Symmetries of the Young lattice and abelian ideals of Borel subalgebras

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joint work with P. Cellini and P. Möseneder Frajria

Starting point: a theorem of Suter *(Europ. J. Comb., 2002)*

For a positive integer n let Y_n be the Hasse graph for the subposet $\mathfrak{Y}_n$ of the Young lattice corresponding to subdiagrams of the staircase diagram for the partition $(n-1, n-2, ..., 1)$ with hook length $\leq n-1$.

Theorem

If $n \geq 3$, the dihedral group of order $2n$ acts faithfully on the (undirected) graph Y_n.

Example

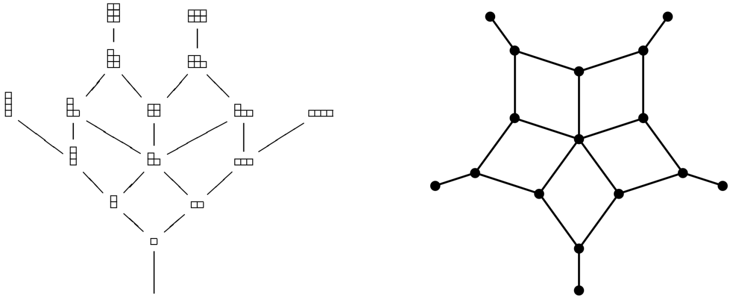


Figure: The Hasse diagram of $\mathcal{Y}_5$ and its underlying graph.

Idea of the proof

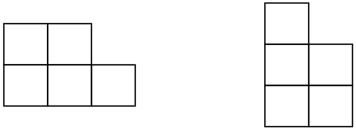
It suffices to makes explicit the action of generators

$$\sigma_n, \tau, o(\sigma_n) = n, o(\tau) = 2$$

of the dihedral group.

Idea of the proof

τ is simply the transposition of the diagram.



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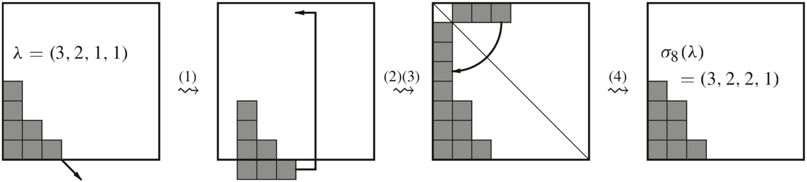

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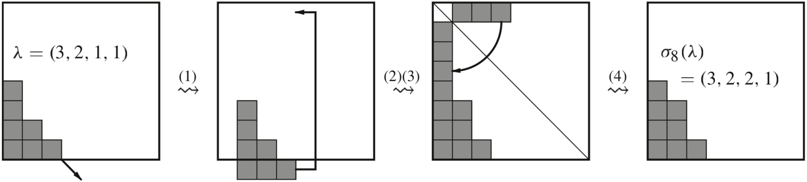

One has to verify that σ_n acts on $\mathcal{Y}_n$, maps edges to edges, and has order n.

This talk

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This result is conceptually due to Suter; details can be found in our (Cellini-Möseneder-P.) paper arXiv:1301.2548, to appear in Journal of Lie Theory

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Remark 2

Suter's dihedral symmetry got recently much attention: see e.g.

Berg, C., Zabrocki M.,*Symmetries on the lattice of k-bounded partitions*, arXiv:1111.2783v2.

Suter R.,*Youngs lattice and dihedral symmetries revisited: Möbius strips & metric geometry*, arXiv:1212.4463v1

Thomas H., N. Willliams, *Cyclic symmetry of the scaled simplex*, arXiv:1207.5240v1

Our paper and its motivations

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1. *If $\mathfrak{g}$ is not of type C_3 then $Aut(\mathfrak{Ab}) \cong Aut(\Pi)$.*
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The result on the dihedral simmetry is a byproduct of our methods.

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- $\mathfrak{b} \subset \mathfrak{g}$ Borel subalgebra
- $\mathfrak{Ab} = \{\mathfrak{i} \subset \mathfrak{b} \mid \mathfrak{i} \text{ ideal}, [\mathfrak{i}, \mathfrak{i}] = 0\}$ set of abelian ideals of $\mathfrak{b}$ regarded as a poset w.r.t. inclusion.

