Automorphisms in Spaces of Functions and Shifts of Coefficients in Infinite Series

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Summary

1. Introduction,
2. Nonlinear Dynamical Systems,
3. Polylogarithms, multiple harmonic sums and polyzêtas,
4. Nonlinear Fuchsian differential equations.

INTRODUCTION

Linear Fuchsian differential equations (LFDE)

$$\dot{q}(z) = [M_0 u_0(z) + M_1 u_1(z)]\, q(z), \quad y(z) = \lambda q(z), \quad q(z_0) = \eta,$$

where $M_0, M_1 \in \mathcal{M}_{n,n}(\mathbb{C}), \lambda \in \mathcal{M}_{1,n}(\mathbb{C}), \eta \in \mathcal{M}_{n,1}(\mathbb{C}), u_0(z), u_1(z) \in \mathcal{C}$.

Example (hypergeometric equation)

$$z(1-z)\ddot{y}(z) + [t_2 - (t_0 + t_1 + 1)z]\dot{y}(z) - t_0 t_1 y(z) = 0.$$

Let $q_1(z) = y(z)$ and $q_2(z) = z(1-z)\dot{y}(z)$. One has

$$\begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} = \left[\begin{pmatrix} 0 & 0 \\ -t_0 t_1 & -t_2 \end{pmatrix} \frac{1}{z} - \begin{pmatrix} 0 & 1 \\ 0 & t_2 - t_0 - t_1 \end{pmatrix} \frac{1}{1-z} \right] \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}.$$

$$\lambda = \begin{pmatrix} 1 & 0 \end{pmatrix}, M_0 = -\begin{pmatrix} 0 & 0 \\ t_0 t_1 & t_2 \end{pmatrix}, M_1 = \begin{pmatrix} 0 & 1 \\ 0 & t_0 + t_1 - t_2 \end{pmatrix}, \eta = \begin{pmatrix} q_1(z_0) \\ q_2(z_0) \end{pmatrix}.$$

For (LFDE), one can base one self on the R. Jungen thesis *"Sur les séries de Taylor n'ayant que des singularités algébrico-logarithmiques sur leur cercle de convergence"* (1931).

But for nonlinear Fuchsian differential equations ?

NONLINEAR DYNAMICAL SYSTEMS

State Representation of Nonlinear Dynamical Systems

Let $(\mathcal{D}, d)$ be a k-commutative associative differential algebra with unit ($\mathrm{ch}(k) = 0$) and $\mathcal{C}$ be a differential subfield of $\mathcal{D}$.

$y(z) = \sum_{n \geq 0} y_n z^n$ is the output of :

$$(NLS) \quad \begin{cases} y(z) &= f(q(z)), \\ \dot{q}(z) &= A_0(q)u_0(z) + A_1(q)u_1(z), \\ q(z_0) &= q_0, \end{cases}$$

where :

- $u_0(z), u_1(z) \in \mathcal{C}$,
- the state $q = (q_1, \ldots, q_N)$ belongs the complex analytic manifold Q of dimension N and q_0 is the initial state,
- the observation $f \in \mathcal{O}$, with $\mathcal{O}$ is the ring of holomorphic functions over Q,
- For $i = 0..1$, $A_i = \sum_{j=1}^{N} A_i^j(q) \frac{\partial}{\partial q_j}$ is an analytic vector field[1] over Q, with $A_i^j(q) \in \mathcal{O}$, for $j = 1, \ldots, N$.

[1] A vector field A_i is said to be linear if the $A_i^j(q), j = 1..N$, are constants.

Examples of Nonlinear Dynamical Systems

Example (harmonic oscillator)

$\dot{y}(z) + k_1 y(z) + k_2 y^2(z) = u_1(t).$

$$
\begin{aligned}
\dot{q}(z) &= A_0(q)u_0(z) + A_1(q)u_1(z) \quad \text{with } u_0(z) \equiv 1, \\
A_0 &= -(k_1 q + k_2 q^2)\frac{\partial}{\partial q}, \\
A_1 &= \frac{\partial}{\partial q}, \\
y(z) &= q(z).
\end{aligned}
$$

Example (Duffing's equation)

