$. Let $(a_1, \dots, a_n)$ be its ascent sequence. We also assume $\lambda_1 \ge \ell = \ell(\lambda)$ without loss of generality. Since the vertices of G can be partitioned in two cliques, acyclic orientations can have at most two sources. The expansion of X_G thus only involves partitions with at most two parts.

4.4.1. *The formulas of Stanley and Harada and Precup.* Harada and Precup [13, Theorem 1.1] gave a proof of Conjecture 4.3. They used the celebrated work of Brosnan and Chow [6] that showed the connection of X_G for any Dyck graph G with the study of Hessenberg varieties. Their result can be readily formulated as follows:

$$X_G = |\mathrm{Acy}_1^q(G)|\, e_n + \sum_{\{i<j\} \notin E} q^{a_i + a_j} X_{G\setminus\{i,j\}}^{+(1,1)}, \tag{28}$$

where $\mathrm{Acy}_1^q(G)$ is the set of acyclic orientations of G with one source, counted according to ascents; and for a symmetric function $f = \sum c_\mu e_\mu$, then $f^{+(a,b,\dots)} := \sum_\mu c_\mu e_{\mu_1+a, \mu_2+b, \dots}$.

Now by iterating the previous equation one easily obtains

$$X_G = \sum_A q^{\#\text{ ascents of } A}\, e_{n-\mathrm{in}(A),\mathrm{in}(A)}, \tag{29}$$

where $\mathrm{in}(A)$ is the length of the run of 2's at the beginning of $\mathrm{ss}(A)$. The case $q = 1$ is due to Stanley in his original paper [25, Theorem 3.4 and Corollary 3.6]. In fact, Stanley's proof can be extended to include q and thus prove (29), which thus gives an independent proof of the result of Harada and Precup.

4.4.2. *The formula of Abreu and Nigro.* A second proof was given by Abreu and Nigro [1, Theorem 1.3]. Their result can be stated as follows:

$$X_G = \sum_{j=0}^{\ell} q^j (j)_q!(n-2j)_q H_j^{n-j-1}(\lambda)\, e_{n-j,j}. \tag{30}$$

Note that we slightly simplified their formula: the coefficient of $e_{n-\ell,\ell}$ in (30) is given in [1] as $(\ell)_q! H_\ell^{n-\ell}(\lambda)$. Let us explain why they coincide: after simplifying by $(\ell)_q!$, this reduces readily to the identity

$$H_\ell^{n-\ell}(\lambda) = q^\ell (n-2\ell)_q H_\ell^{n-\ell-1}(\lambda). \tag{31}$$

Sketch of the proof of (31). Write $N = n-\ell$. Fix a maximal rook configuration C in $R(N-1, N-1, \lambda, \ell)$. Note that, since $\ell = \ell(\lambda)$, all rooks in the top ℓ rows are necessarily inside λ, say in columns $J = \{j_1, \dots, j_\ell\}$. One can extend C to a configuration C' in $R(N, N, \lambda, \ell)$ by inserting a rook in the bottom row in one of the $N-\ell$ columns $[N-1] \setminus J \cup \{N\}$. Tracking the new unattacked cells gives us the coefficient $q^\ell (N-\ell)_q$: there are ℓ new unattacked cells in the top ℓ positions of the last column of C', while $(N-\ell)_q$ comes from the inversions created by the insertion in the last row. □

4.4.3. *Comparison.* It is certainly interesting to connect directly (30) to (28), (29). More precisely, equating the two implies the following result.

Proposition 4.5. *Let G be an abelian Dyck graph. Then the number of acyclic orientations A with $\mathrm{in}(A) = j$ and with weight $q^{\#\text{ ascents of } A}$ is given by $q^j (j)_q!(n-2j)_q H_j^{n-j-1}(\lambda)$.*

Let us sketch a direct bijective proof for $q = 1$: let A be an acyclic orientation with $\mathrm{in}(A) = j$. Let $S = (\{u_1 < v_1\}, \{u_2 < v_2\}, \dots, \{u_j < v_j\})$ be the first j sets in the source sequence decomposition of A. Denote by V the set containing these $2j$ vertices. The orientation A is then entirely characterized by S together with an acyclic orientation A_1 of $G \setminus V$ that has a unique source by the definition of $\mathrm{in}(A)$. Recall that the cells of $\lambda = \lambda(G)$ are in bijection with the non-edges of G. From this it follows that the vertices of V can be represented by j non-attacking roots in the shape λ, and they can be ordered in $j!$ ways. Let $\lambda' \subset (n-2j) \times (n-2j)$ be the shape corresponding to $G \setminus V$: it is obtained by removing the columns and rows occupied by the rooks in λ. Now the number of acyclic orientations of $G \setminus V$ with a unique source is given by $(n-2j)$ times the number $H_0^{n-2j-1}(\lambda')$, and this can be proved bijectively [3, § 9.1].

Putting things together, we get a 1-to-$j!(n-2j)$ map between acyclic orientations of G with $\mathrm{in}(A) = j$, and pairs of rook placements in $R(j, \ell, \lambda, j) \times R(n-2\ell-1, n-2\ell-1, \lambda', 0)$ with λ' as above. These two rook placements can be naturally combined to give a rook placement in $R(n-\ell-1, n-\ell-1, \lambda, j)$, which completes the bijective proof.

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