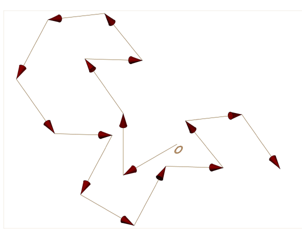
Séminaire Lotharingien de Combinatoire **87** (2023), Article B87c

ON THE PROBLEM OF RANDOM FLIGHTS IN ODD DIMENSIONS

ALEXANDER KOVAČEC AND PEDRO BARATA DE TOVAR SÁ

Abstract. In the first part of this paper we give a procedure to compute the exact probability for a particle starting from the origin of an odd-dimensional Euclidean space to be encountered within a distance r from the start after n random jumps of unit length. In the second part we use a combinatorial identity to deduce for integers $m \geq 0$ and a certain large family of integers $l \geq 0$, detailed information concerning the primitives $\int x^{l-2m}((-1+x+s)(1-x+s)(1+x-s)(1+x+s))^m dx$. This will imply that the density function associated with this random flight problem is piecewise polynomial. The approach is significantly different from the one chosen by García-Pelayo [*J. Math. Phys.* **53** (2012), 103504] who used advanced analytical tools.

0. Introduction



To motivate the investigation, we begin with a simple question. Assume a particle, at instant 0 at the origin of three dimensional Euclidean space jumps at each tick of the clock exactly one unit from its current position into a random direction. (Here the directions are defined as position vectors to uniformly distributed points of the origin-centred unit sphere.)

Question: What is — as a function of r — the probability to encounter the particle after exactly n random jumps within the 0-centred ball $B = B(0, r)$ of radius r?

Thus if R_n is the distance of the particle from the starting place after n steps viewed as a random variable, we want to know the probability distribution $r \mapsto \text{Prob}(R_n \leq r)$ or, equivalently, the density function $r \mapsto f_{R_n}(r)$.

For this particular problem an elementary proof for an elementary result — piecewise polynomiality of f_{R_n} — was given by Treloar in 1945 and solutions which use Fourier transforms and discontinuous factors are also known; a further elementary proof was added by the present authors in [SK] where some of the history and ideas of the other solutions are also explained and many references cited. Later we had access to the 1995 book by B. D. Hughes [H] which has in Chapter 2 a very extensive history on these problems (not completely coincident with the findings in [SK]). For a history up till 1985, see [D]. The problem can of course be reasonably asked for Euclidean spaces of arbitrary dimension and actually would make sense, suitably formulated, for many Riemannian manifolds and even for a given Euclidean space interesting variants can be formulated, but the possibility of elementary or even 'closed form' solutions will be the exception.

In 2012 García-Pelayo [G-P] showed that the direct generalisation of the problem to odd dimensional Euclidean space also leads to piecewise polynomial solutions for the functions f_{R_n}. For uniform random flights in odd dimensional spaces, it seems to have been the first advance over Treloar' s result. Curiously, the problem in even dimensional spaces does

Key words and phrases. Random flights, odd dimensions, piecewise polynomial, combinatorial identity.

not seem to admit exact elementary solutions. Solutions using integrals over expressions with Bessel functions and numerical approximations can be given; see [G-P], [H], or [D].

The article [G-P] uses the Fourier transform, a result of Kingman of 1963 on the behaviour of convolutions under projections; and a generalisation of the Abel transform which is known as a tool to analyse radial (i.e., spherical symmetrical) functions. García-Pelayo stopped short of giving an explicit formula. This was remedied by Borwein and Sinnamon [BS] who based their arguments on García-Pelayo's main result and who gave a formula valid for all n and odd dimensions d.

In this paper we propose an elementary proof for the García-Pelayo result. We need for our reasoning only standard formulas for the area of the spherical cap in dimension d, formulas from elementary probability theory, and a combinatorial identity.

What concerns the organisation of the paper, it consists of eight sections, often short, which we thought better to leave untitled. A short description follows below.

Section 1 (referred to as §1) expresses the distance after $n+1$ jumps as a function in the distance after n jumps; §2 determines the probability for a point $\vec{Z}$ randomly chosen from the hypersphere $\mathbb{S}^{d-1}$ to lie in a cap of height h centred at the north pole. A formula for the density f_Z of the random variable Z (being the last coordinate of $\vec{Z}$) is written down. In §3 we give a general formula for expressing the density function $f_{(R,S)}$ of a joint distribution (R, S) where $S = g(R, Z)$ in terms of f_R, f_Z, and a differentiable g satisfying some mild conditions and work it out in detail in §4 for the case $g(r, z) = \sqrt{r^2 + 2rz + 1}$ (which is indeed the prominent formula of §1). In §5 we give a recurrence expressing $f_{R_{n+1}}$ via an integral over a function in which f_{R_n} occurs, and do the same for functions b_n which are related to f_{R_n} in a very simple way. It can in principle be used for a short code that allows e.g. *Mathematica*© to compute the f_{R_n}; but in §6 we give a piecewise representation for the functions b_n leading to an idea for faster computations; and more importantly we show in §8 that the pieces of the b_n are polynomial functions hereby also establishing the correctness of the algorithm. We prepare for this in §7 by reasoning that needs a combinatorial identity. This is used to analyse the primitives

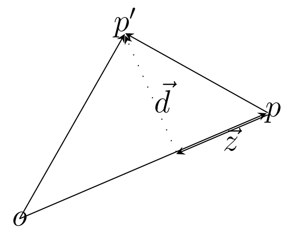
$$\int x^{l-2m}\left((-1+x+s)(1-x+s)(1+x-s)(1+x+s)\right)^m dx$$

for many nonnegative integers l, m and associated definite integrals.

References like 'Theorem 6' will mean the (unique) theorem in §6.

1. We begin with the following lemma, which can be found already in [SK] formulated for 3-space, but is valid in any Euclidean d-space $(E, \langle\,,\,\rangle)$. The 2-norm associated to the usual inner product $\langle\,,\,\rangle$ is denoted $|\cdot|$. Positions p are identified with the vector $\vec{op}$.

Lemma. *Let p and p' be two points at distance 1 in Euclidean d-dimensional space $(E, \langle\,,\,\rangle)$ with origin denoted o. Let $\vec{z}$ be the orthogonal projection of vector $\vec{pp'}$ onto vector $\vec{op}$. Then $|\vec{p'}| = \sqrt{1 + |\vec{p}|^2 + 2\langle\vec{p}, \vec{z}\rangle}$.*



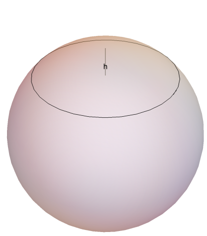
Proof. We may write $p' = p + \vec{pp'} = p + (\vec{z} + \vec{d})$ where $\vec{d} \perp \vec{z}$.It follows that

$$\begin{aligned}
|\vec{p'}|^2 &= \langle \vec{p'}, \vec{p'} \rangle \\
&= \langle \vec{p} + \vec{z} + \vec{d}, \vec{p} + \vec{z} + \vec{d} \rangle \\
&= |\vec{p}|^2 + |\vec{z}|^2 + |\vec{d}|^2 + 2\langle \vec{p}, \vec{z} \rangle + 2\langle \vec{p}, \vec{d} \rangle + 2\langle \vec{z}, \vec{d} \rangle \\
&= |\vec{p}|^2 + 1 + 2\langle \vec{p}, \vec{z} \rangle + 0 + 0
\end{aligned}$$

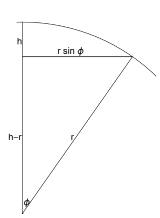
Here we used Pythagoras' theorem, and the parallelism or antiparallelism of $\vec{p}$ with $\vec{z}$ and the perpendicularity of $\vec{d}$ with respect to the latter two vectors. □

We will use this later in the following way: given n, the particle is after n jumps at a certain position p; its distance from the origin $|p|$ is a random variable $R = R_n$. We can write $\langle \vec{p}, \vec{z} \rangle = |p| \langle \frac{\vec{p}}{|p|}, \vec{z} \rangle$. The $n+1$-st jump is also a random variable and hence the inner product $\langle \frac{\vec{p}}{|p|}, \vec{z} \rangle$ is the random variable which expresses the length of the projection of the jump onto the line op. We designate this random variable by Z. We will have to consider, thus, the random variable $\sqrt{R^2 + 2RZ + 1}$.

