

## THE GENERATING FUNCTION OF THE SURVIVAL PROBABILITIES IN A CONE IS NOT RATIONAL

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**Abstract.** We look at multidimensional random walks  $(S_n)_{n \geq 0}$  in convex cones, and address the question of whether two naturally associated generating functions may define rational functions. The first series is the one of the survival probabilities  $\mathbb{P}(\tau > n)$ , where  $\tau$  is the first exit time from a given cone; the second series is that of the excursion probabilities  $\mathbb{P}(\tau > n, S_n = y)$ . Our motivation to consider this question is twofold: first, it goes along with a global effort of the combinatorial community to classify the algebraic nature of the series counting random walks in cones; second, rationality questions of the generating functions are strongly associated with the asymptotic behaviors of the above probabilities, which have their own interest. Using well-known relations between rationality of a series and possible asymptotics of its coefficients, recent probabilistic estimates immediately imply that the excursion generating function is not rational. Regarding the survival probabilities generating function, we propose a short and self-contained proof that it cannot be rational neither.

**Keywords:** random walks in cones, survival probabilities, generating functions, rational functions, Laplace transform, univariate singularity analysis.

### 1. INTRODUCTION

**Main result and our approach.** For a  $d$ -dimensional random walk  $(S_n)_{n \geq 0}$  with integrable and independent increments  $X_n = S_n - S_{n-1}$  having common distribution  $\mu$ , we consider the generating function

$$F(t) = \sum_{n \geq 0} a_n t^n = \sum_{n \geq 0} \mathbb{P}^x(\tau > n) t^n, \quad (1.1)$$

where  $\mathbb{P}^x$  is a probability distribution under which the random walk starts at  $S_0 = x$ , and  $\tau$  denotes the first exit time from a given cone  $K$ , i.e.,

$$\tau = \inf\{n > 0 : S_n \notin K\}.$$

See (1.7) for an explicit computation of (1.1) in a simple one-dimensional example. Our first main result can be stated as follows:

**Theorem 1.1.** *If the drift  $m = \mathbb{E}X_1$  is not interior to the cone  $K$ , and if four further assumptions (A1)–(A4) (to be introduced below) are satisfied, then the generating function  $F(t)$  in (1.1) is not a rational function.*

Let us emphasize that in Theorem 1.1, the distribution of  $X_1$  is not assumed to be a discrete measure. This theorem covers, amongst other cases, the class of walks with small steps in orthants (with arbitrary weights on the steps), a model that has attracted a lot of attention from the combinatorial community, but is much more general.

The non-rationality of the generating function (1.1) is based on the fact that the numbers  $a_n$  don't have an asymptotic behavior that is compatible with the Taylor coefficients of a rational function. More precisely, we identify in Theorem 1.3 a rate  $\rho \in (0, 1]$  such that

$$a_n = \rho^n B_n, \quad (1.2)$$

with  $B_n$  satisfying

- (i)  $\sqrt[n]{B_n} \rightarrow 1$ ,
- (ii)  $B_n \rightarrow 0$ .

Using then classical analytic combinatorics techniques (see in particular Theorem 4.1 and Lemma 4.2), one will directly deduce that the generating function (1.1) cannot be rational.

In this paper, we provide proofs of estimates (i) and (ii) which are self-contained, and as simple and elementary as possible. Only item (ii) is new. Item (i) (in particular the value of the rate  $\rho$  in (1.2)) is already obtained in [11], but for the reader's convenience, we shall give here a detailed proof in a simplified setting, that covers the cases which are relevant to combinatorialists. In a special case (when the drift is, in some sense, directed towards the vertex of the cone), the precise asymptotics of the survival probability (hence in particular items (i) and (ii)) is derived in [8].

**Drift inside of the cone.** In case of a drift  $m = \mathbb{E}X_1$  interior to the cone, the probabilistic behavior is rather constrained as we have  $\mathbb{P}^x(\tau > n) \rightarrow \mathbb{P}^x(\tau = \infty) > 0$ . The positivity of the escape probability is intuitively clear, based on the law of large numbers and the fluctuations of the random walk; see Lemma 3.1 for a precise statement. Equivalently, in the neighborhood of  $t = 1$ , one has

$$F(t) \sim \frac{\mathbb{P}^x(\tau = \infty)}{1 - t},$$

which contains no contradiction with  $F$  being a rational function. However, for one-dimensional walks with bounded jumps, it is proved in [1, Thm 4] that  $\mathbb{P}^x(\tau > n) = \mathbb{P}^x(\tau = \infty) + \frac{c\rho^n}{n^{3/2}} + \dots$ , with  $\rho \in (0, 1)$ , which is not compatible with  $F$  being rational.

