Séminaire Lotharingien de Combinatoire **87B** (2023) *Special issue for the* 9^{th} *International Conference*
Article #8, 25pp. *on Lattice Path Combinatorics and Applications*

THREE FAMILIES OF q-LOMMEL POLYNOMIALS

JANG SOO KIM[1] AND DENNIS STANTON[2]

[1]Department of Mathematics, Sungkyunkwan University (SKKU), Suwon, Gyeonggi-do 16419, South Korea; https://jangsookim.github.io/

[2]School of Mathematics, University of Minnesota, Minneapolis, Minnesota 55455, USA; https://www-users.cse.umn.edu/~stant001/

Abstract. Three q-versions of Lommel polynomials are studied. Included are explicit representations, recurrences, continued fractions, and connections to associated Askey–Wilson polynomials. Combinatorial results are emphasized, including a general theorem when R_I moments coincide with orthogonal polynomial moments. The combinatorial results use weighted Motzkin paths, Schröder paths, and parallelogram polyominoes.

Keywords: Lommel polynomial, Bessel function, orthogonal polynomial.

1. Introduction

Lehmer [27] used the following Bessel function identity to study zeros of Bessel functions

$$\frac{J_{\nu+1}(x)}{J_\nu(x)} = 2\sum_{n=1}^{\infty} \sigma_{2n}(\nu) x^{2n-1}, \tag{1.1}$$

where $\sigma_{2n}(\nu)$ is the $2n^{\text{th}}$ power sum of the inverses of the positive zeros $j_{\nu,k}$ of $J_\nu(x)$:

$$\sigma_{2n}(\nu) = \sum_{k=1}^{\infty} j_{\nu,k}^{-2n}. \tag{1.2}$$

Lehmer noted that $\sigma_{2n}(\nu)$ is a rational function of ν, with a predictable denominator, and a numerator with nonnegative coefficients. Kishore [19] proved Lehmer's positivity conjecture. Lalanne ([25, Proposition 3.6], [26, Theorem 4.7]) proved q-versions of Kishore's result using weighted binary trees and also weighted Dyck paths.

The above series is related to the Lommel polynomials $L_{n,\nu}$, which are orthogonal polynomials with respect to the linear functional

$$\mathcal{L}(P(x)) := 2(\nu+1)\sum_{k=1}^{\infty}\left(P\left(j_{\nu,k}^{-1}\right) + P\left(-j_{\nu,k}^{-1}\right)\right) j_{\nu,k}^{-2},$$

i.e., $\mathcal{L}(L_{n,\nu}(x)L_{m,\nu}(x)) = 0$ if $n \neq m$ and $\mathcal{L}(1) = 1$; see [15, Eq. (6.5.17)]. Thus $\sigma_{2n}(\nu)$ in (1.2) is the $(2n-2)^{\text{th}}$ moment for the Lommel polynomials, while (1.1) is the Lommel moment generating function.

The purpose of this paper is to study three sets of q-Lommel polynomials, whose moment generating functions are quotients of q-Bessel functions. These polynomials were analytically studied by Ismail [15], Koelink and Van Assche [23], and Koelink [21]. In this paper we concentrate on the combinatorial aspect of these three sets of q-Lommel polynomials.

The literature already contains some combinatorial results on the quotient of Bessel functions and the quotient of q-Bessel functions. Delest and Fédou [9] showed that a generating function for parallelogram polyominoes can be written as a ratio of Jackson's third q-Bessel functions. Bousquet-Mélou and Viennot [4] generalized their result by adding one more parameter. A recounting of the history of the combinatorics of the q-analogue of the quotient of Bessel functions may be found in [3, Section 1] (see also [26, Section 4]). It includes results by Klarner and Rivest [20, Eq. (19)], Fédou [12], Lalanne [25, 26], Brak and Guttmann [5], and Barcucci et al. [1, Corollary 3.5], [2, Theorems 4.3 and 5.3].

In this paper we put these results in perspective by relating them to q-Lommel polynomials. The moment generating function has a continued fraction expansion. Using the general theory of orthogonal and type R_I polynomials we give finite versions of the infinite continued fractions. We show that a generating function for bounded diagonal parallelogram polyominoes is given by a ratio of q-Lommel polynomials, which is a finite version of the result of Bousquet-Mélou and Viennot [4].

Even though the Lommel polynomials have a hypergeometric representation as a ${}_2F_3$, they do not appear in the Askey scheme. In this paper we rectify this, by realizing two sets of q-Lommel polynomials as limiting cases of associated Askey–Wilson polynomials. One may ask for an associated Askey scheme which contains this limiting case (see Problem 8.7).

The paper is organized in the following way. In Section 2 we define the three sets of q-Lommel polynomials using three-term recurrence relations. The classical connection between these polynomials and q-Bessel functions is given in Section 3. The associated Askey–Wilson polynomials are reviewed in Section 4, along with explicit limiting cases to the q-Lommel polynomials; see Theorems 4.7 and 4.8. In Section 5 we independently prove the continued fraction expansions for the moment generating functions, and give two surprising equalities of continued fractions in Corollary 5.6 and Theorem 5.12. Combinatorial interpretations of these continued fractions are given in Section 6; see Theorem 6.9 and Corollary 6.11. A general combinatorial result for the concurrence of type R_I moments and orthogonal polynomials moments is given in Section 7; see Theorem 7.2. In Section 8 we propose some open problems.

We use the standard notations ${}_pF_q$ for hypergeometric series and ${}_p\phi_q$ for basic hypergeometric series (also sometimes called q-hypergeometric series) [14].

2. q-Lommel polynomials

In this section we give the defining recurrence relations for the Lommel, the classical q-Lommel, the even-odd q-Lommel, and the type R_I q-Lommel polynomials.

Definition 2.1. The *monic Lommel polynomials* $h_n(x;c)$ are defined by

$$h_{n+1}(x;c) = xh_n(x;c) - \frac{1}{(c+n)(c+n-1)}h_{n-1}(x;c), n \geq 0, \quad h_{-1}(x;c) = 0, h_0(x;c) = 1.$$

We consider three versions of q-Lommel polynomials.

Definition 2.2 ([15, § 14.4])**.** The *classical q-Lommel polynomials* are defined by

$$h_{n+1}(x;c,q) = xh_n(x;c,q) - \lambda_n h_{n-1}(x;c,q), \quad n \geq 0, \quad h_{-1}(x;c,q) = 0, \quad h_0(x;c,q) = 1,$$

$$\text{where} \quad \lambda_n = \frac{cq^{n-1}}{(1-cq^{n-1})(1-cq^n)}.$$

Definition 2.3. The *even-odd q-Lommel polynomials* are defined by

$$p_{n+1}(x;c,q) = xp_n(x;c,q) - \lambda_n p_{n-1}(x;c,q), \quad n \geq 0, \quad p_{-1}(x;c,q) = 0, \quad p_0(x;c,q) = 1,$$

$$\text{where} \quad \lambda_{2n} = \frac{cq^{3n-1}}{(1-cq^{2n-1})(1-cq^{2n})}, \qquad \lambda_{2n+1} = \frac{q^n}{(1-cq^{2n})(1-cq^{2n+1})}.$$

Note that each polynomial h_n and p_n may be considered as a q-analogue of the classical Lommel polynomials since

$$\lim_{q\to 1}(1-q)^n h_n(x/(1-q);q^c,q) = h_n(x;c), \quad \lim_{q\to 1}(1-q)^n p_n(x/(1-q);q^c,q) = h_n(x;c).$$

Definition 2.4. The *type R_I q-Lommel polynomials* are defined by

$$r_{n+1}(x;c,q) = (x-b_n)r_n(x;c,q) - xa_n r_{n-1}(x;c,q), \quad r_{-1}(x;c,q) = 0, \quad r_0(x;c,q) = 1, \tag{2.1}$$

$$\text{where} \quad b_n = \frac{q^n}{1-cq^n}, \qquad a_n = \frac{cq^{2n-1}}{(1-cq^{n-1})(1-cq^n)}.$$

Note that if

$$\hat{r}_n(x;c) = \lim_{q\to 1}(1-q)^{2n} r_n(x/(1-q)^2;q^c,q),$$

then

$$\hat{r}_{n+1}(x;c) = x\hat{r}_n(x;c) - \frac{x}{(c+n-1)(c+n)}\hat{r}_{n-1}(x;c). \tag{2.2}$$

The polynomials $\hat{r}_n(x;c)$ in (2.2) are closely related to the monic Lommel polynomials. For example, it is known that their moments are the same; see (7.2).

Koelink and Van Assche study the even-odd and the type R_I q-Lommel polynomials[1] in [23, Section 4], and Koelink continues this analytic study in [21].

Orthogonality relations for the classical q-Lommel are in [15, Theorem 14.4.3], while those for the even-odd q-Lommel and the type R_I q-Lommel are in [23, Theorem 4.2] and [23, Theorem 3.4].

3. q-Bessel functions and q-Lommel polynomials

In this section we give the recurrence relation which connects q-Bessel functions to the classical q-Lommel polynomials and the type R_I q-Lommel polynomials.

Definition 3.1. The *Bessel functions* $J_\nu(x)$ are defined by

$$J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu+1)} \sum_{n\geq 0} \frac{(-z^2/4)^n}{n!(\nu+1)_n}.$$

Definition 3.2 ([6, p. 188, (6.2)])**.** The *classical Lommel polynomials* $L_{n,\nu}(z)$ are (non-monic) polynomials in z^{-1} defined by $L_{0,\nu}(z) = 1$, $L_{1,\nu}(z) = 2\nu/z$, and

$$L_{n+1,\nu}(z) = \frac{2(n+\nu)}{z} L_{n,\nu}(z) - L_{n-1,\nu}(z), \qquad n \geq 1.$$

Equivalently,

$$h_n(x;c) = L_{n,c}(2/x)/(c)_n.$$

[1] Continued fractions of type R and the corresponding orthogonal polynomials of type R_I or R_{II} were introduced by Ismail and Masson in [16], the notation being a mnemonic for rational interpolation.

The Bessel function satisfy $J_{\nu+1}(z) = \frac{2\nu}{z} J_\nu(z) - J_{\nu-1}(z)$. Iterating this recurrence offers the following connection with Lommel polynomials.

Proposition 3.3 ([6, p. 187])**.** *The Bessel functions and the classical Lommel polynomials are related by the recurrence*

$$J_{\nu+n}(z) = L_{n,\nu}(z) J_\nu(z) - L_{n-1,\nu+1}(z) J_{\nu-1}(z).$$

Definition 3.4. *Jackson's first q-Bessel function* $J_\nu^{(1)}(z;q)$ and *second q-Bessel function* $J_\nu^{(2)}(z;q)$ are defined by

$$J_\nu^{(1)}(z;q) = \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} (z/2)^\nu {}_2\phi_1 \left(0, 0; q^{\nu+1}; q, -z^2/4\right),$$
$$J_\nu^{(2)}(z;q) = \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} (z/2)^\nu {}_0\phi_1 \left(-; q^{\nu+1}; q, -q^{\nu+1} z^2/4\right).$$

In this paper we consider only the first and third q-Bessel function, as the second q-Bessel can be obtained from the first by changing q to q^{-1}. Recall that we consider formal power series in z, and have no restriction on q.