$\ddot{y}(z) + a\dot{y}(z) + by(z) + cy^3(z) = u_1(z).$

$$
\begin{aligned}
\dot{q}(z) &= A_0(q)u_0(z) + A_1(q)u_1(z) \quad \text{with } u_0(z) \equiv 1, \\
A_0 &= -(aq_2 + bq_1^2 + cq_1^3)\frac{\partial}{\partial q_2} + q_2\frac{\partial}{\partial q_1}, \\
A_1 &= \frac{\partial}{\partial q_2}, \\
y(z) &= q_1(z).
\end{aligned}
$$

Our works

Let $X=\{x_0,x_1\}$ with $x_0<x_1$. For any $w=x_{i_1}\cdots x_{i_k}\in X^*$, let

$$\mathcal{A}(1_{X^*})=\mathrm{Id},\qquad \mathcal{A}(w)=A_{i_1}\circ\ldots\circ A_{i_k},$$
$$\alpha_{z_0}^{z}(1_{X^*})=1,\qquad \alpha_{z_0}^{z}(w)=\int_{z_0}^{z}\int_{z_0}^{z_1}\ldots\int_{z_0}^{z_{k-1}}u_{i_1}(z_1)dz_1\ \cdots\ u_{i_k}(z_k)dz_k.$$

Theorem (Deneufchâtel,Duchamp,HNM, Solomon, 2010)

Let $S=\sum_{w\in X^*}\alpha_{z_0}^{z}(w)\,w\in\mathcal{D}\langle\langle X\rangle\rangle$. *The conditions are equivalent :*

i) *The family* $(\alpha_{z_0}^{z}(w))_{w\in X^*}$ *of coefficients of* S *is free over* $\mathcal{C}$.

ii) *The family of coefficients* $(\alpha_{z_0}^{z}(x))_{x\in X\cup\{1_{X^*}\}}$ *is free over* $\mathcal{C}$.

iii) *The family* $(u_x)_{x\in X}$ *is such that, for* $f\in\mathcal{C}$ *and* $\alpha_x\in k$,
$$d(f)=\sum_{x\in X}\alpha_x u_x\Longrightarrow(\forall x\in X)(\alpha_x=0).$$

iv) *The family* $(u_x)_{x\in X}$ *is free over* k *and* $d(\mathcal{C})\cap\mathrm{span}_k((u_x)_{x\in X})=\{0\}$.

Therefore, by successive Picard iterations, one get

$$y(z)=\sum_{w\in X^*}\mathcal{A}(w)\circ f_{|q_0}\ \alpha_{z_0}^{z}(w).$$

Chen-Fliess generating series

- Chen series

$$S_{z_0 \rightsquigarrow z} = \sum_{w \in X^*} \alpha_{z_0}^{z}(w)\, w.$$

Any Chen generating series $S_{z_0 \rightsquigarrow z}$ is group-like, for $\Delta_{\sqcup\!\sqcup}$, and it depends only on the homotopy class of $z_0 \rightsquigarrow z$ (**Ree**).

The product of two Chen generating series $S_{z_1 \rightsquigarrow z_2}$ and $S_{z_0 \rightsquigarrow z_1}$ is the Chen generating series $S_{z_0 \rightsquigarrow z_2} = S_{z_1 \rightsquigarrow z_2} S_{z_0 \rightsquigarrow z_1}$ (**Chen**).

- The generating series of the polysystem $\{A_i\}_{i=0,1}$ and of the observation $f \in \mathcal{O}$ at q is given by

$$\sigma f_{|_q} \quad := \quad \sum_{w \in X^*} \mathcal{A}(w) \circ f_{|q}\, w \quad \in \mathbb{C}\langle\langle X \rangle\rangle.$$

For any $f, g \in \mathcal{O}$ anf for any $\lambda, \mu \in \mathbb{C}$, one has (**Fliess**)

$$\sigma(\nu f + \mu g)_{|q} = \sigma(\nu f)_{|q} + \sigma(\mu g)_{|q} \quad \text{and} \quad \sigma(fg)_{|q} = \sigma f_{|q} \sqcup\!\sqcup \sigma g_{|q}.$$