2. In this section we aim to find the density function of the random variable Z referred to above.



If $\vec{Z}$ is a d-dimensional random vector with uniform distribution over othe unit sphere $\mathbb{S}^{d-1}$, its d-th component defines a real random variable Z. It is clear that for reasons of symmetry, the random variable Z so defined is the same (i.e., has the same distribution) as the random variable Z defined in Section 1. We deduce the functions f_Z as the density functions associated with the distribution function $\mathrm{Prob}(Z \le z)$.



Let $\mathbb{S}^{d-1} \subseteq \mathbb{R}^d$ be the unit sphere in $\mathbb{R}^d$. A standard formula is that $A_d(r) := \mathrm{area}(r\mathbb{S}^{d-1}) = \frac{2\pi^{d/2}}{\Gamma(d/2)} r^{d-1}$. In [L] we find a simple proof for the fact that the area of the hyperspherical cap defined by the colatitude angle ϕ can via the incomplete regularised beta function $I_z(a, b)$ be expressed by

$$A_d(r, \phi) = \tfrac{1}{2} A_d(r) I_{\sin^2 \phi}(\tfrac{d-1}{2}, \tfrac{1}{2}).$$

The figure at the left shows by an application of Pythagoras' theorem, that the height h of the cap and its colatitude are related by $\sin^2 \phi = \frac{2hr - h^2}{r^2}$.

Claim. As a function of height h, the area $A(h)$ of the hyperspherical cap in $\mathbb{S}^{d-1}$ is given via

$$G_{d-2}(1-h) = \int_0^{1-h} (1 - t^2)^{\frac{d-3}{2}}\, dt$$

by

$$A(h) = A_d(1) \cdot \frac{1}{2}\left(1 - \frac{G_{d-2}(1-h)}{G_{d-2}(1)}\right).$$

$\triangleright$ Putting $r = 1$ in $A_d(r, \phi)$ and using the definition of the regularised incomplete beta function we have to prove

$$I_{2h-h^2}(\tfrac{n-1}{2}, \tfrac{1}{2}) = 1 - \frac{G_{n-2}(1-h)}{G_{n-2}(1)},$$

that is

$$\frac{\int_0^{2h-h^2} t^{\frac{n-3}{2}}(1-t)^{-1/2}dt}{\int_0^1 t^{\frac{n-3}{2}}(1-t)^{-1/2}dt} = \frac{\int_{1-h}^1 (1-t^2)^{\frac{n-3}{2}}dt}{\int_0^1 (1-t^2)^{\frac{n-3}{2}}dt}.$$

To see this, use the substitution rule with $u = 1 - t^2$ on the right-hand side integrals. $\triangleleft$

Note: The formula for $A(h)$ is attributed by Wikipedia [Wi] to a 1986 paper by Chudnov. But in fact it cannot be found there.

Since $A_d(1)$ is the area of $\mathbb{S}^{d-1}$ we see that the probability that $\vec{Z}$ falls into the spherical cap of height $h = 1 - z$ is $A(h)/A_d(1)$, so

$$\mathrm{Prob}(Z \le z) = 1 - \mathrm{Prob}(Z \ge z) = 1 - \frac{A(h)}{A_d(1)} = \frac{1}{2}\left(1 + \frac{G_{d-2}(z)}{G_{d-2}(1)}\right),$$

and therefore, for $-1 < z < 1$, we have

$$f_Z(z) = \frac{d}{dz}\mathrm{Prob}(Z \le z) = \frac{1}{2G_{d-2}(1)}\frac{d}{dz}\int\limits_0^z (1-t^2)^{\frac{d-3}{2}}dt = \frac{1}{2G_{d-2}(1)}(1-z^2)^{\frac{d-3}{2}}.$$

For $z \notin [-1, 1]$ it is clear that $f_Z(z) = 0$. It follows that a globally valid formula for the density f_Z is given by

$$f_Z(z) = \frac{1}{2G_{d-2}(1)}(1-z^2)^{\frac{d-3}{2}}\,\mathbb{1}_{[-1,1[}(z), \quad z \in \mathbb{R}.$$

Note: Here and in many other cases later the arguments are indifferent to whether we take half open intervals like $[-1, 1[$ or closed ones like $[-1, 1]$. Sometimes however, like in the representation theorem of Section 7, we have to choose half open ones. We will use left closed, right open intervals wherever possible.

Thanks to an observation of C. Krattenthaler, we can give a formula for $G_d(1)$. Recall that for the double factorial one has $n!! = n(n-2)\cdots(n-2\lfloor\frac{n-1}{2}\rfloor)$, provided $n \in \mathbb{Z}_{\ge 1}$, and $0!! = 1$.

Claim. Letting the odd $d = 2k + 1$, we have

$$G_d(1) = \frac{4^k k!^2}{(2k+1)!} = \frac{(2k)!!}{(2k+1)!!}.$$

$\triangleright$ Using in the following computation successively the definition of G_d, the substitution $t^2 = u$, the definition of the beta function and its well known relation with the gamma function, we have

$$G_d(1) = \int_0^1 (1-t^2)^{\frac{d-1}{2}} = \frac{1}{2}\int_0^1 (1-u)^{\frac{d-1}{2}} u^{-1/2} du$$
$$= \frac{1}{2}B(\tfrac{1}{2}, \tfrac{d+1}{2}) = \frac{1}{2}\frac{\Gamma(1/2)\Gamma(\frac{d+1}{2})}{\Gamma(\frac{d+2}{2})} = \frac{1}{2}\frac{\Gamma(1/2)\Gamma(k+1)}{\Gamma(\frac{2k+3}{2})}.$$

The numerator of this expression is $\sqrt{\pi}\,k!$, while by iterated use of $\Gamma(x+1) = x\Gamma(x)$, we find for the denominator, that

$$\Gamma(\tfrac{2k+3}{2}) = \tfrac{2k+1}{2}\cdot\tfrac{2k-1}{2}\cdots\tfrac{1}{2}\Gamma(\tfrac{1}{2}) = \sqrt{\pi}\frac{(2k+1)!}{2^{k+1}2\cdot 4\cdots(2k)} = \sqrt{\pi}\frac{(2k+1)!}{2\cdot k!\,4^k}.$$

The equality with the other formula given is easy to see. This yields the claim. ⊣

3. We explain in §5 how to find $f_{R_n}(r)$ by repeated marginalisation. As a preparation we deal here with the following question. Assume R, Z are real random variables with known density functions and g is differentiable on an open set containing the range of the random vector (R, Z). Then $S = g(R, Z)$ is a further real random variable and by $f_{(R,S)}$ is denoted the density function of the joint variable (R, S). How to find $f_{(R,S)}$ from f_R? We answer this under mild conditions for g and f_R as implicit in the following reasoning.

The probability that $r - h \le R < r + h$ holds is given by the definition of density functions, by $\int_{r-h}^{r+h} f_R(t)dt$, and hence assuming continuity of f_R, there is a $t_1 = t_1(h)$ such that $\mathrm{Prob}[r - h \le R < r + h] = 2hf_R(t_1)$.