Proposition 3.5 ([15, Eq. (14.4.1)])**.** *The first q-Bessel functions satisfy*

$$q^{n\nu+\binom{n}{2}} J_{\nu+n}^{(1)}(x;q) = L_{n,\nu}^{(1)}(x;q) J_\nu^{(1)}(x;q) - L_{n-1,\nu+1}^{(1)}(x;q) J_{\nu-1}^{(1)}(x;q). \tag{3.1}$$

where $L_{0,\nu}^{(1)}(x;q) = 1$, $L_{1,\nu}^{(1)}(x;q) = 2(1-q^{n+\nu})/x$, *and*

$$\frac{2}{x}(1-q^{n+\nu}) L_{n,\nu}^{(1)}(x;q) = L_{n+1,\nu}^{(1)}(x;q) + q^{n+\nu-1} L_{n-1,\nu}^{(1)}(x;q), \qquad n \geq 1.$$

Again, we need a rescaling to obtain the classical q-Lommel polynomials,

$$h_n(x;c,q) = L_{n,\nu}^{(1)}(2/x;q)/(q^\nu;q)_n, \quad c = q^\nu.$$

Definition 3.6. The *Jackson's third q-Bessel functions* $J_\nu^{(3)}(z;q)$ are defined by

$$J_\nu^{(3)}(z;q) = \frac{(q^{\nu+1};q)_\infty z^\nu}{(q;q)_\infty} {}_1\phi_1 \left(0; q^{\nu+1}; q, qz^2\right).$$

Define the Laurent polynomials $L_{m,\nu}^{(3)}(z;q)$ by

$$L_{m+1,\nu}^{(3)}(z;q) = \left(z + z^{-1}(1-q^{\nu+m})\right) L_{m,\nu}^{(3)}(z;q) - L_{m-1,\nu}^{(3)}(z;q).$$

We rescale these Laurent polynomials to obtain polynomials

$$r_n^{(3)}(x;c,q) := \frac{x^{n/2}}{(q^{-\nu};q^{-1})_n} L_{n,\nu}^{(3)}(x^{-1/2};q^{-1}), \quad c = q^\nu. \tag{3.2}$$

Then $r_n^{(3)}(x;c,q)$ are the type R_I polynomials defined by $r_{-1}^{(3)}(x;c,q) = 0$, $r_0^{(3)}(x;c,q) = 1$, and

$$r_{n+1}^{(3)}(x;c,q) = (x - \hat{b}_n) r_n^{(3)}(x;c,q) - x\hat{a}_n r_{n-1}^{(3)}(x;c,q), \qquad n \geq 0,$$
$$\text{where } \hat{b}_n = \frac{cq^n}{1-cq^n}, \qquad \hat{a}_n = \frac{c^2 q^{2n-1}}{(1-cq^{n-1})(1-cq^n)}.$$

Using the recurrences one can easily check that

$$r_n(x;c,q) = \frac{r_n^{(3)}(cx;c,q)}{c^n},$$

where $r_n(x;c,q)$ are the type R_I q-Lommel polynomials $r_n(x;c,q)$ in Definition 2.4.

Koelink and Swarttouw [22, Eq. (4.12)] showed that the third q-Bessel functions satisfy the following property analogous to (3.3) and (3.1).

Proposition 3.7. *The third q-Bessel functions satisfy*

$$J^{(3)}_{\nu+m}(z;q) = L^{(3)}_{m,\nu}(z;q)J^{(3)}_{\nu}(z;q) - L^{(3)}_{m-1,\nu+1}(z;q)J^{(3)}_{\nu-1}(z;q).$$

Koelink and Swarttouw [22, Eq. (4.24)] also showed that

$$\lim_{m\to\infty} z^m R^{(3)}_{m,\nu}(z;q) = \frac{(q;q)_\infty z^{1-\nu}}{(z^2;q)_\infty} J^{(3)}_{\nu-1}(z;q),$$

which implies

$$\lim_{m\to\infty} \frac{R^{(3)}_{m,\nu+2}(z;q)}{R^{(3)}_{m+1,\nu+1}(z;q)} = \frac{J^{(3)}_{\nu+1}(z;q)}{J^{(3)}_{\nu}(z;q)}. \tag{3.3}$$

By (3.2) and (3.3) we have

$$\frac{J^{(3)}_{\nu+1}(x^{1/2};q^{-1})}{J^{(3)}_{\nu}(x^{1/2};q^{-1})} = \lim_{n\to\infty} \frac{-q^{\nu+1} r^{(3)}_n(x^{-1};q^{\nu+2},q)}{x^{1/2}(1-q^{\nu+1}) r^{(3)}_{n+1}(x^{-1};q^{\nu+1},q)}. \tag{3.4}$$

The q-Bessel function relation for the even-odd q-Lommel polynomials which corresponds to Proposition 3.5 is given in [23, Proposition 4.1].

4. q-Lommel polynomials and the Askey scheme

The q-Lommel polynomials do not appear in the Askey scheme. In this section we realize both the classical q-Lommel and the even-odd q-Lommel polynomials as limiting cases of the associated Askey–Wilson polynomials; see Theorems 4.7 and 4.8. We then use results of Ismail and Masson [16] to give explicit formulas for each polynomial. Finally, we prove that the moments for even-odd q-Lommel and the type R_I q-Lommel agree; see Theorem 4.14.

An explicit formula for the Lommel polynomial $h_n(x;c)$ is

$$h_n(x;c) = x^n {}_2F_3(-n/2,(1-n)/2;c,1-c-n,-n;-4/x^2).$$

In this section we give explicit formulas for our three families of q-Lommel polynomials. The classical q-Lommel polynomials have a corresponding single sum formula [15, Theorem 14.4.1]:

$$h_n(x;c,q) = \frac{1}{(c;q)_n} \sum_{j=0}^{\lfloor n/2\rfloor} \frac{(-1)^j\,(c,q;q)_{n-j}}{(q,c;q)_j\,(q;q)_{n-2j}} x^{n-2j} c^j q^{j(j-1)}.$$

Here are the main results for the even-odd q-Lommel polynomials.

Theorem 4.1. *The even even-odd q-Lommel polynomials have the explicit formula*

$$\begin{aligned} p_{2n}(x;c,q) = (-1)^n \frac{q^{\binom{n}{2}}}{(c;q)_{2n}} \sum_{k=0}^{n} \frac{(q^{-n},cq^n,c;q)_k}{(q;q)_k} q^k x^{2k} \\ \times \sum_{s=0}^{n-k} \frac{(cq^{k-1};q)_s}{(q;q)_s} \frac{1-cq^{k-1+2s}}{1-cq^{k-1}} \frac{(cq^{n+k},q^{k-n},q^k;q)_s}{(q^{-n},cq^n,c;q)_s} c^s q^{-sk+s(s-1)}. \end{aligned}$$

Theorem 4.2. *The odd even-odd q-Lommel polynomials have the explicit formula*

$$p_{2n+1}(x;c,q) = (-c)^n \frac{q^{n^2+\binom{n+1}{2}}}{(cq;q)_{2n}} \sum_{k=0}^{n} \frac{(q^{-n}, cq^{n+1}, cq;q)_k}{(q;q)_k} c^{-k} q^{-k^2} x^{2k+1}$$
$$\times \sum_{s=0}^{n-k} \frac{(cq^k;q)_s}{(q;q)_s} \frac{1-cq^{k+2s}}{1-cq^k} \frac{(cq^{n+k+1}, q^{k-n}, q^{k+1};q)_s}{(q^{-n}, cq^{n+1}, c;q)_s} c^s q^{-(3k+2)s-s(s-1)}.$$

Note that the inner sums in Theorems 4.1 and 4.2 are basic hypergeometric series. The coefficient of x^{2k} in Theorem 4.1 and x^{2k+1} in Theorem 4.2 are terminating basic hypergeometric series.

In order to prove Theorems 4.1 and 4.2, we first write the even even-odd polynomials as orthogonal polynomials in x^2 using the odd-even trick. Then we realize the new polynomials as limiting cases of *associated Askey–Wilson polynomials*, for which explicit formulas are known. The same method will work for the odd even-odd polynomials.

We begin with the associated Askey–Wilson polynomials. First, recall that the *monic Askey–Wilson polynomials* satisfy the following three-term recurrence (which, as claimed by Favard's theorem, is characterizing orthogonal polynomials):

$$p_{n+1}(x) = (x-b_n)p_n(x) - \lambda_n p_{n-1}(x), \quad n \geq 1, \tag{4.1}$$

where

$$b_n = \frac{1}{2}(a + a^{-1} - A_n - C_n), \qquad \lambda_n = \frac{1}{4}A_{n-1}C_n,$$
$$A_n = \frac{(1-abq^n)(1-acq^n)(1-adq^n)(1-abcdq^{n-1})}{a(1-abcdq^{2n-1})(1-abcdq^{2n})},$$
$$C_n = \frac{a(1-q^n)(1-bcq^{n-1})(1-bdq^{n-1})(1-cdq^{n-1})}{(1-abcdq^{2n-2})(1-abcdq^{2n-1})}.$$

Then, the *associated Askey–Wilson polynomials* are defined by the same three-term recurrence relation (4.1), with q^n replaced by αq^n.

Definition 4.3. The *associated Askey–Wilson polynomials* $p_n^{(\alpha)}(x)$ are defined as a solution to

$$p_{n+1}^{(\alpha)}(x) = (x - b_n(\alpha))p_n^{(\alpha)}(x) - \lambda_n(\alpha)p_{n-1}^{(\alpha)}(x), \quad n \geq 1, \tag{4.2}$$

$$b_n(\alpha) = \frac{1}{2}(a + a^{-1} - A_n(\alpha) - C_n(\alpha)), \qquad \lambda_n(\alpha) = \frac{1}{4}A_{n-1}(\alpha)C_n(\alpha),$$

$$A_n(\alpha,q) = \frac{(1-ab\alpha q^n)(1-ac\alpha q^n)(1-ad\alpha q^n)(1-abcd\alpha q^{n-1})}{a(1-abcd\alpha^2 q^{2n-1})(1-abcd\alpha^2 q^{2n})},$$
$$C_n(\alpha,q) = \frac{a(1-\alpha q^n)(1-bc\alpha q^{n-1})(1-bd\alpha q^{n-1})(1-cd\alpha q^{n-1})}{(1-abcd\alpha^2 q^{2n-2})(1-abcd\alpha^2 q^{2n-1})}.$$

There are two linearly independent solutions to (4.2), depending on the initial conditions. Ismail and Rahman [15, Eq. (4.15), Eq. (8.9)] gave these two independent solutions as double sums, the inner sum being a very-well-poised ${}_{10}W_9$.