POLYLOGARITHM-HARMONIC SUM-POLYZETA

Chen series and generating series of polylogarithms

Let $u_0(z) = \frac{1}{z}, u_1(z) = \frac{1}{1-z}$ and $\omega_0(z) = u_0(z)dz, \omega_1(z) = u_1(z)dz$.

$$\begin{array}{rcl} \forall w \in X^* x_1, \quad \alpha_0^z(w) & = & \mathrm{Li}_w(z), \\ \mathrm{P}_w(z) & := & (1-z)^{-1}\,\mathrm{Li}_w(z) = \sum_{n\geq 1} \mathrm{H}_w(n)\, z^n, \\ \mathrm{Li}_{x_0}(z) & := & \log z, \\ \mathrm{L}(z) & := & \sum_{w\in X^*} \mathrm{Li}_w(z)\, w, \\ \mathrm{P}(z) & := & (1-z)^{-1}\mathrm{L}(z). \end{array}$$

Let

$$(DE) \quad dG(z) = [x_0\,\omega_0(z) + x_1\,\omega_1(z)]G(z).$$

Proposition

- $S_{z_0 \rightsquigarrow z}$ *satisfies* (DE) *with* $S_{z_0 \rightsquigarrow z_0} = 1$,
- $\mathrm{L}(z)$ *satisfies* (DE) *with* $\mathrm{L}(z) \underset{z\to 0}{\widetilde{\ }} \exp(x_0 \log z)$.

Hence, $S_{z_0 \rightsquigarrow z} = \mathrm{L}(z)\mathrm{L}(z_0)^{-1}$, *or equivalently,* $\mathrm{L}(z) = S_{z_0 \rightsquigarrow z}\mathrm{L}(z_0)$.

Noncommutative generating series of convergent polyzêtas

Let $X=\{x_0,x_1\}$ (resp. $Y=\{y_i\}_{i\geq 1}$) with $x_0<x_1$ (resp. $y_1>y_2>\ldots$). Let $\mathcal{L}ynX$ (resp. $\mathcal{L}ynX$) be the transcendence basis of $(\mathbb{C}\langle X\rangle, ш)$ (resp. $(\mathbb{C}\langle Y\rangle, ⧢)$) and let $\{\hat{l}\}_{l\in\mathcal{L}ynX}$ (resp. $\{\hat{l}\}_{l\in\mathcal{L}ynY}$) be its dual basis. Then

Theorem (HNM, 2009)

We have $\Delta_{ш}\mathrm{L}=\mathrm{L}\otimes\mathrm{L}$ *and* $\Delta_{⧢}\mathrm{H}=\mathrm{H}\otimes\mathrm{H}$.

Moreover, let $\mathrm{L}_{\mathrm{reg}}(z):=\prod^{\searrow}_{\substack{l\in\mathcal{L}ynX\\ l\neq x_0,x_1}} e^{\mathrm{Li}_l(z)\,\hat{l}}$ *and* $\mathrm{H}_{\mathrm{reg}}(N):=\prod^{\searrow}_{\substack{l\in\mathcal{L}ynY\\ l\neq y_1}} e^{\mathrm{H}_l(N)\,\hat{l}}$.

Then $\mathrm{L}(z)=e^{x_1\log\frac{1}{1-z}}\mathrm{L}_{\mathrm{reg}}(z)e^{x_0\log z}$ *and* $\mathrm{H}(N)=e^{y_1\mathrm{H}_1(N)}\mathrm{H}_{\mathrm{reg}}(N)$.
We put $Z_{ш}:=\mathrm{L}_{\mathrm{reg}}(1)$ *and* $Z_{⧢}:=\mathrm{H}_{\mathrm{reg}}(\infty)$.

Theorem (*à la* Abel theorem, HNM, 2005)

Let $\Pi_Y\mathrm{L}$ *and* $\Pi_Y Z_{ш}$ *be the projections of* L *and* $Z_{ш}$ *over* Y. *Then*

$$\lim_{z\to 1} e^{y_1\log\frac{1}{1-z}}\Pi_Y\mathrm{L}(z)=\lim_{N\to\infty}\exp\left[-\sum_{k\geq 1}\mathrm{H}_{y_k}(N)\frac{(-y_1)^k}{k}\right]\mathrm{H}(N)=\Pi_Y Z_{ш}.$$

Hence, $Z_{ш}$ and $Z_{⧢}$ are group-likes and $Z_{⧢}=e^{-\gamma\, y_1}\Gamma(1+y_1)\Pi_Y Z_{ш}$.