Now take a $\dot r \in [r-h, r+h[$. Then $g_{\dot r}(Z) := g(\dot r, Z)$ is a real valued random variable and $z \mapsto g_{\dot r}(z)$ a differentiable function. We assume furthermore that $g_{\dot r}$ is strictly monotone in an interval containing the range of Z. By the discussion in [B, p. 264], the density of the random variable $Y = g_{\dot r}(Z)$ is then given by $f_{g_{\dot r}(Z)}(y) = f_Z(g_{\dot r}^{-1}(y))|(g_{\dot r}^{-1})'(y)|$. So the probability that $g_{\dot r}(Z)$ assumes values in $[s-h, s+h[$ is given by $\int_{s-h}^{s+h} f_Z(g_{\dot r}^{-1}(y))|(g_{\dot r}^{-1})'(y)|\,dy$ which in turn is equal to $2hf_Z(g_{\dot r}^{-1}(y_1))|(g_{\dot r}^{-1})'(y_1)|$ for some $y_1 = y_1(h, \dot r) \in [s-h, s+h]$. Thus we find for any $h > 0$ and $\dot r \in [r-h, r+h[$, reals $t_1(h) \in [r-h, r+h[$ and $y_1(h, \dot r) \in [s-h, s+h[$ such that

$$\mathrm{Prob}[r-h \le R < r+h,\ s-h \le g(\dot r, Z) < s+h] = 4h^2\overbrace{f_R(t_1)f_Z(g_{\dot r}^{-1}(y_1))|(g_{\dot r}^{-1})'(y_1)|}^{\phi}.$$

The over-braced expression ϕ is a function of

$$(t_1, \dot r, y_1) \in [r-h, r+h[\times[r-h, r+h[\times[s-h, s+h[.$$

Supposing this function to be continuous, we shall have

$$\lim_{h\downarrow 0}\phi(t_1, \dot r, y_1) = \phi(r, r, s) = f_R(r)f_Z(g_r^{-1}(s))|(g_r^{-1})'(s)|.$$

On the other hand, the density $f_{(R,S)}$ is a function which in the rectangle $A = [r-h, r+h[\times[s-h, s+h[$ satisfies

$$\int_A f_{(R,S)}\,d(r,s) = \mathrm{Prob}[r-h \le R < r+h,\ s-h \le g(R,Z) < s+h].$$

So, assuming continuity of $f_{(R,S)}$, for every $h>0$ sufficiently small there exist $(\dot{r},\dot{s})\in A$ such that the left-hand side is $4h^2 f_{(R,S)}(\dot{r},\dot{s})$. Letting h shrink to 0 we find from comparison of the expressions obtained that

$$f_{(R,S)}(r,s) = f_R(r) f_Z(g_r^{-1}(s))|(g_r^{-1})'(s)|.$$

4. The above set-up will serve in many problems as theoretical foundation for random flights since g will in many cases have the properties necessary that make the above arguments work. For Euclidean spaces we know from §2 that

$$f_Z(z) = \frac{1}{2G_{d-2}(1)}(1-z^2)^{\frac{d-3}{2}}\mathbb{1}_{[-1,1[}(z),$$

and

$$g(r,z) = \sqrt{r^2+2rz+1};$$

so

$$S = g(R,Z) = \sqrt{R^2+2RZ+1}.$$

It follows that $g_r^{-1}(s) = \frac{(s^2-r^2-1)}{2r}$ and so $(g_r^{-1})'(s) = s/r$. We also compute $\mathbb{1}_{[-1,1[}(g_r^{-1}(s))$. Since $-1\le g_r^{-1}(s)<1$ if and only if $-2r\le s^2-r^2-1<2r$ if and only if $r^2-2r+1\le s^2<r^2+2r+1$ if and only if $|-1+r|\le s\le 1+r$, we get $\mathbb{1}_{[-1,1[}(g_r^{-1}(s)) = \mathbb{1}_{[|-1+r|,1+r[}(s)$. Note that $[|-1+r|,1+r]\neq\emptyset$ if and only if $r\ge 0$ and that then $s\in[|-1+r|,1+r]$ if and only if $r\in[|-1+s|,1+s]$. Thus we find

$$\begin{aligned} f_{(R,S)}(r,s) &= f_R(r)\frac{|s/r|}{2G_{d-2}(1)}\left(1-(g_r^{-1}(s))^2\right)^{\frac{d-3}{2}}\mathbb{1}_{[|-1+r|,1+r]}(s) \\ &= f_R(r)\frac{|s/r|}{2G_{d-2}(1)}\left(1-(\tfrac{s^2-r^2-1}{2r})^2\right)^{\frac{d-3}{2}}\mathbb{1}_{[|-1+s|,1+s]}(r). \end{aligned}$$

5. If one knows the density function $f_{(R,S)}$ of a joint distribution (R,S) one finds the density of random variable S by marginalisation: $f_S(s) = \int_{-\infty}^{\infty} f_{(R,S)}(r,s)\,dr$. We know that the distribution of the random variable R_1, that is, the distance of the particle after one jump, is given, trivially, by $F_{R_1}(r) = \mathrm{Prob}(R_1<r) = \mathbb{1}_{[1,\infty[}(r)$; that is, it is given by 0 if $r<1$ and 1 if $r\ge 1$. Therefore the probability density f_{R_1} is modelled by a shift of the Dirac delta function: $f_{R_1}(r)=\delta(1-r)$. Thus by the previous section we know $f_{(R_1,R_2)}$ and hence by marginalisation (in principle) f_{R_2}. This then gives us $f_{(R_2,R_3)}$ and thus by marginalisation f_{R_3}, etc. The density f_{R_2} is direct because if a function f satisfies some mild conditions then

$$\int_{-\infty}^{\infty}\delta(1-r)f(r)\,dr = f(1).$$

So

$$\begin{aligned} f_{R_2}(s) &= \frac{1}{2^{d-2}G_{d-2}(1)}s^{d-2}(4-s^2)^{\frac{d-3}{2}}\mathbb{1}_{[0,2[}, \\ f_{R_{n+1}}(s) &= \int_{-\infty}^{\infty} f_{R_n}(r)\frac{|s/r|}{2G_{d-2}(1)}\left(1-(\tfrac{s^2-r^2-1}{2r})^2\right)^{\frac{d-3}{2}}\mathbb{1}_{[|-1+s|,1+s[}(r)\,dr. \end{aligned}$$

In order to strip this formula down to the essentials, note that in the inductive definition of the f_{R_n} each integration will introduce one more multiplication with $1/(2^{d-2}G_{d-2}(1))$. Furthermore $|s/r|$ can be substituted by s/r since we are only interested in values $s\ge 0$

and since $[|-1+s|, 1+s[\subseteq \mathbb{R}_{\geq 0}$ implies that the integrations can be restricted to the realm of the nonnegative real numbers. We can then put s to the left of the integral sign, and divide both sides by s seeing that we can formulate a recursion for the function $f_{R_n}(s)/s$. We introduce the functions $e(r,s)$, $e_2(r,s)$, and, by induction, b_n as follows:

$$\begin{aligned}
e(r,s) &= (-1+r+s)(1-r+s)(1+r-s)(1+r+s)\\
&= (-1+2r^2-r^4+2s^2+2r^2s^2-s^4),\\
e_2(r,s) &= e(r,s)/r^2,\\
b_2(s) &= \left(s^2(4-s^2)\right)^{\frac{d-3}{2}} \mathbb{1}_{[0,2[}(r),\\
b_{n+1}(s) &= \int_{-\infty}^{\infty} b_n(r) \left(\frac{e(r,s)}{r^2}\right)^{\frac{d-3}{2}} \mathbb{1}_{[|-1+s|,1+s[}(r)\,dr\\
&= \int_{-\infty}^{\infty} b_n(r)(e_2(r,s))^{\frac{d-3}{2}} \mathbb{1}_{[|-1+s|,1+s[}(r)\,dr.
\end{aligned}$$

Then the functions f_{R_n} and b_n are connected by the formula

$$f_{R_n}(s) = (1/(2^{d-2}G_{d-2}(1)))^{n-1} \cdot s \cdot b_n(s).$$

The inductive formulas just found for b_n and f_{R_n}can be directly translated into a short *Mathematica* code using e.g. the `UnitBox` command to take care of the functions $\mathbb{1}_{[|-1+s|,1+s[}(r)$ above. Experimenting a little with this code, one soon confirms that the functions b_n (and hence the f_{R_n}) are piecewise polynomial. Since the integration of polynomials in *Mathematica* is in principle very fast it could come as a surprise, that *Mathematica* slows down notably if n or d become a little larger. The authors suspect that a large part of the time is consumed treating the complicated codification and decodification of piecewise polynomial functions. A better code can be written using the representation theorem we prove in the next section.