Theorem 4.4. *Two linearly independent solutions $\psi_n^{(\alpha,\epsilon)}(x,q), \epsilon = 1,2$ to (4.2) are given by*

$$\psi_n^{(\alpha,\epsilon)}(x;q) = K_n \sum_{k=0}^{n} \frac{(q^{-n}, abcd\alpha^2 q^{n-1}, abcd\alpha^2/q, az, a/z; q)_k}{(q, ab\alpha, ac\alpha, ad\alpha, abcd\alpha/q; q)_k} q^k$$
$$\times\, {}_{10}W_9\left(abcd\alpha^2 q^{k-2}; \alpha, bc\alpha/q, bd\alpha/q, cd\alpha/q, S, abcd\alpha^2 q^{n+k-1}, q^{k-n}; q; T\right)$$

where

$$K_n = (2a)^{-n} \frac{(ab\alpha, ac\alpha, ad\alpha, abcd\alpha/q; q)_n}{(abcd\alpha^2 q^{n-1}, abcd\alpha^2/q; q)_n}$$

and the two choices for ϵ correspond to

$$(S,T) = (q^{k+1}, a^2), \text{ for } \epsilon = 1,\ (S,T) = (q^k, qa^2), \text{ for } \epsilon = 2.$$

We next explain how Theorem 4.1 follows from Theorem 4.4. First we rewrite the recurrence relation [6] in terms of polynomials in x^2. Let $t_n(x)$ and $s_n(x)$ be the polynomials satisfying

$$p_{2n}(x;c,q) = t_n(x^2),$$
$$p_{2n+1}(x;c,q) = x s_n(x^2).$$

Proposition 4.5. *We have*

$$t_{n+1}(x) = (x - B_n)t_n(x) - \Lambda_n t_{n-1}(x), \quad t_{-1} = 0, \quad t_0(x) = 1.$$

where

$$B_0 = \frac{1}{(1-c)(1-cq)},$$
$$B_n = \lambda_{2n} + \lambda_{2n+1}, \quad n \ge 1,$$
$$\Lambda_n = \lambda_{2n-1}\lambda_{2n}, \quad n \ge 1.$$

Proposition 4.6. *We have*

$$s_{n+1}(x) = (x - B_n)s_n(x) - \Lambda_n s_{n-1}(x), \quad s_{-1} = 0, \quad s_0(x) = 1.$$

where

$$B_n = \lambda_{2n+2} + \lambda_{2n+1}, \quad n \ge 1,$$
$$\Lambda_n = \lambda_{2n+1}\lambda_{2n}, \quad n \ge 1.$$

We shall obtain the recurrence relations in Propositions 4.5 and 4.6 by an appropriate limiting case of Theorem 4.4. Our goal is to obtain $(A_n, C_n) = (\lambda_{2n+1}, \lambda_{2n})$ for $t_n(x)$ and $(A_n, C_n) = (\lambda_{2n+2}, \lambda_{2n+1})$ for $s_n(x)$. Then we match the initial conditions to find the correct linear combination of the two solutions.

First choosing $a = c^{-1}q^{-1}\alpha$, $b = c = d = 1/\alpha$, we obtain

$$A_n(\alpha, 1/q) = \frac{\alpha(1 - cq^{n+1}/\alpha)^3(1 - \alpha c q^n)}{cq(1 - cq^{2n})(1 - cq^{2n+1})},$$
$$C_n(\alpha, 1/q) = \frac{cq(1 - q^n/\alpha)(1 - \alpha q^{n-1})^3}{\alpha(1 - cq^{2n-1})(1 - cq^{2n})}.$$

By rescaling x by $B\alpha^2 x/2$, i.e., $\hat{p}_n(x) = 2^n \alpha^{-2n} B^{-n} p_n^{(\alpha)}(B\alpha^2 x/2)$, we have

$$\hat{p}_{n+1}(x) = (x - \hat{b}_n(\alpha))\hat{p}_n(x) - \hat{\lambda}_n(\alpha)\hat{p}_{n-1}(x),$$

$$\hat{b}_n(\alpha) = \frac{1}{B\alpha^2}\left(\frac{cq}{\alpha} + \frac{\alpha}{cq} - A_n(\alpha, 1/q) - C_n(\alpha, 1/q)\right),$$
$$\hat{\lambda}_n(\alpha) = \frac{1}{B^2\alpha^4} A_n(\alpha, 1/q) C_n(\alpha, 1/q).$$

If $\alpha \to \infty$, the first two terms in $\hat{b}_n(\alpha)$ vanish. Choosing $B = 1/q$, we obtain the desired values for Proposition 4.5

$$\lim_{\alpha\to\infty} \frac{-1}{B\alpha^2} A_n(\alpha, 1/q) = \frac{q^n}{(1-cq^{2n})(1-cq^{2n+1})} = \lambda_{2n+1},$$
$$\lim_{\alpha\to\infty} \frac{-1}{B\alpha^2} C_n(\alpha, 1/q) = \frac{cq^{3n-1}}{(1-cq^{2n-1})(1-cq^{2n})} = \lambda_{2n}.$$

The first degree limiting polynomial matches the second Ismail–Rahman solution in Theorem 4.4 with $(a, b, c, d) = (\alpha/cq, 1/\alpha, 1/\alpha, 1/\alpha)$,

$$x - \frac{1}{(1-c)(1-cq)}$$

so that

$$\lim_{\alpha\to\infty} \hat{p}_n(x) = \lim_{\alpha\to\infty} \psi_n^{(\alpha,2)}(x; 1/q),$$

which is the stated explicit formula in Theorem 4.1.

For the odd even-odd polynomials in Proposition 4.6, we choose $(a, b, c, d) = (cq^2\alpha, 1/\alpha, 1/\alpha, 1/\alpha)$,

$$A_n(\alpha, q) = \frac{(1-c\alpha q^{n+2})^3(1-cq^{n+1}/\alpha)}{\alpha cq^2(1-cq^{2n+1})(1-cq^{2n+2})},$$
$$C_n(\alpha, q) = \frac{\alpha cq^2(1-\alpha q^n)(1-q^{n-1}/\alpha)^3}{(1-cq^{2n})(1-cq^{2n+1})}.$$

As before choosing $\hat{p}_n(x) = 2^n \alpha^{-2n} B^{-n} p_n^{(\alpha)}(B\alpha^2 x/2)$ and $B = -cq^2$ we find

$$\lim_{\alpha\to\infty} \frac{1}{B\alpha^2} A_n(\alpha, q) = \frac{cq^{3n+2}}{(1-cq^{2n+1})(1-cq^{2n+2})} = \lambda_{2n+2},$$
$$\lim_{\alpha\to\infty} \frac{1}{B\alpha^2} C_n(\alpha, q) = \frac{q^n}{(1-cq^{2n})(1-cq^{2n+1})} = \lambda_{2n+1}.$$

The first degree limiting polynomial matches the first Ismail–Rahman solution in Theorem 4.4 with $(a, b, c, d) = (cq^2\alpha, 1/\alpha, 1/\alpha, 1/\alpha)$,

$$x - \frac{1+cq}{(1-c)(1-cq^2)}$$

so that

$$\lim_{\alpha\to\infty} \hat{p}_n(x) = \lim_{\alpha\to\infty} \psi_n^{(\alpha,1)}(x; q),$$

which is the stated explicit formula in Theorem 4.2.

We summarize these limits for the even-odd q-Lommel polynomials.

Theorem 4.7. *The even-odd q-Lommel polynomials are the following limits of associated Askey–Wilson polynomials*

$$p_{2n}(x;c,q) = \lim_{\alpha\to\infty} \frac{(2q)^n}{\alpha^{2n}} \psi_n^{(\alpha,2)}(\alpha^2x^2/2q;1/q), \quad (a,b,c,d) = (\alpha/cq, 1/\alpha, 1/\alpha, 1/\alpha),$$

$$p_{2n+1}(x;c,q) = x \lim_{\alpha\to\infty} \frac{(-2/cq^2)^n}{\alpha^{2n}} \psi_n^{(\alpha,1)}(-cq^2\alpha^2x^2/2;q), \; (a,b,c,d) = (cq^2\alpha, 1/\alpha, 1/\alpha, 1/\alpha).$$

For the classical q-Lommel polynomials $h_n(x;c;q)$, for the even polynomials choose

$$(a,b,c,d) = (1, q/\alpha^2, c, 1)$$

and for the odd polynomials choose

$$(a,b,c,d) = (1, q^2/\alpha^2, c, 1).$$

Similar calculations to the proof of Theorem 4.7 show the next result.

Theorem 4.8. *The classical q-Lommel polynomials are the following limits of associated Askey–Wilson polynomials*

$$h_{2n}(x;c,q) = \lim_{\alpha\to\infty} \frac{(-2)^n}{\alpha^{2n}} \psi_n^{(\alpha,2)}(-\alpha^2x^2/2;q), \quad (a,b,c,d) = (1, q/\alpha^2, c, 1),$$

$$h_{2n+1}(x;c,q) = x \lim_{\alpha\to\infty} \frac{(-2q)^n}{\alpha^{2n}} \psi_n^{(\alpha,1)}(-\alpha^2x^2/2q;q), \quad (a,b,c,d) = (1, q^2/\alpha^2, c, 1).$$

Theorem 4.9 is [15, Theorem 14.4.1].

Theorem 4.9. *The classical q-Lommel polynomials are*

$$h_n(x;c,q) = \sum_{k=0}^{n/2} \begin{bmatrix} n-k \\ k \end{bmatrix}_q \frac{(-c)^k q^{k^2-k}}{(c;q)_k (cq^{n-1};q^{-1})_k} x^{n-2k}.$$

Proof. We consider the even case, the proof for the odd case is similar. The inner sum becomes an evaluable very-well-poised ${}_6W_5$

$${}_6W_5\left(cq^{k-1}; q^k, cq^{n+k}, q^{k-n} \;\middle|\; q; q^{-2k}\right) = \frac{(cq^{n-1};q^{-1})_k (q^{n+1};q)_k}{(c;q)_k (q^{k+1};q)_k} q^{-k(n-k)}.$$

By considering the coefficient of x^{2n-2k}, we arrive at Theorem 4.9 with n replaced by $2n$. The odd case actually gives the same result. □

For the type R_I q-Lommel polynomials there is a simple generating function which gives an explicit expression.

Proposition 4.10. *The type R_I q-Lommel polynomials have the generating function*

$$\sum_{n=0}^{\infty} (c^{-1};q^{-1})_n \; r_n(x;c,q) t^n = \sum_{k=0}^{\infty} \frac{(-xt/c)^k q^{-\binom{k}{2}}}{(t/c, tx; q^{-1})_{k+1}}.$$

Proof. If $G(x,t)$ is the generating function on the left side, then Definition 2.4 implies

$$G(x,t) - 1 = (x + 1/c)tG(x,t) - xt/c\; G(x,tq^{-1}) - xt^2/c\; G(x,t),$$

$$G(x,t) = \frac{1}{(1-xt)(1-t/c)} - \frac{xt/c}{(1-xt)(1-t/c)} G(x,tq^{-1})$$

whose iterate is the result. □

Theorem 4.11. *The type R_I q-Lommel polynomials have the explicit formula*

$$r_n(x;c,q) = \frac{1}{(c^{-1};q^{-1})_n} \sum_{k=0}^{n} \sum_{a=0}^{n-k} (-x/c)^k q^{-\binom{k}{2}} \begin{bmatrix} k+a \\ a \end{bmatrix}_{q^{-1}} c^{-a} \begin{bmatrix} n-a \\ k \end{bmatrix}_{q^{-1}} x^{n-k-a}.$$

Proof. Apply the q^{-1}-binomial theorem to Proposition 4.10 to find the resulting coefficient of t^n. □

Proposition 4.12. *We have the connection coefficient relation*

$$r_n(x^2;c,q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{c^k q^{n^2-(n-k)^2}}{(cq^{n-1}, cq^{2n-k}; q^{-1})_k} p_{2n-2k}(x;c,q).$$

Proof. Induction on n using the three-term relations. □

Proposition 4.13. *If $\mathcal{L}_p$ is the linear functional for the even-odd polynomials $p_n(x;c,q)$, then*

$$\mathcal{L}_p(r_n(x^2;c,q)) = \frac{c^n q^{n^2}}{(c,cq;q)_n}.$$

Proof. Apply $\mathcal{L}_p$ to both sides of Proposition 4.12. By orthogonality, $\mathcal{L}_p(p_j(x)) = 0$ for $j > 0$, so only the $k = n$ term survives. □

Theorem 4.14. *The moments of the type R_I q-Lommel polynomials are equal to the even moments of the even-odd q-Lommel polynomials,*

$$\mathcal{L}_r(x^m) = \mathcal{L}_p(x^{2m}), \quad m \geq 0.$$

Proof. Using the three-term recurrence (2.1), the type R_I moments $\mathcal{L}_r(x^m)$ are recursively determined by [18, Corollary 3.15]

$$\mathcal{L}_r(r_n(x;c,q)) = a_1 a_2 \cdots a_n = \frac{c^n q^{n^2}}{(c,cq;q)_n}, \quad n \geq 0.$$

By Proposition 4.13 the moments $\mathcal{L}_p(x^{2m})$ satisfy the same recurrence. □

For completeness, we give the inverse relation to Proposition 4.12.