Successive derivations of L

For any $w = x_{i_1} \ldots x_{i_k} \in X^*$ and for any derivation multi-index $\mathbf{r} = (r_1, \ldots, r_k)$ of degree $\deg \mathbf{r} = |w| = k$ and of weight $\mathrm{wgt}\, \mathbf{r} = k + r_1 + \ldots + r_k$, let us define the monomial $\tau_{\mathbf{r}}(w)$ by

$$\tau_{\mathbf{r}}(w) = \tau_{r_1}(x_{i_1}) \ldots \tau_{r_k}(x_{i_k}) = [u_{i_1}^{(r_1)}(z) \ldots u_{i_k}^{(r_k)}(z)]\, x_{i_1} \ldots x_{i_k}.$$

In particular, for any integer r

$$\tau_r(x_0) = u_0^{(r)}(z)\, x_0 = \frac{-r!x_0}{(-z)^{r+1}},$$

$$\text{and} \qquad \tau_r(x_1) = u_1^{(r)}(z)\, x_1 = \frac{r!x_1}{(1-z)^{r+1}}.$$

Theorem (HNM, 2003)

For any $n \in \mathbb{N}$, we have, $\mathrm{L}^{(n)}(z) = P_n(z)\mathrm{L}(z)$, where

$$P_n(z) = \sum_{wgt\, \mathbf{r}=n} \sum_{w \in X^n} \prod_{i=1}^{\deg \mathbf{r}} \binom{\sum_{j=1}^{i} r_j + j - 1}{r_i} \tau(w) \in \mathcal{D}\langle X \rangle.$$

NONLINEAR FUCHSIAN DIFFERENTIAL EQUATIONS

Nonlinear Fuchsian differential equations (NLFDE)

$y(z) = \sum_{n\geq 0} y_n z^n$ is the output of :

$$(NLFDE) \begin{cases} y(z) &=& f(q(z)), \\ \dot{q}(z) &=& \dfrac{A_0(q)}{z} + \dfrac{A_1(q)}{1-z}, \\ q(z_0) &=& q_0, \end{cases}$$

$(\rho, \check{\rho}, C_f)$ and $(\rho, \check{\rho}, C_i)$, for $i = 0, .., m$, are convergence modules of f and $\{A_i^j\}_{j=1,..,n}$ respectively at $q \in \mathrm{CV}(f) \Cap_{i=0,..,m}^{j=1,..,n} \mathrm{CV}(A_i^j)$.

$\sigma f_{|_{q_0}} = \sum_{w\in X^*} \mathcal{A}(w) \circ f_{|_{q_0}} \; w$ satisfies the $\chi-$growth condition.

Computation of the output

The duality between $\sigma f_{|_{q_0}}$ and $S_{z_0 \rightsquigarrow z}$ consists on the convergence (precisely speaking, the convergence of a duality pairing) of the Fliess' fundamental formula which is extended as follows

$$y(z) = \langle \sigma f_{|_{q_0}} \| S_{z_0 \rightsquigarrow z} \rangle = \sum_{w \in X^*} \mathcal{A}(w) \circ f_{|_{q_0}} \langle S_{z_0 \rightsquigarrow z} | w \rangle.$$

The output y admits then the following expansions

$$\begin{aligned} y(z) &= \sum_{w \in X^*} g_w(z)\, \mathcal{A}(w) \circ f_{|_{q_0}}, \\ &= \sum_{k \geq 0} \sum_{n_1, \ldots, n_k \geq 0} g_{x_0^{n_1} x_1 \ldots x_0^{n_k} x_1}(z)\ \mathrm{ad}_{A_0}^{n_1} A_1 \ldots \mathrm{ad}_{A_0}^{n_k} A_1 e^{\log z A_0} \circ f_{|_{q_0}}, \\ &= \exp\left(\sum_{w \in X^*} g_w(z)\, \mathcal{A}(\pi_1(w)) \circ f_{|_{q_0}} \right), \\ &= \prod_{l \in \mathcal{L}ynX} \exp\left(g_l(z)\, \mathcal{A}(\hat{l}) \circ f_{|_{q_0}} \right), \end{aligned}$$

where, for any $w \in X^*$, $g_w \in \mathrm{LI}_{\mathcal{C}}$ and

$$\pi_1(w) = \sum_{k \geq 1} \frac{(-1)^{k-1}}{k} \sum_{v_1, \cdots, v_k \in X^* \setminus \{1_{X^*}\}} \langle w | v_1 ⧢ \cdots ⧢ v_k \rangle\, v_1 \cdots v_k.$$