6. The following representation theorem will be used foremost to show in §8 that the functions below called $\mathrm{pol}_i(s)$ and $\widetilde{\mathrm{pol}}_i(s)$ are in fact polynomials.

Theorem. *Assume $d \geq 3$ is an odd integer. Then the functions $b_n(s)$, inductively defined for $n \geq 2$ by*

$$b_2(s) = (s^2(4-s^2))^{\frac{d-3}{2}} \mathbb{1}_{[0,2[}(s), \quad b_{n+1}(s) = \int_{-\infty}^{\infty} b_n(r) e_2(r,s)^{\frac{d-3}{2}} \mathbb{1}_{[|-1+s|,1+s[}(r)\,dr,$$

admit piecewise representations

$$b_n(s) = \begin{cases} \sum_{i=0}^{\dot{n}-1} \mathrm{pol}_i(s) \mathbb{1}_{[2i,2i+2[}(s), & \text{if } n = 2\dot{n},\\ \sum_{i=0}^{\dot{n}} \widetilde{\mathrm{pol}}_i(s) \mathbb{1}_{[(2i-1)^+,2i+1[}(s), & \text{if } n = 1+2\dot{n}, \end{cases}$$

in which we can express the functions $\widetilde{\mathrm{pol}}_i(s)$ *from the functions* pol_i *by*

$$\widetilde{\mathrm{pol}}_i(s) = \begin{cases} \int\limits_{1-s}^{1+s} \mathrm{pol}_0(r) e_2(r,s)^{\frac{d-3}{2}}\, dr, & \text{if } i = 0, \\ \int\limits_{-1+s}^{2i} \mathrm{pol}_{i-1}(r) e_2(r,s)^{\frac{d-3}{2}}\, dr + \int\limits_{2i}^{1+s} \mathrm{pol}_i(r) e_2(r,s)^{\frac{d-3}{2}}\, dr, & \text{if } i = 1,\dots,\dot{n}-1, \\ \int\limits_{-1+s}^{2\dot{n}} \mathrm{pol}_{\dot{n}-1}(r) e_2(r,s)^{\frac{d-3}{2}}\, dr, & \text{if } i = \dot{n}, \end{cases}$$

and the functions pol_i *pertaining to the case* $n = 2(\dot{n}+1)$ *from the functions* $\widetilde{\mathrm{pol}}_i(s)$ *pertaining to the case* $n = 1 + 2\dot{n}$ *by*

$$\mathrm{pol}_i(s) = \begin{cases} \int\limits_{|-1+s|}^{1} \widetilde{\mathrm{pol}}_0(r) e_2(r,s)^{\frac{d-3}{2}}\, dr + \int\limits_{1}^{1+s} \widetilde{\mathrm{pol}}_1(r) e_2(r,s)^{\frac{d-3}{2}}\, dr, & \text{if } i = 0, \\ \int\limits_{s-1}^{2i+1} \widetilde{\mathrm{pol}}_i(r) e_2(r,s)^{\frac{d-3}{2}}\, dr + \int\limits_{2i+1}^{s+1} \widetilde{\mathrm{pol}}_{i+1}(r) e_2(r,s)^{\frac{d-3}{2}}\, dr, & \text{if } i = 1,\dots,\dot{n}-1, \\ \int\limits_{s-1}^{2\dot{n}+1} \widetilde{\mathrm{pol}}_{\dot{n}}(r) e_2(r,s)^{\frac{d-3}{2}}\, dr, & \text{if } i = \dot{n}. \end{cases}$$

(The index n *on which* pol_i *and* $\widetilde{\mathrm{pol}}_i$ *depend of course is for lightness of notation here suppressed.)*

Proof. For the case $n = 2, \dot{n} = 1$, the representation given is evidently of the type claimed since then $\sum_{i=0}^{\dot{n}-1} \mathrm{pol}_i(s) \mathbb{1}_{[2i,2i+2[}(s)$ collapses to $\mathrm{pol}_0(s)\mathbb{1}_{[0,2[}$, with $\mathrm{pol}_0(s) = (s^2(4-s^2))^{\frac{d-3}{2}}$. Now fix n even, $n = 2\dot{n}$, say, and compute from b_n the function b_{n+1} as defined above. We get

$$\begin{aligned} b_{n+1}(s) &= \int_{-\infty}^{\infty} \sum_{i=0}^{\dot{n}-1} \mathrm{pol}_i(r) \mathbb{1}_{[2i,2i+2[}(r)\; e_2(r,s)^{\frac{d-3}{2}}\; \mathbb{1}_{[|-1+s|,1+s]}(r)\, dr \\ &= \sum_{i=0}^{\dot{n}-1} \int_{-\infty}^{\infty} \mathrm{pol}_i(r) e_2(r,s)^{\frac{d-3}{2}} \mathbb{1}_{[2i,2i+2[\cap[|-1+s|,1+s[}(r)\, dr. \end{aligned}$$

We show first the formula for $\widetilde{\mathrm{pol}}_0(s)$, which, by its definition, is to describe $b_{n+1} = b_{1+2\dot{n}}$ in the interval $[0,1[$. So assume $0 \le s < 1$. Then $[|-1+s|, 1+s[= [1-s, 1+s[\subseteq [0,2[$ and so of the intervals $\{[2i, 2i+2[\}_{i=0}^{\dot{n}-1}$ only one intersects $[|-1+s|, 1+s[$, namely the one associated with $i = 0$. So we see for these s that $b_{n+1}(s) = \int_{1-s}^{1+s} \mathrm{pol}_0(r) e_2(r,s)^{\frac{d-3}{2}}\, dr$ proving the formula for $\widetilde{\mathrm{pol}}_0(s)$.

Now consider $\widetilde{\mathrm{pol}}_l$, with $l \in \{1, 2, \dots, \dot{n}-1\}$. The range in which $\widetilde{\mathrm{pol}}_l$ is to describe b_{n+1} is $[2l-1, 2l+1[$. So assume $2l-1 \le s < 2l+1$. Since $2l-1 \ge 1$, we see $|-1+s| = s-1$ and $[s-1, s+1[\subseteq [2l-2, 2l+2[= [2l-2, 2l[\uplus [2l, 2l+2[$, so that $[s-1, s+1[$ intersects typically only two adjacent of the intervals $\{[2i, 2i+2[\}_{i=0}^{\dot{n}-1}$, which figure in the defining formula for b_{n+1}, namely those pertaining to $i = l-1$ and $i = l$; the intersections yield the intervals in $[-1+s, 2l[$ and $[2l, 1+s[$. So for all $i \ne l-1, l,\quad \mathbb{1}_{[2i,2i+2[\cap[|-1+s|,1+s[} = 0$. and we get the claim concerning $\widetilde{\mathrm{pol}}_l$.

Finally consider $\widetilde{\mathrm{pol}}_{\dot n}$, that is, $s \in [2\dot n - 1, 2\dot n + 1[$. In this case, reasoning as before, but taking into account that the interval $[s-1, s+1[\subseteq [2\dot n - 2, 2\dot n + 2[$ intersects then only the last of the intervals figuring in the formula for b_{n+1}, leads to the last of the formulas of the first part of the theorem.