Proposition 4.15. *We have the connection coefficient relation*

$$p_{2n}(x;c,q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-c)^k q^{2nk-\binom{k+1}{2}}}{(cq^{n-1}, cq^{2n-1}; q^{-1})_k} r_{n-k}(x^2;c,q).$$

Proposition 4.16. *The even-odd q-Lommel polynomials have the explicit expressions*

$$p_{2n}(x;c,q) = \frac{1}{(c;q)_{2n}} \sum_{k=0}^{n} (-1)^k x^{2n-2k} (cq^k;q)_{2n-2k} \sum_{j=0}^{k} \begin{bmatrix} n-j \\ k-j \end{bmatrix}_q \begin{bmatrix} n-k+j-1 \\ j \end{bmatrix}_q c^j q^{jn+\binom{k}{2}},$$

$$p_{2n+1}(x;c,q) = \frac{1}{(c;q)_{2n+1}} \sum_{k=0}^{n} (-1)^k x^{2n-2k+1} (cq^k;q)_{2n-2k+1} \sum_{j=0}^{k} \begin{bmatrix} n-j \\ k-j \end{bmatrix}_q \begin{bmatrix} n-k+j \\ j \end{bmatrix}_q c^j q^{jn+\binom{k}{2}}.$$

Proof. This follows from Definition 2.3, by considering the coefficients of $x^{2n-2k-1}$. □

5. Moments and continued fractions

In this section we review the known facts which connect continued fractions to moment generating functions. We independently prove the continued fractions for the moment generating functions of each of the three q-Lommel polynomials.

Definition 5.1. Take a sequence of orthogonal polynomials $p_n(x)$ which satisfy $p_{-1}(x) = 0$, $p_0(x) = 1$, and

$$p_{n+1}(x) = (x - b_n)p_n(x) - \lambda_n p_{n-1}(x), \quad n \geq 0,$$

and whose linear functional for orthogonality is $\mathcal{L}_p$. Define

$$\mu_n(\{b_k\}_{k\geq 0}, \{\lambda_k\}_{k\geq 0}) = \mathcal{L}_p(x^n).$$

The moment generating function for $\mathcal{L}_p$ is

$$\sum_{n=0}^{\infty} \mathcal{L}_p(x^n)t^n = \sum_{n=0}^{\infty} \mu_n(\{b_k\}_{k\geq 0}, \{\lambda_k\}_{k\geq 0})t^n.$$

A Jacobi continued fraction also exists, converging as formal power series in t:

$$\sum_{n=0}^{\infty} \mathcal{L}_p(x^n)t^n = \cfrac{1}{1 - b_0 t - \cfrac{\lambda_1 t^2}{1 - b_1 t - \cfrac{\lambda_2 t^2}{1 - \ddots}}}. \tag{5.1}$$

Definition 5.2 ([18])**.** For general type R_I orthogonal polynomials

$$r_{n+1}(x) = (x - b_n)r_n(x) - (a_n x + \lambda_n)r_{n-1}(x), \quad n \geq 0,$$

with linear functional $\mathcal{L}_r$, define

$$\mu_n(\{b_k\}_{k\geq 0}, \{a_k\}_{k\geq 0}, \{\lambda_k\}_{k\geq 0}) = \mathcal{L}_r(x^n). \tag{5.2}$$

The corresponding continued fraction for the type R_I moment generating function is [18, Corollary 3.7]

$$\sum_{n=0}^{\infty} \mathcal{L}_r(x^n)t^n = \cfrac{1}{1 - b_0 t - \cfrac{a_1 t + \lambda_1 t^2}{1 - b_1 t - \cfrac{a_2 t + \lambda_2 t^2}{1 - \ddots}}}. \tag{5.3}$$

Note that both continued fractions in (5.1) and (5.3) are explicitly given in terms of the three-term recurrence coefficients. We shall evaluate the continued fractions as quotients of basic hypergeometric series, namely q-Bessel functions, using contiguous relations.

For the Lommel polynomials $h_n(x;c)$, it is known that the moment generating function is a quotient of Bessel functions, with $\lambda_n = 1/(c+n-1)(c+n)$,

$$\sum_{n=0}^{\infty} \mathcal{L}_h(x^n)t^n = \frac{{}_0F_1(c+1; -t^2)}{{}_0F_1(c; -t^2)} = \cfrac{1}{1 - \cfrac{\lambda_1 t^2}{1 - \cfrac{\lambda_2 t^2}{1 - \ddots}}}.$$

The moment generating function for the classical q-Lommel polynomials is a quotient of q-Bessel functions. In this section we shall see that a corresponding result holds for our other two q-Lommel polynomials, and in fact their moment generating functions are equal.

Theorem 5.3 ([15, Theorem 14.4.3])**.** *The moment generating function for the classical q-Lommel polynomials $h_n(x;c,q)$ is a quotient of Jackson's first q-Bessel functions*

$$\sum_{n=0}^{\infty}\mathcal{L}_h(x^n)t^n=\frac{{}_2\phi_1\left(0,0;cq;q,-t^2\right)}{{}_2\phi_1\left(0,0;c;q;-t^2\right)}=\cfrac{1}{1-\cfrac{\lambda_1t^2}{1-\cfrac{\lambda_2t^2}{1-\ddots}}},$$

where, as in Definition 2.2, $\lambda_n=cq^{n-1}/((1-cq^{n-1})(1-cq^n))$.

Theorem 5.4. *The moment generating function for the even-odd q-Lommel polynomials $p_n(x;c,q)$ is a quotient of Jackson's third q-Bessel functions*

$$\sum_{n=0}^{\infty}\mathcal{L}_p(x^n)t^n=\frac{{}_1\phi_1\left(0;cq;q;qt^2\right)}{{}_1\phi_1\left(0;c;q;t^2\right)}=\cfrac{1}{1-\cfrac{\lambda_1t^2}{1-\cfrac{\lambda_2t^2}{1-\ddots}}},$$

where, as in Definition 2.3,

$$\lambda_{2n}=\frac{cq^{3n-1}}{(1-cq^{2n-1})(1-cq^{2n})},\qquad \lambda_{2n+1}=\frac{q^n}{(1-cq^{2n})(1-cq^{2n+1})}.$$

Theorem 5.5. *The moment generating function for the type R_I q-Lommel polynomials $r_n(x;c,q)$ is a quotient of Jackson's third q-Bessel functions*

$$\sum_{n=0}^{\infty}\mathcal{L}_r(x^n)z^n=\frac{{}_1\phi_1\left(0;cq;q;qz\right)}{{}_1\phi_1\left(0;c;q;z\right)}=\cfrac{1}{1-b_0z-\cfrac{a_1z}{1-b_1z-\cfrac{a_2z}{1-b_2z-\ddots}}},$$

where, as in Definition 2.4,

$$a_n=\frac{cq^{2n-1}}{(1-cq^{n-1})(1-cq^n)},\qquad b_n=\frac{q^n}{1-cq^n}.$$

Theorem 4.14 implies that the two continued fractions in Theorems 5.4 and 5.5 with $z=t^2$ are equal.

Corollary 5.6. *We have the equality of continued fractions*

$$\cfrac{1}{1-b_0z-\cfrac{a_1z}{1-b_1z-\cfrac{a_2z}{1-b_2z-\ddots}}}=\cfrac{1}{1-\cfrac{\lambda_1z}{1-\cfrac{\lambda_2z}{1-\ddots}}},$$

where a_n, b_n, and λ_n are defined as in Theorems 5.4 and 5.5.

Theorems 5.3, 5.4, and 5.5 may all be proven using contiguous relations for hypergeometric and basic hypergeometric series. To prove Theorems 5.3 and 5.4 we use Heine's contiguous relation [10, Eq. 17.6.19] which is

$$ {}_2\phi_1\left(aq, b; cq; q, z\right) - {}_2\phi_1\left(a, b; c; q, z\right) = \frac{(1-b)(a-c)z}{(1-c)(1-cq)} {}_2\phi_1\left(aq, bq; cq^2; q, z\right). $$

Equivalently,

$$ \frac{{}_2\phi_1\left(aq, b; cq; q, z\right)}{{}_2\phi_1\left(a, b; c; q, z\right)} = \frac{1}{1 - \dfrac{(1-b)(a-c)z}{(1-c)(1-cq)} \cdot \dfrac{{}_2\phi_1\left(bq, aq; cq^2; q, z\right)}{{}_2\phi_1\left(b, aq; cq; q, z\right)}}. \tag{5.4} $$

Applying (5.4) iteratively, we obtain Heine's continued fraction, which is a q-analogue of Gauss's continued fraction.

Lemma 5.7 (Heine's fraction)**.** *We have*

$$ \frac{{}_2\phi_1\left(aq, b; cq; q, z\right)}{{}_2\phi_1\left(a, b; c; q, z\right)} = \cfrac{1}{1 - \cfrac{\beta_1 z}{1 - \cfrac{\beta_2 z}{1 - \ddots}}}, $$

where

$$ \beta_{2n+1} = \frac{(1-bq^n)(a-cq^n)q^n}{(1-cq^{2n})(1-cq^{2n+1})}, \qquad \beta_{2n} = \frac{(1-aq^n)(b-cq^n)q^{n-1}}{(1-cq^{2n-1})(1-cq^{2n})}. $$

Theorem 5.3 is the special case $a = b = 0$ and $z = -t^2$ of Lemma 5.7. Theorem 5.4 is also the limiting case $z = -t^2/a$, $b = 0$, $a \to \infty$ of Lemma 5.7.

For Theorem 5.5 we need the q-Nörlund fraction [8, Eq. (19.2.7)]. However, to simplify the expressions we need some notation for continued fractions.

Definition 5.8. For sequences a_i and b_i, let

$$ \mathop{\mathbf{K}}_{i=0}^{m}\left(\frac{a_i}{b_i}\right) = \cfrac{a_0}{b_0 + \cfrac{a_1}{b_1 + \ddots + \cfrac{a_m}{b_m}}}, \qquad \mathop{\mathbf{K}}_{i=0}^{\infty}\left(\frac{a_i}{b_i}\right) = \cfrac{a_0}{b_0 + \cfrac{a_1}{b_1 + \ddots}}. $$

The following lemma will be used later.

Lemma 5.9. *For any sequences* $\{a_i : 0 \le i \le m\}$, $\{b_i : 0 \le i \le m\}$, *and* $\{c_i : -1 \le i \le m\}$, *we have*

$$ \mathop{\mathbf{K}}_{i=0}^{m}\left(\frac{a_i}{b_i}\right) = \frac{1}{c_{-1}} \mathop{\mathbf{K}}_{i=0}^{m}\left(\frac{a_i c_{i-1} c_i}{b_i c_i}\right). $$

Proof. By multiplying the numerator and denominator of the i^{th} fraction by c_i, we obtain

$$ \cfrac{a_0}{b_0 + \cfrac{a_1}{b_1 + \ddots + \cfrac{a_m}{b_m}}} = \cfrac{a_0 c_0}{b_0 c_0 + \cfrac{a_1 c_0 c_1}{b_1 c_1 + \ddots + \cfrac{a_m c_{m-1} c_m}{b_m c_m}}}, $$

which is equivalent to the equation in the lemma. □

Lemma 5.10 (q-Nörlund fraction)**.** *We have*

$$\frac{{}_2\phi_1\left(a,b;c;q,z\right)}{{}_2\phi_1\left(aq,bq;cq;q,z\right)} = \frac{1-c-(a+b-ab-abq)z}{1-c} + \frac{1}{1-c}\mathop{\mathbf{K}}_{m=1}^{\infty}\left(\frac{c_m(z)}{e_m+d_m z}\right),$$

where

$$\begin{aligned} c_m(z) &= (1-aq^m)(1-bq^m)(cz-abq^m z^2)q^{m-1},\\ e_m &= 1-cq^m,\\ d_m &= -(a+b-abq^m-abq^{m+1})q^m. \end{aligned}$$

The q-Nörlund fraction can be restated in the form of a continued fraction for type R_I orthogonal polynomials.