Asymptotics of the output

The output y of nonlinear differential equation with three singularities is then combination of the elements belonging the $\mathrm{LI}_{\mathcal{C}}$.

For $z_0 = \varepsilon \to 0^+$, the asymptotic behaviour of the output y at $z = 1$ is given by

$$y(1) \underset{\varepsilon \to 0^+}{\widetilde{\quad}} \langle \sigma f_{|_{q_0}} \| S_{\varepsilon \rightsquigarrow 1-\varepsilon} \rangle = \sum_{w \in X^*} \mathcal{A}(w) \circ f_{|_{q_0}} \langle S_{\varepsilon \rightsquigarrow 1-\varepsilon} | w \rangle,$$

with $S_{\varepsilon \rightsquigarrow 1-\varepsilon} \underset{\varepsilon \to 0^+}{\widetilde{\quad}} e^{-x_1 \log \varepsilon} Z_{⧢} \; e^{-x_0 \log \varepsilon}$.

If $y(z) = \sum_{n \geq 0} y_n z^n$ then, the coefficients of its ordinary Taylor expansion belong the harmonic algebra and there exist algorithmically computable coefficients $a_i \in \mathbb{Z}$, $b_i \in \mathbb{N}$ and c_i belong a completion of the $\mathbb{C}$-algrebra generated by $\mathcal{Z}$ and by the Euler's γ constant, such that

$$y_n \underset{n \to \infty}{\widetilde{\quad}} \sum_{i \geq 0} c_i n^{a_i} \log^{b_i} n.$$

Finite parts of the output

Definition

For any $f \in \mathcal{O}$ such that

$$\langle \sigma f_{|_{q_0}} \| S_{z_0 \rightsquigarrow z} \rangle = \sum_{n \geq 0} y_n z^n$$

and for $z_0 = \varepsilon \to 0^+$, let

$$\phi(f_{|_{q_0}}) \underset{z \to 1}{\widetilde{\quad}} \text{f.p. } y(z) \quad \text{in the scale} \quad \{(1-z)^a \log(1-z)^b\}_{a \in \mathbb{Z}, b \in \mathbb{N}}$$
$$\psi(f_{|_{q_0}}) \underset{n \to \infty}{\widetilde{\quad}} \text{f.p. } y_n \quad \text{in the scale} \quad \{n^a \log^b(n)\}_{a \in \mathbb{Z}, b \in \mathbb{N}}.$$

Proposition

For any $f, g \in \mathcal{O}$ anf for any $\lambda, \mu \in \mathbb{C}$, one has

$$\phi((\nu f + \mu g)_{|_{q_0}}) = \phi(\nu f_{|_{q_0}}) + \phi(\mu g_{|_{q_0}}) \quad \textit{and} \quad \phi(fg_{|_{q_0}}) = \phi(f_{|_{q_0}})\phi(g_{|_{q_0}}),$$
$$\psi((\nu f + \mu g)_{|_{q_0}}) = \psi(\nu f_{|_{q_0}}) + \psi(\mu g_{|_{q_0}}) \quad \textit{and} \quad \psi(fg_{|_{q_0}}) = \psi(f_{|_{q_0}})\psi(g_{|_{q_0}}).$$

Residual calculus and derivations

Let S and $P \in \mathbb{Q}\langle X\rangle$. The left (resp. right) residual of S by P, is the formal power series $P \triangleleft S$ (resp. $S \triangleright P$) in $\mathbb{Q}\langle\langle X\rangle\rangle$ defined by :

$$\langle P \triangleleft S | w\rangle = \langle S | wP\rangle \quad (\textit{resp.} \quad \langle S \triangleright P | w\rangle = \langle S | Pw\rangle).$$