The second part of the theorem follows largely the same pattern of reasoning. In this case we start from the assumption that $b_n = b_{1+2\dot n}$ is given as above and deduce via the defining formula the representation of $b_{2+2\dot n} = b_{2(1+2\dot n)}$. So now we have

$$\begin{aligned} b_{n+1}(s) &= \int_{-\infty}^{\infty} \sum_{i=0}^{\dot n} \widetilde{\mathrm{pol}}_i(r) \mathbb{1}_{[(2i-1)^+, 2i+1[}(r) e_2(r,s)^{\frac{d-3}{2}} \mathbb{1}_{[|-1+s|, 1+s[}(r)\, dr \\ &= \sum_{i=0}^{\dot n} \int_{-\infty}^{\infty} \widetilde{\mathrm{pol}}_i(r) e_2(r,s)^{\frac{d-3}{2}} \mathbb{1}_{[(2i-1)^+, 2i+1[\cap[|-1+s|, 1+s[}(r)\, dr. \end{aligned}$$

Now $\mathrm{pol}_0(s)$ is to represent $b_{n+1}(s)$ for s in the interval $[0,2[$. Of all the intervals $\{[(2i-1)^+, 2i+1[\}_{i=0}^{\dot n}$ the only ones intersected by $[|-1+s|, 1+s[\subseteq [0,3[$, are $[0,1[, [1,3[$ in $[|-1+s|, 1[, [1, 1+s[$, respectively. This gives the formula for $\mathrm{pol}_0(s)$. Now let $l \in \{1, \dots, \dot n - 1\}$. Then pol_l is to describe $b_{1+n} = b_{2(1+\dot n)}$ for $s \in [2l, 2l+2]$. Then

$$[|-1+s|, 1+s[= [s-1, s+1[\subseteq [2l-1, 2l+3[= [2l-1, 2l+1[\uplus [2l+1, 2l+3[,$$

so that $[s-1, s+1[$ intersects the two intervals $[2l-1, 2l+1[$, $[2l+1, 2l+3[$ in $[s-1, 2l+1[, [2l+1, s+1[$, respectively. So from the defining formula above we get the formula for pol_l.

Finally, if $l = \dot n$, we have to look at $s \in [2\dot n, 2\dot n + 2[$. Then $[|-1+s|, 1+s[= [s-1, s+1[\subseteq [2\dot n - 1, 2\dot n + 3[$ intersects among the intervals referred to in b_n only $[2\dot n - 1, 2\dot n + 1[$, namely in $[s-1, 2\dot n + 1[$. This yields the last formula of the second part. □

This result suggests to represent the functions b_n as lists; namely the functions $b_{2\dot n}$ as a list $\{\mathrm{pol}_0, \mathrm{pol}_1, \dots, \mathrm{pol}_{-1+\dot n}\}$ representing the function on $\{[0,2[, [2,4[, \dots, [2\dot n - 2, 2\dot n[\}$ respectively; and $b_{1+2\dot n}$ as a list $\{\widetilde{\mathrm{pol}}_0, \widetilde{\mathrm{pol}}_1, \dots, \widetilde{\mathrm{pol}}_{\dot n}\}$ representing the function on $\{[0,1[, [1,3[, \dots, [2\dot n - 1, 2\dot n + 1[\}$ respectively.

Since these functions pol, $\widetilde{\mathrm{pol}}$ happen to be polynomials as we shall prove below, they are symbolically easily integrable. *Mathematica* code based on the idea to represent the functions b_n as lists is indeed much faster than code using the direct approach at the end of §5.

7. To prepare for the proof of polynomiality of the functions pol_i and $\widetilde{\mathrm{pol}}_i$ we need the following lemma and theorem, respectively. The proof of the lemma below is based on an idea suggested by C. Krattenthaler [K]; the authors' original proof was more complicated.

Lemma. *Assume nonnegative integers $m, \dot s, s, \ddot s$ satisfy* i or ii*:*

i. $m \geq \dot s + \lceil \frac{1+s}{2} \rceil + \ddot s$. ii. *$s$ is odd and $\ddot s = 0$.*

Then we have

$$\sum_{\substack{a,b,c \geq 0 \\ a+b+c=m}} \frac{(-2)^b}{a!\,b!\,c!} b^{\dot s} (a-c)^s c^{\ddot s} = 0. \tag{$*$}$$

Proof. First note the following chain of equations:

$$\begin{aligned}
\sum_{s,\dot{s},\ddot{s}\geq 0} \frac{x_1^s x_2^{\dot{s}} x_3^{\ddot{s}}}{s!\,\dot{s}!\,\ddot{s}!} \sum_{a,b,c\geq 0} t^{a+b+c} \frac{(-2)^b}{a!\,b!\,c!} b^{\dot{s}} (a-c)^s c^{\ddot{s}} \\
&= \sum_{a,b,c\geq 0} t^{a+b+c} \frac{(-2)^b}{a!\,b!\,c!} \sum_{s,\dot{s},\ddot{s}\geq 0} \frac{(x_1(a-c))^s}{s!} \frac{(x_2 b)^{\dot{s}}}{\dot{s}!} \frac{(x_3 c)^{\ddot{s}}}{\ddot{s}!} \\
&= \sum_{a,b,c\geq 0} t^{a+b+c} \frac{(-2)^b}{a!\,b!\,c!} e^{x_1(a-c)+x_2 b + x_3 c} \\
&= \sum_{a,b,c\geq 0} \frac{(te^{x_1})^a}{a!} \frac{(t(-2)e^{x_2})^b}{b!} \frac{(te^{(x_3-x_1)})^c}{c!} \\
&= \exp(te^{x_1}) \exp(-2te^{x_2}) \exp(te^{x_3-x_1}) \\
&= \exp(t(e^{x_1} - 2e^{x_2} + e^{x_3-x_1})) \\
&= \sum_{m\geq 0} \frac{t^m}{m!} (e^{x_1} - 2e^{x_2} + e^{x_3-x_1})^m.
\end{aligned}$$

Thus,

$$\text{left-hand side of } (*) = \frac{1}{m!} \cdot \text{ coefficient of } \frac{x_1^s x_2^{\dot{s}} x_3^{\ddot{s}}}{s!\,\dot{s}!\,\ddot{s}!} \text{ in } (e^{x_1} - 2e^{x_2} + e^{x_3-x_1})^m.$$

If $\ddot{s} = 0$ we get one of the coefficients occurring in the power series $(e^{x_1} - 2e^{x_2} + e^{0-x_1})^m$. This is evidently invariant under the operation $x_1 \mapsto -x_1$, which means that there occur only terms $x_1^s x_2^{\dot{s}}$ in which s is even. So we have proven that if s is odd and $\ddot{s} = 0$ then the coefficient is 0, showing that ii implies $(*)$.