Proposition 5.11 (q-Nörlund fraction restated)**.** *We have*

$$\frac{{}_2\phi_1\left(aq,bq;cq;q,z\right)}{{}_2\phi_1\left(a,b;c;q,z\right)} = \cfrac{1}{1-b_0 z-\cfrac{a_1 z+\lambda_1 z^2}{1-b_1 z-\cfrac{a_2 z+\lambda_2 z^2}{1-b_2 z-\ddots}}},$$

where

$$\begin{aligned} b_m &= \frac{(a+b-abq^m-abq^{m+1})q^m}{1-cq^m},\\ a_m &= -\frac{(1-aq^m)(1-bq^m)cq^{m-1}}{(1-cq^{m-1})(1-cq^m)},\\ \lambda_m &= \frac{(1-aq^m)(1-bq^m)abq^{2m-1}}{(1-cq^{m-1})(1-cq^m)}. \end{aligned}$$

Proof. By taking the inverse on each side of the equation in Lemma 5.10, we obtain

$$\frac{{}_2\phi_1\left(aq,bq;cq;q,z\right)}{{}_2\phi_1\left(a,b;c;q,z\right)} = \frac{1-c}{c_0(z)}\mathop{\mathbf{K}}_{m=0}^{\infty}\left(\frac{c_m(z)}{e_m+d_m z}\right).$$

Applying Lemma 5.9 with $c_i = 1/(1-cq^i)$ and $m\to\infty$ yields

$$\frac{{}_2\phi_1\left(aq,bq;cq;q,z\right)}{{}_2\phi_1\left(a,b;c;q,z\right)} = \frac{(1-cq^{-1})(1-c)}{c_0(z)}\mathop{\mathbf{K}}_{m=0}^{\infty}\left(\frac{c_m(z)/(1-cq^{m-1})(1-cq^m)}{e_m/(1-cq^m)+d_m z/(1-cq^m)}\right),$$

which is the same as the desired identity. □

Proof of Theorem 5.5. Replace z by z/b, put $a=0$, and let $b\to\infty$ in Proposition 5.11. The result is Theorem 5.5. □

Note that when $b=0$ both Lemma 5.7 and Proposition 5.11 give a continued fraction expression for

$$\frac{{}_2\phi_1\left(a,0;c;q,z\right)}{{}_2\phi_1\left(aq,0;cq;q,z\right)}.$$

Therefore we obtain the following theorem.

Theorem 5.12. *We have the equality of continued fractions*

$$\cfrac{1}{1-b_0z-\cfrac{a_1z}{1-b_1z-\cfrac{a_2z}{1-b_2z-\ddots}}}=\cfrac{1}{1-\cfrac{\lambda_1z}{1-\cfrac{\lambda_2z}{1-\ddots}}},$$

where

$$a_n=\frac{(aq^n-1)cq^{n-1}}{(1-cq^{n-1})(1-cq^n)},\qquad b_n=\frac{aq^n}{1-cq^n},$$

$$\lambda_{2n}=\frac{-cq^{2n-1}(1-aq^n)}{(1-cq^{2n-1})(1-cq^{2n})},\qquad \lambda_{2n+1}=\frac{(a-cq^n)q^n}{(1-cq^{2n})(1-cq^{2n+1})}.$$

When Theorem 5.12 is interpreted as an equality for moment generating functions, we find the following generalization of Theorem 4.14 which holds for q-Lommel polynomials.

Corollary 5.13. *Let λ_n, a_n and b_n be given by Theorem 5.12. The $2n^{\text{th}}$ moment of the orthogonal polynomials defined by $p_{n+1}(x)=xp_n(x)-\lambda_np_{n-1}(x)$ is equal to the n^{th} moment of the type R_I polynomials defined by $r_{n+1}(x)=(x-b_n)r_n(x)-a_nxr_{n-1}(x)$.*

6. Combinatorics of moments of type R_I q-Lommel polynomials

The moment generating function for type R_I polynomials is given by the continued fraction in (5.3). For type R_I q-Lommel polynomials we give in this section a general combinatorial interpretation for this infinite continued fraction in terms of parallelogram polyominoes. We also interpret the finite continued fraction and give an explicit rational expression using q-Lommel polynomials. To be specific we give a combinatorial interpretation for the ratio

$$r_n^{(3)}(x^{-1};q^{\nu+2},q)/r_{n+1}^{(3)}(x^{-1};q^{\nu+1},q)$$

of (rescaled) type R_I q-Lommel polynomials, Theorem 6.9. This is a finite version of the result of Bousquet-Mélou and Viennot [4]. The $n\to\infty$ limit of Theorem 6.9 yields a quotient of q-Bessel functions,

$$J_{\nu+1}^{(3)}(x^{1/2};q^{-1})/J_{\nu}^{(3)}(x^{1/2};q^{-1})$$

which is the moment generating function for the type R_I q-Lommel polynomials. This material appears in our unpublished manuscript [17, Section 5].

We shall need several definitions related to parallelogram polyominoes and Motzkin paths.

Definition 6.1. An *NE-path* is a lattice path from $(0,0)$ to (a,b) for some positive integers a,b consisting of north steps $(0,1)$ and east steps $(1,0)$. A *parallelogram polyomino* is a set of unit squares enclosed by two NE-paths with the same ending points that do not intersect except the starting and ending points. Denote by $\mathcal{P}$ the set of parallelogram polyominoes.

For a parallelogram polyomino $\alpha\in\mathcal{P}$ let $U(\alpha)$ be the upper boundary path and $D(\alpha)$ the lower boundary path; see Figure 1. A *diagonal* of α is the set of squares in α whose centers are on the line $x+y=i$ for some integer i. The *size* of a diagonal is the number of squares in it. See Figure 2.

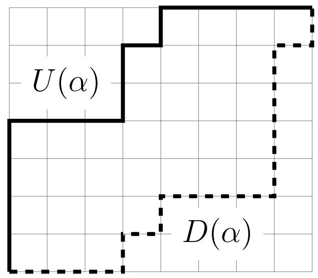

FIGURE 1. The boundary paths $U(\alpha)$ and $D(\alpha)$ for a parallelogram polyomino.

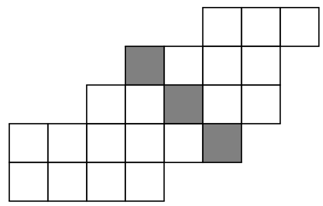

FIGURE 2. A diagonal with size 3 in a parallelogram polyomino.

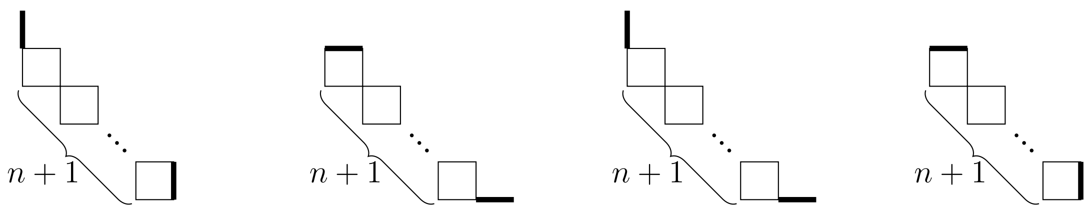

FIGURE 3. From left to right are shown an NN-diagonal, EE-diagonal, NE-diagonal, and EN-diagonal of size $n+1$ whose weights are, respectively, a_n, b_n, c_n, and d_n.

Definition 6.2. We denote by $\mathcal{P}^{\le k}$ the set of parallelogram polyominoes in which every diagonal has size at most k.

Consider $\alpha \in \mathcal{P}$ and a diagonal τ of α. Let u (resp. d) be the northwest (resp. southeast) corner of the topmost (resp. bottommost) square of τ. There are four cases for d. We say that d is an *NN-diagonal* (resp. *NE-diagonal*, *EN-diagonal*, and *EE-diagonal*) if the step in $U(\alpha)$ starting at u is a north (resp. north, east, and east) step and the step in $D(\alpha)$ starting at d is a north (resp. east, north, and east) step. See Figure 3.

For sequences $\{a_n\}_{n\ge0}, \{b_n\}_{n\ge0}, \{c_n\}_{n\ge0}$, and $\{d_n\}_{n\ge0}$, define the *weight* $\mathrm{wt}(\alpha; a, b, c, d)$ of $\alpha \in \mathcal{P}$ to be the product of a_n (resp. b_n, c_n, and d_n) for each NN-diagonal (resp. EE-diagonal, NE-diagonal, and EN-diagonal) of size $n+1$.

Now we review Flajolet's theory [13] on continued fraction expressions for Motzkin path generating functions.

Definition 6.3. A *Motzkin path* is a lattice path from $(0,0)$ to $(n,0)$ consisting of up steps $(1,1)$, down steps $(1,-1)$, and horizontal steps $(1,0)$ that never goes below the x-axis. A *2-Motzkin path* is a Motzkin path in which every horizontal step is colored red or blue. The *height* of a 2-Motzkin path is the largest integer y for which (x,y) is a point in the path.

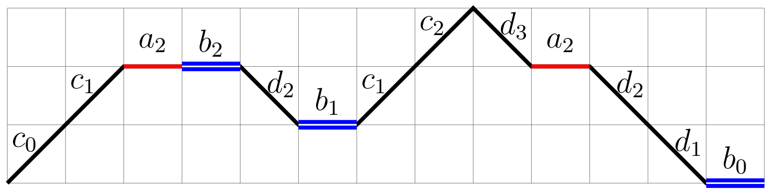

FIGURE 4. A 2-Motzkin path p in $\mathrm{Motz}_2^{\le 3}$ with $\mathrm{wt}(p;a,b,c,d) = a_2^2 b_0 b_1 b_2 c_0 c_1^2 c_2 d_1 d_2^2 d_3$. The blue horizontal edges are represented by double edges.

Denote by Motz_2 the set of all 2-Motzkin paths and by $\mathrm{Motz}_2^{\le m}$ the set of all 2-Motzkin paths with height at most m.

For sequences $\{a_n\}_{n\ge0}, \{b_n\}_{n\ge0}, \{c_n\}_{n\ge0}$, and $\{d_n\}_{n\ge0}$, define the *weight* $\mathrm{wt}(p;a,b,c,d)$ of a 2-Motzkin path p to be the product of a_n (resp. b_n, c_n, and d_n) for each red horizontal step (resp. blue horizontal step, up step, and down step) starting at height n; see Figure 4.

Flajolet's theory [13] proves the following lemma for a finite continued fraction.