Setup

Abelian ideals of Borel subalgebras appeared a long time ago in Kostant's work on the structure of $\bigwedge \mathfrak{g}$ as a $\mathfrak{g}$-module.

Some fifteen years ago they got renewed interest, and proved to provide application to very different fields such as number theory (Kostant), invariant theory (Witten, Kumar), representation theory of affine and vertex algebras (Kac-Möseneder-P.).

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A natural generalization of abelian ideals, the *ad*-nilpotents ideals of $\mathfrak{b}$, show very interesting connections with combinatorics (Panyushev, Andrews-Krattenthaler-Orsina-P.).

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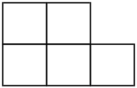
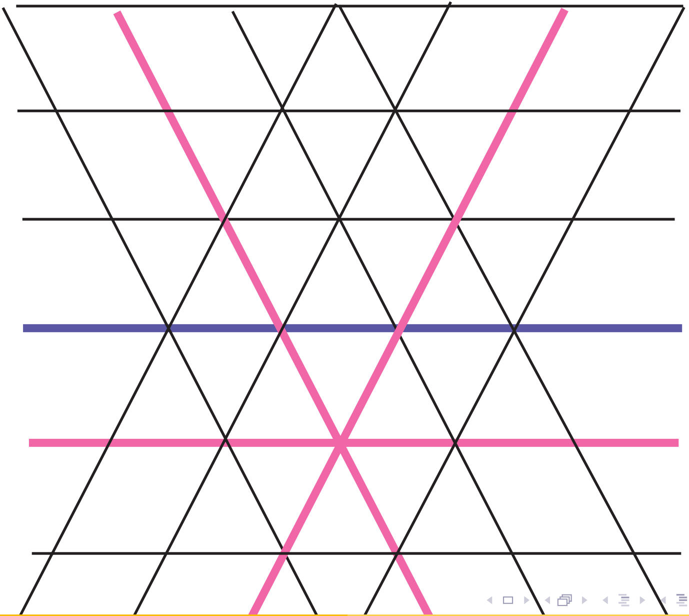
$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 \end{pmatrix}$$



Root system notation

- $\mathfrak{h}$ Cartan component of $\mathfrak{b}$
- Δ^+ positive system of the root system Δ of $(\mathfrak{g}, \mathfrak{h})$ corresponding to $\mathfrak{b}$.
- $\Pi = \{\alpha_1, \dots, \alpha_n\}$ simple roots.
- $A = (a_{ij})$ Cartan matrix.
- $\widehat{A} = (\hat{a}_{ij})$ extended Cartan matrix.
- W Weyl group of Δ
- $\widehat{W}$ affine Weyl group
- $\widehat{\Delta}^+ = (\Delta^+ + \mathbb{Z}_{\geq 0}\delta) \bigcup (-\Delta^+ + \mathbb{N}\delta)$ positive affine system

Example: type A_2



$\widehat{W}$ is a Coxeter group with generating set the reflections in the affine roots

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A fundamental domain for the action of $\widehat{W}$ on V is

$$C_1 = \{\lambda \in V \mid (\alpha, \lambda) \geq 0 \,\forall\, \alpha \in \Delta^+, \, (\theta, \lambda) \leq 1\}.$$

Preliminary definitions

- Recall that Δ^+ is a poset in a natural way:

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- Also, for $w \in \widehat{W}$, define

$$N(w) = \{\alpha \in \widehat{\Delta}^+ \mid w^{-1}(\alpha) \in -\widehat{\Delta}^+\}.$$

Abelian ideals

Proposition

The following sets are in bijection with $\mathfrak{Ab}$:

1. *the set of abelian dual order ideals in* Δ^+;
2. *the set of alcoves contained in* $2C_1$;
3. *the set of weights of abelian ideals.*

Idea of proof: abelian ideals $\leftrightsquigarrow$ abelian dual order ideals of Δ^+

By basic structure theory, if $\mathfrak{i} \in \mathfrak{Ab}$ then

$$\mathfrak{i} = \bigoplus_{\alpha \in \Phi_{\mathfrak{i}}} \mathfrak{g}_\alpha.$$

The fact that $\mathfrak{i}$ is an abelian ideal of $\mathfrak{b}$ translates into the fact that $\Phi_{\mathfrak{i}}$ is a dual order ideal of the root poset $(\Delta^+, \leq)$.