What concerns i, use the series expansion of e^x to see that

$$(e^{x_1} - 2e^{x_2} + e^{x_3-x_1})^m = (-2x_2 + x_3 + \text{higher order terms})^m.$$

Allow to identify a monomial $x_1^\alpha x_2^\beta x_3^\gamma$ also with the triple (α, β, γ). In this sense, the monomials occurring in the parentheses $(\dots)$ of the right-hand side are all triples of nonnegative integers from the list $\{(0,1,0),(0,0,1)\} \cup \{\text{triples of sum of entries} \geq 2\}$. In order that a monomial $x_1^s x_2^{\dot{s}} x_3^{\ddot{s}}$ occurs in the right-hand side, it is necessary to find monomials M_i in $(\dots)$ such that the a product $M_1 M_2 \cdots M_m = x_1^s x_2^{\dot{s}} x_3^{\ddot{s}}$. This means to choose m triples from the mentioned list whose vector-sum is $(s, \dot{s}, \ddot{s})$. Assume we have chosen a triples $(0,1,0)$ and b triples $(0,0,1)$; then $m-a-b$ of the chosen triples have sum of entries≥ 2. The vector-sum has then sum of entries $s + \dot{s} + \ddot{s} \geq a + b + 2(m-a-b) = 2m-a-b$. From nonnegativity of the triples it also follows that $a \leq \dot{s}$ and $b \leq \ddot{s}$. Therefore we get $s + \dot{s} + \ddot{s} \geq 2m - \dot{s} - \ddot{s}$ or $m \leq \frac{s}{2} + \dot{s} + \ddot{s}$ as a necessary condition for the occurrence of $x_1^s x_2^{\dot{s}} x_3^{\ddot{s}}$ in the m-th power $(\dots)^m$. Thus the inequality $m \geq \lceil \frac{1+s}{2} \rceil + \dot{s} + \ddot{s}$ is a sufficient condition for non-occurrence of $x_1^s x_2^{\dot{s}} x_3^{\ddot{s}}$, proving the lemma. □

The functions $e(x,s) = (-1+x+s)(1-x+s)(1+x-s)(1+x+s)$ and $e_2(x,s) = e(x,s)/x^2$ were encountered in §5 in order to define the functions $b_n(s)$. The following theorem provides detailed information about the primitives $\int x^l(-e_2(x,s))^m dx$.

Theorem. *For integers $m \geq 0$ and $l \in \{0,2,4,\dots,2m-2\} \cup Z_{\geq 2m}$ there exists a primitive $F_{l,m}(x,s) \in \int x^l(-e_2(x,s))^m dx$, so that:*

a. *$F_{l,m}(x,s)$ is element of $\mathbb{R}[x,x^{-1},s]$ with $\deg_s F_{l,m} \le 4m$; and:*
   i. *$F_{l,m}(x,s)$ is even in s. So $F_{l,m}(x,s) = F_{l,m}(x,-s)$.*
   ii. *For odd $l > 2m$, $F_{l,m}(x,s)$ is an even polynomial also in x.*
   iii. *For even l, $F_{l,m}(x,s)$ is odd in x.*

b. *Furthermore $F_{l,m}(1+s,s)$ has these properties:*
   i. *It is a polynomial in s of degree $\le 1+l+2m$.*
   ii. *It has $(1+s)^{1+l}$ as a factor.*
   iii. *It is reciprocal: writing*
$$F_{l,m}(1+s,s) = \sum_{\nu=0}^{1+l+2m} a_\nu s^\nu,$$
one finds $a_\nu = a_{1+l+2m-\nu}$ for $\nu = 0,1,\dots,1+l+2m$.
   iv. *The coefficients a_ν with $\nu \in \{1,3,\dots,2m-1\} \cup \{l+2, l+4, \dots, l+2m\}$ are zeros. In particular the exponents which occur in $F_{l,m}(1+s,s)$ again lie all in $\{0,2,4,\dots,2m-2\} \cup Z_{\ge 2m}$.*

Proof. Expansion of e_2 gives $-e_2(x,s) = x^2 - 2(1+s^2) + (1-s^2)^2 x^{-2}$, and the multinomial theorem yields
$$x^l(-e_2(x,s))^m = \sum_{a,b,c\ge 0} (-2)^b \binom{m}{a,b,c} (1+s^2)^b x^{l+2(a-c)} (1-s^2)^{2c}.$$
By definition of multinomial coefficients, in this sum occur only terms associated with nonnegative integer triples (a,b,c) for which $a+b+c=m$. Therefore $2(a-c)$ is an even integer in the interval $[-2m,2m]$, while, if l is an integer satisfying the hypothesis, then $l \in \{\text{even integers in } [0,2m-2]\} \cup \mathbb{Z}_{\ge 2m}$. Thus $l+2(a-c) \ne -1$. Consequently the indefinite integral $\int x^l(-e_2(x,s))^m dx$ has no logarithmic term and one of the primitive functions of $x^l(-e_2(x,s))^m$ is given by
$$F_{l,m}(x,s) = \sum_{a,b,c\ge 0} \frac{(-2)^b \binom{m}{a,b,c}}{(1+l+2(a-c))} (1+s^2)^b x^{1+l+2(a-c)} (1-s^2)^{2c}.$$
This is evidently a polynomial in even powers of s (and coefficients that are Laurent polynomials in x). Its degree in s is not larger than the maximal value which $2b+4c$ assumes when (a,b,c) range over nonnegative integers of sum m. This value is evidently $4m$. Part a of the theorem is now evident. Replacing x by $(1+s)$ and again using the definition of multinomial coefficients, we get
$$F_{l,m}(1+s,s) = m! \sum_{\substack{a,b,c\ge 0\\ a+b+c=m}} \frac{(-2)^b}{a!\,b!\,c!(1+l+2(a-c))} \underbrace{(1+s^2)^b(1+s)^{1+l+2(a-c)}(1-s^2)^{2c}}_{=:\varphi(s)}.$$
Writing $(1-s^2)^{2c} = (1-s)^{2c}(1+s)^{2c}$, we find
$$\varphi(s) = (1+s)^{1+l}(1+s^2)^b(1+s)^{2a}(1-s)^{2c},$$
which has leading term $s^{1+l+2b+2a+2c} = s^{1+l+2m}$. Also since $(1-s)^{2c} = (1-2s+s^2)^c$ is reciprocal, $\varphi(s)$ is a product of reciprocal polynomials and therefore reciprocal. Independent of a,b,c, $\varphi(s)$ has constant term 1 and leading coefficient 1 and (as seen) degree $1+l+2m$. Thus $F_{l,m}(1+s,s)$ as a linear combination of reciprocal polynomials $\varphi(s)$ is reciprocal in the sense of b.iii. Now b.ii is also clear.

We still need to prove iv. From the original definition of $\varphi(s)$ we find

$$\varphi(s) = \sum_{k\geq 0}\binom{b}{k}s^{2k}\sum_{i\geq 0}\binom{1+l+2(a-c)}{i}s^i\sum_{j\geq 0}\binom{2c}{j}(-1)^j s^{2j}.$$

By writing binomial coefficients via falling factorials (using the notation $n^{\underline{k}} = n(n-1)\cdots(n-k+1)$ of [GKP]), we see

$$\begin{aligned}
&(\text{coefficient of } s^t \text{ in } \varphi(s))\\
&\quad= \sum_{i+2(k+j)=t}\binom{b}{k}\binom{1+l+2(a-c)}{i}\binom{2c}{j}(-1)^j\\
&\quad\overset{\text{if } t \text{ odd}}{=}(1+l+2(a-c))\sum_{i+2(j+k)=t}(i!\,j!\,k!)^{-1}(l+2(a-c))^{\underline{i-1}}\,b^{\underline{k}}\,(-1)^j(2c)^{\underline{j}}.
\end{aligned}$$

Here the 'if t is odd' condition guarantees that $i-1\geq 0$. Thus, if t is odd,

$$\begin{aligned}
&\text{coefficient of } s^t \text{ in } F_{l,m}(1+s,s)\\
&\quad= m!\sum_{\substack{a,b,c\geq 0\\ a+b+c=m}}\frac{(-2)^b}{a!\,b!\,c!}\sum_{i+2(j+k)=t}(i!\,j!\,k!)^{-1}(l+2(a-c))^{\underline{i-1}}\,b^{\underline{k}}\,(-1)^j(2c)^{\underline{j}}\\
&\quad= m!\sum_{i+2(j+k)=t}(i!\,j!\,k!)^{-1}(-1)^j\sum_{\substack{a,b,c\geq 0\\ a+b+c=m}}\frac{(-2)^b}{a!\,b!\,c!}(l+2(a-c))^{\underline{i-1}}\,b^{\underline{k}}\,(2c)^{\underline{j}}.
\end{aligned}$$