Lemma 6.4. *Given sequences $\{a_n\}_{n\ge0}, \{b_n\}_{n\ge0}, \{c_n\}_{n\ge0}$, and $\{d_n\}_{n\ge0}$, we have*

$$\sum_{p\in\mathrm{Motz}_2^{\le m}} \mathrm{wt}(p;a,b,c,d) = \cfrac{1}{1-a_0-b_0-\cfrac{c_0d_1}{1-a_1-b_1-\ddots-\cfrac{c_{m-1}d_m}{1-a_m-b_m}}}.$$

There is a well-known bijection between 2-Motzkin paths and parallelogram polyominoes.

Definition 6.5 (The map $\phi : \mathrm{Motz}_2^{\le m} \to \mathcal{P}^{\le m+1}$)**.** Let $p \in \mathrm{Motz}_2^{\le m}$. Then $\phi(p) = \alpha$ is the parallelogram polyomino whose upper and lower boundary paths U, D are constructed by the following algorithm.

(1) The first step of U (resp. D) is a north (resp. east) step.
(2) For $i = 1, 2, \dots, n$, where n is the number of steps in p, the $(i+1)^{\mathrm{st}}$ steps of U and D are defined as follows.
  (a) If the i^{th} step of p is an up step, then the $(i+1)^{\mathrm{st}}$ step of U (resp. D) is a north (resp. east) step.
  (b) If the i^{th} step of p is a down step, then the $(i+1)^{\mathrm{st}}$ step of U (resp. D) is a east (resp. north) step.
  (c) If the i^{th} step of p is a red horizontal step, then the $(i+1)^{\mathrm{st}}$ steps of U and D are both north steps.
  (d) If the i^{th} step of p is a blue horizontal step, then the $(i+1)^{\mathrm{st}}$ steps of U and D are both east steps.
(3) Finally, the last step of U (resp. D) is an east (resp. north) step.

For example, if p is the 2-Motzkin path in Figure 4, then $\phi(p)$ is the parallelogram polyomino α in Figure 1.

It is easy to see from the construction that $\phi : \mathrm{Motz}_2^{\le m} \to \mathcal{P}^{\le m+1}$ is a bijection such that if $\phi(p) = \alpha$, then $\mathrm{wt}(\alpha;a,b,c,d) = d_0\,\mathrm{wt}(p;a,b,c,d)$.

Therefore we obtain the following proposition from Lemma 6.4, which changes the weighted 2-Motzkin paths into weighted parallelogram polyominoes.

Proposition 6.6. *Given sequences* $\{a_n\}_{n\ge0}, \{b_n\}_{n\ge0}, \{c_n\}_{n\ge0}$, *and* $\{d_n\}_{n\ge0}$, *we have*

$$\sum_{\alpha\in\mathcal{P}^{\le m+1}} \mathrm{wt}(\alpha;a,b,c,d) = \cfrac{d_0}{1-a_0-b_0-\cfrac{c_0d_1}{1-a_1-b_1-\ddots-\cfrac{c_{m-1}d_m}{1-a_m-b_m}}}.$$

As a special case in Proposition 6.6, if $\{a_n\}_{n\ge0}, \{b_n\}_{n\ge0}, \{c_n\}_{n\ge0}$, and $\{d_n\}_{n\ge0}$ are the sequences given by $a_n = q^{n+1}Y$, $b_n = q^{n+1}X$, $c_n = q^{n+1}XY$, and $d_n = q^{n+1}$, then one can easily check that

$$XY\cdot \mathrm{wt}(\alpha;a,b,c,d) = X^{\mathrm{col}(\alpha)}Y^{\mathrm{row}(\alpha)}q^{\mathrm{area}(\alpha)}.$$

Thus we obtain the following corollary.

Corollary 6.7. *We have*

$$\sum_{\alpha\in\mathcal{P}^{\le m+1}} X^{\mathrm{col}(\alpha)}Y^{\mathrm{row}(\alpha)}q^{\mathrm{area}(\alpha)} = \cfrac{qXY}{1-q(X+Y)-\cfrac{q^3XY}{1-q^2(X+Y)-\ddots-\cfrac{q^{2m+1}XY}{1-q^{m+1}(X+Y)}}}.$$

For the rest of this section we will find a finite version of the following result due to Bousquet-Mélou and Viennot [4].

Theorem 6.8 ([9] for $\nu=0$ and [4] for general ν)**.** *The trivariate generating function for parallelogram polyominoes is*

$$\sum_{\alpha\in\mathcal{P}} (q^\nu x)^{\mathrm{col}(\alpha)}(q^\nu)^{\mathrm{row}(\alpha)}q^{\mathrm{area}(\alpha)} = -q^\nu x^{1/2}\frac{J^{(3)}_{\nu+1}(x^{1/2};q^{-1})}{J^{(3)}_{\nu}(x^{1/2};q^{-1})}.$$

In fact Delest and Fédou [9] (for $\nu=0$), and Bousquet-Mélou and Viennot [4] state their results in the following equivalent form:

$$\sum_{\alpha\in\mathcal{P}} x^{\mathrm{col}(\alpha)}y^{\mathrm{row}(\alpha)}q^{\mathrm{area}(\alpha)} = \frac{qxy}{1-qy}\cdot\frac{{}_1\phi_1\left(0;q^2y;q,q^2x\right)}{{}_1\phi_1\left(0;qy;q,qx\right)}.$$

Bousquet-Mélou and Viennot [4] also showed that

$$\sum_{\alpha\in\mathcal{P}} x^{\mathrm{col}(\alpha)}y^{\mathrm{row}(\alpha)}q^{\mathrm{area}(\alpha)} = \cfrac{qxy}{1-q(x+y)-\cfrac{q^3xy}{1-q^2(x+y)-\cfrac{q^5xy}{\ddots}}}. \tag{6.1}$$

We note that in [4, Corollary 4.6] the sequence of the coefficients of $(x+y)$ in the continued fraction (6.1) was inadvertently written $q, q^3, q^5, \dots$, where the correct sequence is $q, q^2, q^3, \dots$ We also note that there are similar results in [1].

For a sequence $s=\{s_n\}_{n\ge0}$, define $\delta s = \{s_{n+1}\}_{n\ge0}$. Kim and Stanton [18, Eq. (5.4)] showed that for given sequences $b=\{b_n\}_{n\ge0}$, $a=\{a_n\}_{n\ge0}$, and $\lambda=\{\lambda_n\}_{n\ge0}$, and for a nonnegative integer k,

$$\frac{x^mP_m(x^{-1};\delta b,\delta a,\delta\lambda)}{x^{m+1}P_{m+1}(x^{-1};b,a,\lambda)} = \frac{1}{-a_0x-\lambda_0x^2}\,\mathop{\mathbf{K}}_{i=0}^{m}\left(\frac{-a_ix-\lambda_ix^2}{1-b_ix}\right). \tag{6.2}$$

Now we are ready to prove a finite version of Theorem 6.8.

Theorem 6.9. *The trivariate generating function for bounded diagonal parallelogram polyominoes is*

$$\sum_{\alpha\in\mathcal{P}^{\le m+1}} (q^\nu x)^{\mathrm{col}(\alpha)}(q^\nu)^{\mathrm{row}(\alpha)}q^{\mathrm{area}(\alpha)} = \frac{q^{2\nu+1}}{1-q^{\nu+1}}\cdot\frac{r_m^{(3)}(x^{-1};q^{\nu+2},q)}{r_{m+1}^{(3)}(x^{-1};q^{\nu+1},q)}. \tag{6.3}$$

Proof. Let $b=\{b_i\}_{i\ge0}$, $a=\{a_i\}_{i\ge0}$, and $\lambda=\{\lambda_i\}_{i\ge0}$, where

$$b_i=\frac{q^{\nu+i+1}}{1-q^{\nu+i+1}},\qquad a_i=\frac{q^{2\nu+2i+1}}{(1-q^{\nu+i})(1-q^{\nu+i+1})},\qquad \lambda_i=0.$$

Then $P_m(x;b,a,\lambda)=r_m^{(3)}(x;q^{\nu+1},q)$ and $P_m(x;\delta b,\delta a,\delta\lambda)=r_m^{(3)}(x;q^{\nu+2},q)$. By (6.2),

$$\frac{r_m^{(3)}(x^{-1};q^{\nu+2},q)}{xr_{m+1}^{(3)}(x^{-1};q^{\nu+1},q)}=\frac{x^mP_m(x^{-1};\delta b,\delta a,\delta\lambda)}{x^{m+1}P_{m+1}(x^{-1};b,a,\lambda)}=\frac{1}{-a_0x}\mathop{\mathbf{K}}_{i=0}^{m}\left(\frac{-a_ix}{1-b_ix}\right).$$

By Lemma 5.9 with $c_i=1-q^{\nu+i+1}$,

$$\begin{aligned}\frac{1}{-a_0x}\mathop{\mathbf{K}}_{i=0}^{m}\left(\frac{-a_ix}{1-b_ix}\right)&=\frac{1}{-a_0x}\frac{1}{c_{-1}}\mathop{\mathbf{K}}_{i=0}^{m}\left(\frac{-a_ic_{i-1}c_ix}{c_i-b_ic_ix}\right)\\&=\frac{1-q^{\nu+1}}{-q^{2\nu+1}x}\mathop{\mathbf{K}}_{i=0}^{m}\left(\frac{-q^{2\nu+2i+1}x}{1-q^{\nu+i+1}-q^{\nu+i+1}x}\right).\end{aligned}$$

Letting $X=q^\nu x$ and $Y=q^\nu$, and combining the above equations, we obtain

$$\frac{q^{2\nu+1}}{1-q^{\nu+1}}\cdot\frac{r_m^{(3)}(x^{-1};q^{\nu+2},q)}{r_{m+1}^{(3)}(x^{-1};q^{\nu+1},q)}=-\mathop{\mathbf{K}}_{i=0}^{m}\left(\frac{-q^{2i+1}XY}{1-q^{i+1}(X+Y)}\right).$$

Corollary 6.7 then completes the proof. □

Remark 6.10. It is well known that the generating function for bounded Motzkin paths is given by a ratio of orthogonal polynomials; see e.g. Flajolet [13], Viennot [28, Ch. V, Eq. (27)], or Krattenthaler [24, Theorem 10.11.1]. The argument in this section implies that the left-hand side of (6.3) is the generating function for certain weighted bounded Motzkin paths. Theorem 6.9 shows that this generating function is also equal to a ratio of type R_I polynomials.

By (3.4), taking the limit $m\to\infty$ in Theorem 6.9 we obtain Theorem 6.8. We may also use Theorem 4.11 to write the finite continued fraction as an explicit rational function.

Corollary 6.11. *The trivariate generating function for bounded diagonal parallelogram polyominoes is*

$$\sum_{\alpha\in\mathcal{P}^{\le n+1}} x^{\mathrm{col}(\alpha)}y^{\mathrm{row}(\alpha)}q^{\mathrm{area}(\alpha)}=-\frac{x\sum_{k=0}^{n}\sum_{a=0}^{n-k}(-1)^kx^ay^{-k-a}q^{-\binom{k}{2}-2k}\begin{bmatrix}k+a\\a\end{bmatrix}_{q^{-1}}\begin{bmatrix}n-a\\k\end{bmatrix}_{q^{-1}}}{\sum_{k=0}^{n+1}\sum_{a=0}^{n+1-k}(-1)^kx^ay^{-k-a}q^{-\binom{k}{2}-k}\begin{bmatrix}k+a\\a\end{bmatrix}_{q^{-1}}\begin{bmatrix}n+1-a\\k\end{bmatrix}_{q^{-1}}}.$$

Cigler and Krattenthaler [7] found a different finite version of Theorem 6.8.