We straightforwardly get, for any $P, Q \in \mathbb{Q}\langle X\rangle$:

$$P \triangleleft (Q \triangleleft S) = PQ \triangleleft S, (S \triangleright P) \triangleright Q = S \triangleright PQ, (P \triangleleft S) \triangleright Q = P \triangleleft (S \triangleright Q).$$

In case $x, y \in X$ and $w \in X^*$, we get :

$$x \triangleleft (wy) = \delta_x^y w \quad \text{and} \quad xw \triangleright y = \delta_x^y w.$$

Thus, "$x\triangleleft$" and "$\triangleright x$" are derivations on $(\mathbb{Q}\langle\langle X\rangle\rangle, ш)$:

$$x \triangleleft (u ш v) = (x \triangleleft u) ш v + u ш (x \triangleleft v),$$
$$(u ш v) \triangleright x = (u \triangleright x) ш v + u ш (v \triangleright x).$$

Consequently, for any Lie series Q, the linear maps "$Q\triangleleft$" and "$\triangleright Q$" are derivations on $(\mathbb{Q}[\mathcal{L}ynX], ш)$.

Successive derivations of the output

Let $n \in \mathbb{N}$,

$$\begin{aligned} y^{(n)}(z) &= \langle \sigma f_{|_{q_0}} \| \frac{d^n}{dz^n} S_{z_0 \rightsquigarrow z} \rangle \\ &= \langle \sigma f_{|_{q_0}} \| \mathrm{L}^{(n)}(z) \mathrm{L}(z_0)^{-1} \rangle \\ &= \langle \sigma f_{|_{q_0}} \| P_n(z) \mathrm{L}(z) \mathrm{L}(z_0)^{-1} \rangle \\ &= \langle \sigma f_{|_{q_0}} \triangleright P_n(z) \| \mathrm{L}(z) \mathrm{L}(z_0)^{-1} \rangle \\ &= \langle \sigma f_{|_{q_0}} \triangleright P_n(z) \| S_{z_0 \rightsquigarrow z} \rangle, \end{aligned}$$

where the polynomial $P_n(z) \in \mathcal{D}\langle X \rangle$ is defined as follows

$$P_n(z) = \sum_{\mathrm{wgt}\, \mathbf{r} = n} \sum_{w \in X^n} \prod_{i=1}^{\deg \mathbf{r}} \binom{\sum_{j=1}^{i} r_j + j - 1}{r_i} \tau(w).$$

Therefore, $\sigma f_{|_{q_0}} \triangleright P_n(z) \in \mathcal{D}\langle\langle X \rangle\rangle$ is the non commutative generating series of $y^{(n)}$.

Asymptotics of the successive derivation of the output

Let $k \in \mathbb{N}$, the successive derivation $y^{(k)}$ of the output of nonlinear differential equation with three singularities is then combination of the elements g belonging the polylogarithm algebra.
For $z_0 = \varepsilon \to 0^+$, the asymptotic behaviour of the output y at $z = 1$ is given by

$$\begin{aligned} y^{(k)}(1) & \underset{\varepsilon \to 0^+}{\widetilde{\quad}} \langle \sigma f_{|_{q_0}} \| P_k(1-\varepsilon) S_{\varepsilon \rightsquigarrow 1-\varepsilon} \rangle \\ & = \sum_{w \in X^*} \mathcal{A}(w) \circ f_{|_{q_0}} \langle P_k(1-\varepsilon) S_{\varepsilon \rightsquigarrow 1-\varepsilon} | w \rangle. \end{aligned}$$

If $y^{(k)}(z) = \sum_{n \geq 0} y_n^{(k)} z^n$ then, the coefficients of its ordinary Taylor expansion belong the harmonic algebra and there exist algorithmically computable coefficients $a_i \in \mathbb{Z}$, $b_i \in \mathbb{N}$ and c_i belong a completion of the $\mathbb{C}$-algrebra generated by $\mathcal{Z}$ and by the Euler's γ constant, such that

$$y_n^{(k)} \underset{n \to \infty}{\widetilde{\quad}} \sum_{i \geq 0} c_i n^{a_i} \log^{b_i} n.$$

THANK YOU FOR YOUR ATTENTION