Now evidently $(l+2(a-c))^{\underline{i-1}}\,b^{\underline{k}}\,(2c)^{\underline{j}}$ can be expanded into a linear combination of products $b^{\dot{s}}(a-c)^s c^{\ddot{s}}$ in each of which we will have $(\dot{s},s,\ddot{s})\leq(k,i-1,j)$ in componentwise order. Now assume additionally $t\leq 2m-1$. Since t, and hence i, are odd we then get $\dot{s}+\lceil\frac{1+s}{2}\rceil+\ddot{s}\leq k+\lceil\frac{i}{2}\rceil+j=(t+1)/2\leq m$. So the claim that the coefficient of an s^t with $t\in\{1,3,\ldots,2m-1\}$ in $F_{l,m}(1+s,s)$ is 0 follows from the preceding lemma. That the monomials s^t with $t\in\{l+2,l+4,\ldots,l+2m\}$ cannot occur either is then a consequence of the fact that $F_{l,m}(1+s,s)$ is reciprocal. $\square$

We conclude this section with examples of the functions $F_{l,m}(x,s)$.

$$\begin{aligned}
F_{2,3}(1+s,s) &= \tfrac{1024}{63} - \tfrac{1024s^2}{35} - \tfrac{1024s^7}{35} + \tfrac{1024s^9}{63} \\
&= \tfrac{1024}{315}(1+s)^3\left(5-15s+21s^2-23s^3+21s^4-15s^5+5s^6\right), \\
F_{2,3}(2,s) &= \tfrac{11491}{504} - \tfrac{8297s^2}{140} + \tfrac{459s^4}{8} - \tfrac{141s^6}{2} + \tfrac{163s^8}{8} + \tfrac{13s^{10}}{4} - \tfrac{s^{12}}{24}, \\
F_{4,5}(1+s,s) &= -\tfrac{262144}{2145} + \tfrac{524288s^2}{1287} - \tfrac{262144s^4}{693} - \tfrac{262144s^{11}}{693} + \tfrac{524288s^{13}}{1287} - \tfrac{262144s^{15}}{2145}, \\
F_{2,3}(x,s) &= \left(\tfrac{x^9}{9} - \tfrac{6x^7}{7} + 3x^5 - \tfrac{20x^3}{3} - \tfrac{1}{3x^3} + 15x + \tfrac{6}{x}\right)s^0 \\
&\quad + \left(-\tfrac{6x^7}{7} + \tfrac{18x^5}{5} - 4x^3 + \tfrac{2}{x^3} - 12x - \tfrac{18}{x}\right)s^2 \\
&\quad + \left(3x^5 - 4x^3 - \tfrac{5}{x^3} - 6x + \tfrac{12}{x}\right)s^4 \\
&\quad + \left(-\tfrac{20x^3}{3} + \tfrac{20}{3x^3} - 12x + \tfrac{12}{x}\right)s^6 \\
&\quad + \left(-\tfrac{5}{x^3} + 15x - \tfrac{18}{x}\right)s^8 + \left(\tfrac{2}{x^3} + \tfrac{6}{x}\right)s^{10} + \left(-\tfrac{1}{3x^3}\right)s^{12}.
\end{aligned}$$

8. We now show that the functions $\mathrm{pol}_i, \widetilde{\mathrm{pol}}_i$ occurring in Theorem 6 are polynomials. In view of the formula at the end of §5, this will complete the proof that the functions f_{R_n} are piecewise polynomials. Partly for the convenience of the reader the following is a slightly modified copy of the long statement of Theorem 6.

Theorem. *Assume $d \geq 3$ is an odd integer and let $m = \frac{d-3}{2}$. Then the functions $b_n(s)$, inductively defined for $n \geq 2$ by*

$$b_2(s) = (s^2(4-s^2))^m \mathbb{1}_{[0,2[}(r), \quad b_{n+1}(s) = \int_{-\infty}^{\infty} b_n(r) e_2(r,s)^m \mathbb{1}_{[|-1+s|,1+s[}(r)\,,dr;$$

admit, depending on the parity of index n, the piecewise representations

$$b_{2\dot{n}} = \sum_{i=0}^{\dot{n}-1} \mathrm{pol}_i(s) \mathbb{1}_{[2i,2i+2[}(s)$$

$$b_{1+2\dot{n}} = \sum_{i=0}^{\dot{n}} \widetilde{\mathrm{pol}}_i(s) \mathbb{1}_{[(2i-1)^+,2i+1[}(s),$$

in which we can express the functions $\widetilde{\mathrm{pol}}_i(s)$ pertaining to indices $1+2\dot{n}$ from the functions pol_i *pertaining to index $2\dot{n}$ by*

$$\widetilde{\mathrm{pol}}_i(s) = \begin{cases} \int_{1-s}^{1+s} \mathrm{pol}_0(r) e_2(r,s)^m\,dr, & \textit{if } i = 0, \\ \int_{-1+s}^{2i} \mathrm{pol}_{i-1}(r) e_2(r,s)^m\,dr + \int_{2i}^{1+s} \mathrm{pol}_i(r) e_2(r,s)^m\,dr, & \textit{if } i = 1,\dots,\dot{n}-1, \\ \int_{-1+s}^{2\dot{n}} \mathrm{pol}_{\dot{n}-1}(r) e_2(r,s)^m\,dr, & \textit{if } i = \dot{n}, \end{cases}$$

and the functions pol_i *pertaining to the case* $n = 2(\dot{n}+1)$ *from the functions* $\widetilde{\mathrm{pol}}_i(s)$ *pertaining to the case* $n = 1+2\dot{n}$ *by*

$$\mathrm{pol}_i(s) = \begin{cases} \int\limits_{|-1+s|}^{1} \widetilde{\mathrm{pol}}_0(r) e_2(r,s)^m \, dr + \int\limits_{1}^{1+s} \widetilde{\mathrm{pol}}_1(r) e_2(r,s)^m \, dr, & \text{if } i = 0, \\ \int\limits_{s-1}^{2i+1} \widetilde{\mathrm{pol}}_i(r) e_2(r,s)^m \, dr + \int\limits_{2i+1}^{s+1} \widetilde{\mathrm{pol}}_{i+1}(r) e_2(r,s)^m \, dr, & \text{if } i = 1, \dots, \dot{n}-1, \\ \int\limits_{s-1}^{2\dot{n}+1} \widetilde{\mathrm{pol}}_{\dot{n}}(r) e_2(r,s)^m \, dr, & \text{if } i = \dot{n}. \end{cases}$$

Proof. The proof will follow from observations $\underline{0}, \underline{1}, \underline{2}, \underline{3}, \underline{4}$ below, which rely heavily on the fact that by part a of the theorem in §7, we have for positive reals a, b and $l \in \{2, 4, \dots, 2m-2\} \cup \mathbb{Z}_{\geq 2m}$, that the definite integral $\int_a^b x^l(-e_2(x,s))^m dx = F_{l,m}(b,s) - F_{l,m}(a,s)$; hence if $p(x)$ is a polynomial in which only monomials x^l with exponents in $l \in \{2, 4, \dots, 2m-2\} \cup \mathbb{Z}_{\geq 2m}$, occur, then $\int_a^b p(x) e_2(x,s)^m dx$ is a real linear combination of expressions $F_{l,m}(b,s) - F_{l,m}(a,s)$ with such l.

Looking at the definition of $b_2(s)$, we find:

$\underline{0}$. The expression pol_0 associated with $\dot{n} = 1$ is the polynomial $(s^2(4-s^2))^m$ and hence is even and has only monomials s^t with $t \geq 2m$.