Theorem 6.12 ([7, Corollary 55]). *For any integer $k \geq 1$, we have*

$$\sum_{\alpha \in \mathcal{P}_1^{\leq k}} x^{\mathrm{col}(\alpha)} y^{\mathrm{row}(\alpha)} q^{\mathrm{area}(\alpha)} = -\frac{y \sum_{j=1}^{k} (-1)^j x^j q^{\binom{j+1}{2}} \sum_{i=0}^{k-j} (yq)^i \begin{bmatrix} k-i-1 \\ j-1 \end{bmatrix}_q \begin{bmatrix} i+j-1 \\ j-1 \end{bmatrix}_q}{\sum_{j=0}^{k} (-1)^j x^j q^{\binom{j+1}{2}} \sum_{i=0}^{k-j} (yq)^i \begin{bmatrix} k-i \\ j \end{bmatrix}_q \begin{bmatrix} i+j-1 \\ j-1 \end{bmatrix}_q},$$

where $\mathcal{P}_1^{\leq k}$ is the set of parallelogram polyominoes such that each column has length at most k and $\begin{bmatrix} i \\ -1 \end{bmatrix}_q = \delta_{i,-1}$.

Remark 6.13. The second odd-even trick (7.1) with $\lambda_{2k-1} = q^k y$ and $\lambda_{2k} = q^k$ gives

$$1 + \cfrac{qy}{1 - q(x+y) - \cfrac{q^3 xy}{1 - q^2(x+y) - \cfrac{q^5 xy}{\ddots}}} = \cfrac{1}{1 - \cfrac{qy}{1 - \cfrac{qx}{1 - \cfrac{q^2 y}{1 - \cfrac{q^2 x}{\ddots}}}}}.$$

Remark 6.14. There are also finite versions of Theorem 6.9 for the classical q-Lommel polynomials and the even-odd q-Lommel polynomials. The rational function is again a quotient of orthogonal polynomials while the weights on $\mathcal{P}^{\leq m+1}$ depend upon the diagonals.

Here are the infinite continued fractions for these two cases. For the classical q-Lommel polynomials, Theorem 5.3 becomes

$$\frac{{}_2\phi_1(0,0;q^2y;q;-qx)}{{}_2\phi_1(0,0;qy;q;-qx)} = \cfrac{1-qy}{1 - qy - \cfrac{q^2 xy}{1 - q^2 y - \cfrac{q^3 xy}{1 - q^3 y - \cfrac{q^4 xy}{\ddots}}}}.$$

For the even-odd q-Lommel polynomials, Theorem 5.4 becomes

$$\frac{{}_1\phi_1(0;q^2y;q;q^2x)}{{}_1\phi_1(0;qy;q;qx)} = \cfrac{1-qy}{1 - qy - \cfrac{A_1}{1 - q^2 y - \cfrac{A_2}{1 - q^3 y - \cfrac{A_3}{\ddots}}}}$$

where $A_{2k-1} = xq^k$ and $A_{2k} = xyq^{3k/2+1}$.

7. Concurrence of moments

Recall the notation (5.2) for the moments $\mu_n\Big(\{b_k\}_{k\geq 0}, \{a_k\}_{k\geq 0}, \{\lambda_k\}_{k\geq 0}\Big)$, which we also write $\mu_n(\{b_k\}, \{a_k\}, \{\lambda_k\})$, or $\mu_n(\{b_k\}, \{\lambda_k\})$ when $a_k = 0$. There is a concurrence of moments (see Propositions 4.5 and 4.6), which we call the first and second *odd-even tricks*

$$\begin{aligned} \mu_{2n}(\{0\}, \{\lambda_k\}) &= \mu_n(\{\lambda_{2k} + \lambda_{2k+1}\}, \{\lambda_{2k}\lambda_{2k-1}\}), \\ \mu_{2n+2}(\{0\}, \{\lambda_k\}) &= \lambda_1 \mu_n(\{\lambda_{2k+2} + \lambda_{2k+1}\}, \{\lambda_{2k}\lambda_{2k+1}\}). \end{aligned} \tag{7.1}$$

The classical orthogonal polynomial moments are a special case of type R_I moments

$$\mu_n(\{b_k\},\{0\},\{\lambda_k\}) = \mu_n\left(\{b_k\},\{\lambda_k\}\right).$$

There is another concurrence of moments, which follows from [18, Corollary 3.7]

$$\mu_{2n}(\{0\},\{a_k\}) = \mu_n(\{0\},\{a_k\},\{0\}). \tag{7.2}$$

It is known [18] that a type R_I moment $\mu_n(\{b_k\},\{a_k\},\{\lambda_k\})$ is a nonnegative polynomial in the recurrence coefficients. Besides (7.2) Theorem 4.14 is another example of classical orthogonal polynomial moments being equal to type R_I moments

$$\mu_{2n}\left(\{0\},\{\Lambda_k\}\right) = \mu_n\left(\{b_k\},\{a_k\},\{0\}\right). \tag{7.3}$$

The main result in this section is Theorem 7.2, which expresses the Λ_k as a function of the sequences a_k and b_k, thereby providing the concurrence (7.3).

To prove Theorem 7.2 we need to recall a classical result and notation. The Hankel determinant [6, Theorem 4.2] will be used:

$$\det(\mu_{i+j}\left(\{b_k\}_{k\geq 0},\{\lambda_k\}_{k\geq 0}\right))_{i,j=0}^n = \lambda_1^n\lambda_2^{n-1}\cdots\lambda_n^1.$$

Recall that for a sequence $a=\{a_k\}_{k\geq 0}$ we write $\delta a=\{a_{k+1}\}_{k\geq 0}$. We also define $\delta^{-1}a = \{a_{k-1}\}_{k\geq 0}$, where $a_{-1}=1$ (the value of a_{-1} is irrelevant for our purpose).

Definition 7.1. A *Schröder path* is a lattice path from $(r,0)$ to $(s,0)$, for some integers r,s, consisting of northeast steps $(1,1)$, east steps $(1,0)$, and south steps $(0,-1)$ that never goes below the x-axis. Given sequences $b=\{b_k\}_{k\geq 0}$ and $a=\{a_k\}_{k\geq 0}$, the *weight* $\mathrm{wt}(P)$ of a Schröder path P is the product of b_i for each east step starting at height i and a_i for each south step starting at height i.

Our main theorem of this section is the next theorem.

Theorem 7.2. *Suppose that sequences $b=\{b_k\}_{k\geq 0}$, $a=\{a_k\}_{k\geq 0}$, and $\Lambda=\{\Lambda_k\}_{k\geq 0}$ satisfy*

$$\mu_{2n}\left(\{0\},\{\Lambda_k\}\right) = \mu_n\left(\{b_k\},\{a_k\},\{0\}\right).$$

Then

$$\Lambda_1\Lambda_2\cdots\Lambda_{2n} = \frac{f_n(a,b)}{f_{n-1}(a,b)},$$

where

$$f_n(a,b) = \sum_p \mathrm{wt}(p),$$

and the sum is over all n-tuples $p=(P_0,P_1,\dots,P_n)$ of non-intersecting Schröder paths, $P_k:(-k,0)\to(k,0), 0\leq k\leq n$. Moreover,

$$\Lambda_1\Lambda_2\cdots\Lambda_{2n-1} = a_0^{-1}\frac{f_n(\delta^{-1}a,\delta^{-1}b)}{f_{n-1}(\delta^{-1}a,\delta^{-1}b)},$$

and if $a_k=b_k=1$ then

$$f_n(\{1\},\{1\}) = 2^{\binom{n+1}{2}}.$$

Proof. Let

$$\begin{aligned}\rho_n &:= \mu_{2n}\left(\{0\},\{\Lambda_k\}\right) = \mu_n\left(\{b_k\},\{a_k\},\{0\}\right),\\ \Delta_n &:= \det(\rho_{i+j})_{0\le i,j\le n}.\end{aligned}$$

Using the odd-even trick $B_n = \Lambda_{2n+1}+\Lambda_{2n}$ and $\Theta_n = \Lambda_{2n-1}\Lambda_{2n}$, we have

$$\rho_n = \mu_{2n}\left(\{0\},\{\Lambda_k\}\right) = \mu_n(\{B_k\},\{\Theta_k\}).$$

Therefore

$$\Delta_n = \det(\mu_{i+j}(\{B_k\},\{\Theta_k\}))_{0\le i,j\le n} = \Theta_1^n\Theta_2^{n-1}\cdots\Theta_n^1 = \Lambda_1^n\Lambda_2^n\Lambda_3^{n-1}\Lambda_4^{n-1}\cdots\Lambda_{2n-1}^1\Lambda_{2n}^1,$$

which shows $\Lambda_1\Lambda_2\cdots\Lambda_{2n} = \Delta_n/\Delta_{n-1}$.

Kim and Stanton [18, Corollary 3.7] showed that $\mu_n\left(\{b_k\},\{a_k\},\{0\}\right)$ is the sum of weights of all Schröder paths from $(0,0)$ to $(n,0)$. Since $\Delta_n = \det(\mu_{i+j}\left(\{b_k\},\{a_k\},\{0\}\right))_{0\le i,j\le n}$, the $(n+1)\times(n+1)$ determinant Δ_n is the signed generating function for $(n+1)$-tuples of Schröder paths $(P_0,\dots,P_n)$, $P_k:(-k,0)\to(\sigma(k),0)$, for some permutation σ of $\{0,1,\dots,n\}$. Because there are no SE edges ($\lambda_k=0$), any two paths which intersect do so at integer coordinates. Thus we may apply the Lindström–Gessel–Viennot lemma of tail swapping to reduce this sum to non-intersecting paths, $\sigma=$ identity, $P_k:(-k,0)\to(k,0)$. Thus $\Delta_n = f_n(a,b)$ and we obtain the identity for $\Lambda_1\Lambda_2\cdots\Lambda_{2n}$.

Now using the second odd-even trick $B_n' = \Lambda_{2n+2}+\Lambda_{2n+1}$ and $\Lambda_n' = \Lambda_{2n+1}\Lambda_{2n}$, we have

$$\rho_{n+1} = \mu_{2n+2}\left(\{0\},\{\Lambda_k\}\right) = \Lambda_1\mu_n(\{B_k'\},\{\Lambda_k'\}).$$

Then

$$\begin{aligned}\Delta_n' := \det(\rho_{i+j+1})_{0\le i,j\le n-1} &= \Lambda_1^n\det(\mu_{i+j}(\{B_k'\},\{\Lambda_k'\}))_{0\le i,j\le n-1}\\ &= \Lambda_1^n\Lambda_2^{n-1}\Lambda_3^{n-1}\cdots\Lambda_{2n-2}^1\Lambda_{2n-1}^1,\end{aligned}$$

so

$$\Lambda_1\Lambda_2\cdots\Lambda_{2n-1} = \Delta_n'/\Delta_{n-1}'.$$

As in the even case, $\Delta_n' = \det(\mu_{i+j+1}\left(\{b_k\},\{a_k\},\{0\}\right))_{0\le i,j\le n-1}$ is the generating function for n-tuples non-intersecting Schröder paths $p'=(P_1',\dots,P_n')$, $P_k':(-k+1,0)\to(k,0)$. For $1\le k\le n$, let P_k be the path from $(-k,-1)$ to $(k,-1)$ obtained from P_k' by adding a northeast step at the beginning and a south step at the end, and let P_0 be the empty path from $(0,-1)$ to $(0,-1)$. This gives a bijection from n-tuples non-intersecting Schröder paths $p'=(P_1',\dots,P_n')$, $P_k':(-k+1,0)\to(k,0)$ to $(n+1)$-tuples non-intersecting Schröder paths $p=(P_0,P_1,\dots,P_n)$, $P_k:(-k,-1)\to(k,-1)$. Note that the starting point of P_k has height -1, which shifts the indices of a_k and b_k down by one. This shows that

$$\Delta_n' = a_0^{-n}\det(\mu_{i+j}\left(\{b_{k-1}\},\{0\},\{a_{k-1}\}\right))_{0\le i,j\le n} = a_0^{-n}f_n(\delta^{-1}a,\delta^{-1}b),$$

and we obtain the identity for $\Lambda_1\Lambda_2\cdots\Lambda_{2n-1}$.