Idea of proof: abelian ideals $\leftrightsquigarrow$ alcoves in $2C_1$

Lemma (Peterson)

If $\mathfrak{i} \in \mathfrak{Ab}$, the set

$$-\Phi_{\mathfrak{i}} + \delta \subset \widehat{\Delta}^+$$

is biconvex, hence there exists a unique $w_{\mathfrak{i}} \in \widehat{W}$ such that

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The elements $w_{\mathfrak{i}}$ are precisely those $w \in \widehat{W}$ such that

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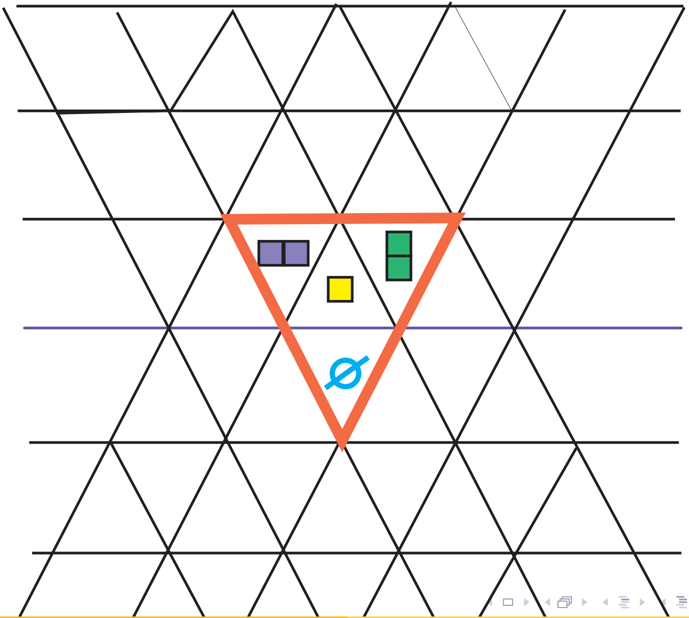
Notice that this implies at once that $|\mathfrak{Ab}| = 2^n$.

Example



Idea of proof: abelian ideals $\leftrightsquigarrow$ weights of ideals

The weight of $\mathfrak{i} \in \mathfrak{Ab}$ is is by definition

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Theorem

The map $\mathfrak{i} \mapsto \langle \mathfrak{i} \rangle$ is injective.

This is an old result of Kostant.

Symmetries

Set
$$Aut(\widehat{\Pi}) = \{\sigma : \widehat{\Pi} \leftrightarrow \widehat{\Pi} \mid \hat{a}_{ij} = \hat{a}_{\sigma(i)\sigma(j)}\}$$

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Proposition

Set $Z = \{Id_V, t_{\varpi_i} w_0^i w_0 \mid i \in J\} \subset \widehat{W}^e$. *Then*

$$Aut(\widehat{\Pi}) \cong I(C_1) = LI(C_1) \ltimes Z.$$

Symmetries

Recall that $Z = \{Id_V, t_{\varpi_i} w_0^i w_0 \mid i \in J\}$. Set

$$Z_2 = \{Id_V, t_{2\varpi_i} w_0^i w_0 \mid i \in J\}.$$

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So:

- $Aut(\widehat{\Pi}) \cong I(2C_1)$ acts on the set of alcoves in $2C_1$, hence on $\mathfrak{Ab}$.
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Proposition

If $\mathfrak{i} \in \mathfrak{Ab}$ *and* $x \in Aut(\widehat{\Pi})$, *then*

$$\langle x \cdot \mathfrak{i} \rangle = x(\langle \mathfrak{i} \rangle).$$

Corollary

In particular, if $x = t_{\varpi_i} w_0^i w_0$, then

$$\langle x \cdot \mathfrak{i} \rangle = w_0^i w_0(\langle \mathfrak{i} \rangle) + h^\vee \omega_i. \tag{1}$$

Application: Suter's dihedral symmetries

Specialize to $\mathfrak{g} = sl(n, \mathbb{C})$. In this case $\widehat{\Pi}$ is, as a graph, a cycle of length n, hence $Aut(\widehat{\Pi})$ is dihedral.
We claim that the action of

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given by (1) coincides with the action of σ_n combinatorially defined at the beginning of the talk.

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This is just a straightforward calculation, once the correct identifications have been done.