$\underline{1}$. If pol_0, associated with $b_{2\dot{n}}$ is a polynomial all whose exponents l are in $\{2, 4, \dots, 2m-2\} \cup \mathbb{Z}_{\geq 2m}$, then $\widetilde{\mathrm{pol}}_0$ associated with $b_{1+2\dot{n}}$ is an odd polynomial of order $\geq 2m$.

$\triangleright$ For l in $\{2, 4, \dots, 2m-2\} \cup \mathbb{Z}_{\geq 2m}$, by its definition, the polynomial $\widetilde{\mathrm{pol}}_0(s)$ is a linear combination of differences

$$F_{l,m}(1+s,s) - F_{l,m}(1-s,s) = F_{l,m}(1+s,s) - F_{l,m}(1-s,-s)$$

for different l. Here we used that, by Theorem 7 a.i, $F(x,s) = F(x,-s)$. We see now that these differences turn into their negatives as we replace s by $-s$, so they are odd polynomials. By Theorem 7 b.iv, the exponents$\leq 2m-1$ occurring in $F_{l,m}(1+s,s)$ are even; so the same holds for $F_{l,m}(1-s,-s)$. So the exponents occurring in the differences all must be$\geq 2m$. This yields the claim. $\triangleleft$

$\underline{2}$. If $l \geq 2m$ is odd, then $\int_{|-1+s|}^1 x^l e_2(x,s)^m dx = \int_{1-s}^1 x^l e_2(x,s)^m dx$.

$\triangleright$ Recall that $e_2(x,s) = -(1-s^2)^2 x^{-2} + 2(1+s^2) - x^2$. So in $e_2(x,s)^m$ the powers of x occurring are all even and$\geq -2m$. Hence $x^l e_2^m$ has only odd powers which are all ≥ 0. Now if x^o is such an odd power of x, then

$$\int_{|-1+s|}^1 x^o dx = (1+o)^{-1}(1 - |-1+s|^{1+o}) = (1+o)^{-1}(1-(1-s)^{1+o}) = \int_{1-s}^1 x^o dx,$$

since $1+o$ is even. The claim follows. $\triangleleft$

$\underline{3}$. All the expressions $F_{l,m}(a,s)$, with a being one of the s-independent lower or upper bounds in the integrals occurring for the $\mathrm{pol}_i, \widetilde{\mathrm{pol}}_i$, are (for given l) even polynomials in s.

$\triangleright$ This is immediate from Theorem 7 a.i. $\triangleleft$

$\underline{4}$. Concerning the integrals below at the left for the computation of the functions pol_0 associated with $b_2, b_4, b_6, \ldots$, we have

$$\int_{|-1+s|}^{1} \widetilde{\mathrm{pol}}_0(r) e_2(r,s)^m \, dr = \int_{1-s}^{1} \widetilde{\mathrm{pol}}_0(r) e_2(r,s)^m \, dr.$$

$\triangleright$ The expression $\mathrm{pol}_0(s)$ associated with b_2 is by observation 0 even and has order $\geq 2m$. It satisfies hence the hypothesis of observation $\underline{1}$ for $\dot{n} = 1$ and that observation yields that $\widetilde{\mathrm{pol}}_0$ associated with b_3 is an odd polynomial of order $\geq 2m$. Thus by observation $\underline{2}$, we may replace the integral $\int_{|-1+s|}^{1} \cdots$ by $\int_{1-s}^{1} \cdots$. If we do so we get that the expression pol_0 pertaining to b_4 is given by $\mathrm{pol}_0(s) = \int_{1-s}^{1} \mathrm{pol}_0(r) e_2(r,s)^m \, dr + \int_{1}^{1+s} \widetilde{\mathrm{pol}}_1(r) e_2(r,s)^m \, dr$. This expression is hence a linear combination of differences $F_{l,m}(1,s) - F_{l,m}(1-s,s)$ and $F_{l,m}(1+s,s) - F_{l,m}(1,s)$ for various ls and so by $\underline{3}$ and Theorem 7 b.iv we see this $\mathrm{pol}_0(s)$ has only exponents in $\{2, 4, \ldots, 2m-2\} \cup \mathbb{Z}_{\geq 2m}$. Then $\underline{1}$ gives us that $\widetilde{\mathrm{pol}}_0(s)$ associated with $b_5 = b_{1+2\cdot 2}$ is odd and has order $\geq 2m$. This in turn, as above, by $\underline{2}$ allows us again a replacement of $\int_{|-1+s|}^{1} \cdots$ by $\int_{1-s}^{1} \cdots$ and so for the calculation of pol_0 associated with b_6 to write the same formula as before we did for the pol_0 associated with b_4. Repeating this type of reasoning, we see by induction that indeed we can write the formulas for pol_0 for all polynomials $b_{2\dot{n}}$ with integrals $\int_{1-s}^{1} \cdots$ instead of $\int_{|-1+s|}^{1} \cdots$. $\triangleleft$

Conclusion of the proof: We realise that all the integrals figuring in the computations of the $\mathrm{pol}_i(s)$ and $\widetilde{\mathrm{pol}}_i(s)$ can be written with upper and lower bounds which are reals $a > 0$ (independent of s) or are of the forms $s-1, 1-s, 1+s$. Due to the observations done up till here, we find that indeed the expressions $\mathrm{pol}_i(s)$ and $\widetilde{\mathrm{pol}}_i(s)$ are polynomials and we are done. $\square$

Concluding this paper, we think that a closer analysis of the reasoning in Section 7 would allow to insert the results of that section into a more general framework and it is also our hope that above analysis can serve as a basis to give more natural and somewhat simpler explicit formulas than those presented in [BS]. Furthermore perhaps it is useful to remark that integrals of the type we investigated are formally close to integrals that occur in certain moment computations and that, provided its factors are positive, the function $e(r,s)$ is a sixteenth of the squared area of a triangle with side length $1, r, s$.

Acknowledgement

A.K. was supported by Centro de Matemática da Universidade de Coimbra – UID/MAT /00324/2019, funded by the Portuguese Government through FCT/MEC and co-funded by the European Regional Development Fund through the Partnership Agreement PT2020.

P.S. thanks the Gulbenkian Foundation for support via their 'Novos Talentos em Matemática' programme.

References

[B] P. Billingsley, *Probability and Measure*, Wiley 1986.

[BS] J.M. Borwein and C.W. Sinnamon, *A closed form for the density functions of random walks in odd dimension*, Bull. Aust. Math. Soc. 93, no. 2, 330–339 (2016).

[D] J.C. Dutka, *On the problem of Random Flights*, Arch. History of Exact Sciences 32, no. 3–4, 351–375 (1985).

[G-P] R. García-Pelayo, *Exact solutions for isotropic random flights in odd dimensions*, J. Math. Phys. 53, No. 10, 103504, 15 p. (2012).

[GKP] R. Graham, D. Knuth, and O. Patashnik, *Concrete Mathematics*, Addison Wesley 1994.

[H] B. D. Hughes, *Random Walks and Random Environments, Vol. 1: Random Walks*, Oxford Univ. Press, 1995.

[K] C. Krattenthaler, private communication.

[L] Shengquiao Li, *Concise formulas for the area and volume of a hyperspherical cap*, Asian Journal of Mathematics and Statistics 4(1): 66–70 (2011).

[SK] P. Sá and A. Kovačec, *On the probability to be after n random jumps in space within a distance r from the origin*, DMUC preprint, pp. 19–33, 2019. (Electronically available at `www.mat.uc.pt/preprints/ps/p1933.pdf`).

[Wi] `https://en.wikipedia.org/wiki/Spherical_cap` , accessed September 1, 2022.

CMUC, Department of Mathematics, Faculty of Science and Technology, University of Coimbra, 3001-501 Coimbra, Portugal
Email address: `kovacec@mat.uc.pt`

Department of Mathematics, Faculty of Science and Technology, University of Coimbra, 3001-501 Coimbra, Portugal
Email address: `pbtcs2@gmail.com`