Finally, the fact that $\Delta_n = 2^{\binom{n+1}{2}}$ and $\Delta_n' = 2^{\binom{n+1}{2}}$ if $a_k=b_k=1$ for all k follows from [18, Theorem 6.15, $A=B=1, C=0$]. $\square$

The first few values of $\Lambda_1\cdots\Lambda_k$ in Theorem 7.2 are

$$\begin{aligned}
\Lambda_1 &= a_0^{-1}\frac{f_1(\delta^{-1}a,\delta^{-1}b)}{f_0(\delta^{-1}a,\delta^{-1}b)} = \frac{a_1+b_0}{1},\\
\Lambda_1\Lambda_2 &= \frac{f_1(a,b)}{f_0(a,b)} = a_1\frac{a_2+b_1}{1},\\
\Lambda_1\Lambda_2\Lambda_3 &= a_0^{-1}\frac{f_2(\delta^{-1}a,\delta^{-1}b)}{f_1(\delta^{-1}a,\delta^{-1}b)}\\
&= a_1\frac{a_1a_2a_3+a_2^2b_0+a_2a_3b_0+2a_2b_0b_1+b_0b_1^2+a_1a_2b_2+a_2b_0b_2}{a_1+b_0},\\
\Lambda_1\Lambda_2\Lambda_3\Lambda_4 &= \frac{f_2(a,b)}{f_1(a,b)}\\
&= a_1a_2\frac{a_2a_3a_4+a_3^2b_1+a_3a_4b_1+2a_3b_1b_2+b_1b_2^2+a_2a_3b_3+a_3b_1b_3}{a_2+b_1}.
\end{aligned}$$

Remark 7.3. Eu and Fu [11] used the idea relating Δ_n and Δ'_{n-1} in the proof of Theorem 7.2 to give a simple proof of the Aztec diamond theorem, which is equivalent to the result $\Delta_n = 2^{\binom{n+1}{2}}$ when $a_k = b_k = 1$.

8. Open problems

Recall that Kishore's theorem is a statement about the power series coefficients of the ratio $J_{\nu+1}(x)/J_\nu(x)$ of two Bessel functions.

Theorem 8.1 (Kishore [19])**.** *We have*

$$\frac{J_{\nu+1}(z)}{J_\nu(z)} = \sum_{n=1}^{\infty}\frac{N_{n,\nu}}{D_{n,\nu}}\left(\frac{z}{2}\right)^{2n-1},$$

where

$$D_{n,\nu} = \prod_{k=1}^{n}(k+\nu)^{\lfloor n/k\rfloor},$$

and $N_{n,\nu}$ is a polynomial in ν with nonnegative integer coefficients.

We conjecture the following finite version of Kishore's theorem on a ratio of Lommel polynomials $L_{m,\nu}(x)$ defined in Section 3.

Conjecture 8.2. *Let*

$$\frac{L_{m,\nu+2}(x)}{L_{m+1,\nu+1}(x)} = \sum_{n=0}^{\infty}\frac{N_{n,\nu}^{(m)}}{D_{n,\nu}^{(m)}}\left(\frac{x}{2}\right)^{2n+1},$$

where

$$D_{n,\nu}^{(m)} = \prod_{k=0}^{m}(\nu+k+1)^{f(m,n,k)},$$

$$f(m,n,k) = \begin{cases}\max\left(\left\lfloor\dfrac{n+1}{k+1}\right\rfloor, \left\lfloor\dfrac{n+m-2k+1}{m-k+1}\right\rfloor\right), & \text{if } k\neq m/2,\\ 1, & \text{if } k=m/2.\end{cases}$$

Then $N_{n,\nu}^{(m)}$ is a polynomial in ν with nonnegative integer coefficients.

In Section 5 we saw that the ratio

$$\frac{J_{\nu+1}^{(3)}(z;q^{-1})}{J_{\nu}^{(3)}(z;q^{-1})} = \frac{-q^{\nu+1}z}{1-q^{\nu+1}} \cdot \frac{{}_1\phi_1\left(0;q^{\nu+2};q,q^{\nu+2}z^2\right)}{{}_1\phi_1\left(0;q^{\nu+1};q,q^{\nu+1}z^2\right)}$$

has two generalizations, the q-Nörlund continued fraction and Heine's continued fraction. These two generalizations seem to have a similar property as follows.

Conjecture 8.3. *Let*

$$\sum_{n\geq 0}\gamma_n(a,b,c)z^n = \frac{{}_2\phi_1\left(aq,bq;cq;q,z\right)}{{}_2\phi_1\left(a,b;c;q,z\right)}.$$

Then

$$\frac{\gamma_n(a,b,c)}{1-c} = \frac{P_n(a,b,c)}{\prod_{k=0}^{n}(1-cq^k)^{\lfloor\frac{n+1}{k+1}\rfloor}},$$

for some polynomial $P_n(a,b,c)$ in a,b,c,q with integer coefficients.

Conjecture 8.4. *Let*

$$\sum_{n\geq 0}\gamma'_n(a,b,c)z^n = \frac{{}_2\phi_1\left(aq,b;cq;q,z\right)}{{}_2\phi_1\left(a,b;c;q,z\right)}.$$

Then

$$\frac{\gamma'_n(a,b,c)}{1-c} = \frac{P'_n(a,b,c)}{\prod_{k=0}^{n}(1-cq^k)^{\lfloor\frac{n+1}{k+1}\rfloor}},$$

for some polynomial $P'_n(a,b,c)$ in a,b,c,q with integer coefficients.

Note that, between these two conjectures, only the second argument of the ${}_2\phi_1$ in the numerator differs (namely, bq vs b).

Problem 8.5. *Find a combinatorial proof of Theorem 5.12.*

Problem 8.6. *Find a combinatorial proof of Theorem 6.9, which contains the Bousquet-Mélou–Viennot result Theorem 6.8.*

Problem 8.7. *Find an Askey scheme whose top element is the associated Askey–Wilson polynomial which contains the q-Lommel polynomials. As an alternative, a referee has suggested that an Askey scheme with non-polynomial entries may exist which contains the q-Lommel polynomials.*

Acknowledgments. The first author thanks the National Research Foundation for its support, via the grants NRF #2022R1A2C101100911 and NRF #2016R1A5A1008055.

References

[1] Elena Barcucci, Alberto Del Lungo, Jean-Marc Fédou, and Renzo Pinzani. Steep polyominoes, q-Motzkin numbers and q-Bessel functions. *Discrete Mathematics*, 189(1-3):21–42, 1998.

[2] Elena Barcucci, Alberto Del Lungo, Elisa Pergola, and Renzo Pinzani. Some permutations with forbidden subsequences and their inversion number. *Discrete Mathematics*, 234(1-3):1–15, 2001.

[3] Mireille Bousquet-Mélou and Jean-Marc Fédou. The generating function of convex polyominoes: the resolution of a q-differential system. *Discrete Mathematics*, 137(1-3):53–75, 1995.

[4] Mireille Bousquet-Mélou and Xavier Gérard Viennot. Empilements de segments et q-énumération de polyominos convexes dirigés. *Journal of Combinatorial Theory, Series A*, 60(2):196–224, 1992.

[5] Richard Brak and Anthony J. Guttmann. Exact solution of the staircase and row-convex polygon perimeter and area generating function. *Journal of Physics A.*, 23(20):4581–4588, 1990.

[6] Theodore S. Chihara. *An Introduction to Orthogonal Polynomials*. Gordon and Breach Science Publishers, 1978. Mathematics and its Applications, Vol. 13.

[7] Johann Cigler and Christian Krattenthaler. Bounded Dyck paths, bounded alternating sequences, orthogonal polynomials, and reciprocity. *European Journal of Combinatorics*, 2023. To appear.

[8] Annie Cuyt, Vigdis Brevik Petersen, Brigitte Verdonk, Haakon Waadeland, and William B. Jones. *Handbook of Continued Fractions for Special Functions*. Springer, 2008.

[9] Marie-Pierre Delest and Jean-Marc Fédou. Enumeration of skew Ferrers diagrams. *Discrete Mathematics*, 112(1-3):65–79, 1993.

[10] *NIST Digital Library of Mathematical Functions*. http://dlmf.nist.gov/, Release 1.1.10 of 2023-06-15. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, eds.

[11] Sen-Peng Eu and Tung-Shan Fu. A simple proof of the Aztec diamond theorem. *Electronic Journal of Combinatorics*, 12:Research Paper 18, 8, 2005.

[12] Jean-Marc Fédou. Combinatorial objects enumerated by q-Bessel functions. *Reports on Mathematical Physics*, 34(1):57–70, 1994.

[13] Philippe Flajolet. Combinatorial aspects of continued fractions. *Discrete Mathematics*, 32(2):125–161, 1980.

[14] George Gasper and Mizan Rahman. *Basic Hypergeometric Series*, volume 96 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, second edition, 2004. With a foreword by Richard Askey.

[15] Mourad E. H. Ismail. *Classical and Quantum Orthogonal Polynomials in One Variable*, volume 98 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, 2009.

[16] Mourad E. H. Ismail and David R. Masson. Generalized orthogonality and continued fractions. *Journal of Approximation Theory*, 83(1):1–40, 1995.

[17] Jang Soo Kim and Dennis Stanton. Ratios of Hahn–Exton q-Bessel functions and q-Lommel polynomials. *arXiv*, 2021.

[18] Jang Soo Kim and Dennis Stanton. Combinatorics of orthogonal polynomials of type R_I. *Ramanujan Journal*, 61(2):329–390, 2023.

[19] Nand Kishore. The Rayleigh polynomial. *Proceedings of the American Mathematical Society*, 15:911–917, 1964.

[20] David A. Klarner and Ronald L. Rivest. Asymptotic bounds for the number of convex n-ominoes. *Discrete Mathematics*, 8:31–40, 1974.

[21] Hendrik T. Koelink. Some basic Lommel polynomials. *Journal of Approximation Theory*, 96(2):345–365, 1999.

[22] Hendrik T. Koelink and René F. Swarttouw. On the zeros of the Hahn–Exton q-Bessel function and associated q-Lommel polynomials. *Journal of Mathematical Analysis and Applications*, 186(3):690–710, 1994.

[23] Hendrik T. Koelink and Walter Van Assche. Orthogonal polynomials and Laurent polynomials related to the Hahn–Exton q-Bessel function. *Constructive Approximation*, 11(4):477–512, 1995.

[24] Christian Krattenthaler. Lattice path enumeration. In *Handbook of Enumerative Combinatorics*, pages 589–678. CRC Press, 2015.

[25] Jean-Claude Lalanne. q-Bessel, arbres, et chemins valués. *Journal of Combinatorial Theory. Series A*, 60(2):225–245, 1992.

[26] Jean-Claude Lalanne. Polyominos parallélogrammes à franges et fonctions de Bessel. *Discrete Mathematics*, 115(1-3):217–230, 1993.

[27] Derrick H. Lehmer. Zeros of the Bessel Function $J_n(x)$. *Mathematical Tables and Other Aids to Computation*, 1(10):405–407, 1945.

[28] Gérard Viennot. *Une Théorie combinatoire des polynômes orthogonaux généraux*. UQAM, 1983.