One of the simplest examples for which the rationality of  $F$  in (1.1) was an open question before the present paper is the following: in the quarter plane  $K = \mathbb{N}^2$ , take a uniform distribution  $\mu$  on  $\{(1, 0), (0, -1), (-1, 0), (0, 1), (1, 1)\}$ . Here, we answer this question and, more generally, solve the problem for the orthant  $K = [0, \infty)^d$  and any (weighted) small step walk, i.e., random walk with increments  $X_k$  that belong to  $\{-1, 0, 1\}^d$  almost surely. If  $\mathbb{P}(X_k \in K) = 1$ , then the random walk is *trapped* forever in  $K$  and  $a_n = \mathbb{P}^x(\tau > n) = 1$  for all  $n$ , so that  $F(t) = \frac{1}{1-t}$  is a rational function. Let us say the walk is *not trapped* if  $\mathbb{P}(X_k \notin K) > 0$ . Our second main result is the following:

**Theorem 1.2.** *For all  $d$ -dimensional weighted small step walks with a drift interior to the orthant  $K = [0, \infty)^d$ , not trapped and satisfying (A2), the generating function  $F(t)$  in (1.1) is not rational.*

Here again, the non-rationality of the generating  $F(t)$  is obtained as a consequence of estimates on  $a_n = \mathbb{P}^x(\tau > n)$ . More precisely, in Theorem 1.4, we prove that

$$\mathbb{P}^x(\tau > n) = \mathbb{P}^x(\tau = \infty) + \Theta(\rho^n B_n), \quad (1.3)$$

where  $\rho \in (0, 1)$  and  $B_n$  satisfies  $\sqrt[n]{B_n} \rightarrow 1$  and  $B_n \rightarrow 0$ , and the notation  $f_n = \Theta(g_n)$  means that there exist constants  $0 < c < C$  such that  $cg_n \leq f_n \leq Cg_n$ .

We could have unified the presentation of the interior and non-interior drift estimates, since  $\mathbb{P}^x(\tau = \infty) = 0$  when the drift is not in  $K^\circ$ . However, we choose not to do so, because the last double-sided estimate (1.3) is obtained only in the small step walk setting. We leave open the general case of this interesting interior drift problem.

In the papers [12, 13] (see also [7]), the authors prove the non-rationality of  $F(t)$ , for certain models of singular walks in the quarter plane, by proving that  $F(t)$  admits infinitely many poles.

**Combinatorial motivations.** Up to a scaling of the  $t$ -variable, our framework is equivalent to a more combinatorial question, related to the enumeration of walks. More precisely, in case  $\mu$  is a uniform distribution on a finite set  $\mathcal{S}$  (with cardinality  $|\mathcal{S}|$ ), one has

$$F(|\mathcal{S}|t) = \sum_{n \geq 0} q_n t^n,$$

where  $q_n$  denotes the number of walks starting from  $x$ , having length  $n$  and staying in the cone  $K$ . More generally, when  $\mu$  is any distribution, the series  $F(t)$  counts the numbers of  $\mu$ -weighted walks of length  $n$  staying in the cone  $K$ . Accordingly, all our results admit direct combinatorial interpretations.

Recently, in the combinatorial literature, the seminal paper [3] inspired the following question, which has attracted a lot of attention: given an orthant  $K = \mathbb{N}^d = \{0, 1, \dots\}^d$  and a distribution  $\mu$  on  $\mathbb{Z}^d$  (a step set in the combinatorial terminology), is the generating function (1.1), or its refined version

$$F(x_1, \dots, x_d; t) = \sum_{n \geq 0} \sum_{(n_1, \dots, n_d) \in \mathbb{N}^d} \mathbb{P}^x(\tau > n, S_n = (n_1, \dots, n_d)) x_1^{n_1} \cdots x_d^{n_d} t^n \quad (1.4)$$

a rational function? An algebraic function? A function satisfying a linear (or non-linear) differential equation? A hypertranscendental function, meaning that like Euler's  $\Gamma$  function it does not satisfy any differential equation? In the present article, we look at the possible rationality of the generating function.

Notice the following relation between (1.1) and (1.4):

$$F(1, \dots, 1; t) = F(t).$$

On the other hand,  $F(0, \dots, 0; t)$  is the generating function of the excursion sequence

$$F(0, \dots, 0; t) = \sum_{n \geq 0} \mathbb{P}^x(\tau > n, S_n = (0, \dots, 0)) t^n,$$

which will be studied (based on earlier literature [5]) in Section 5.

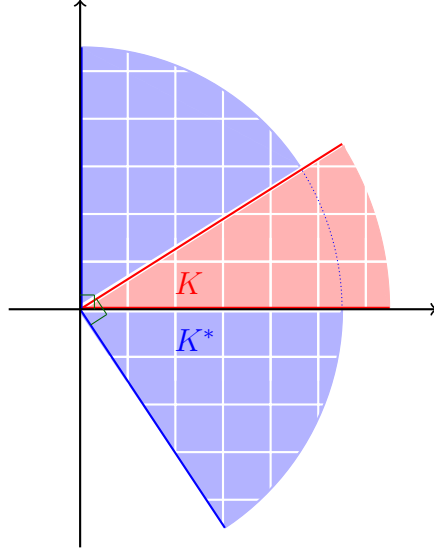


FIGURE 1. A cone  $K$  (in red) and its dual cone  $K^*$  (in blue)

**Technical assumptions.** In order to present the hypotheses in the statement of our main results, we need to introduce two objects, through which the exponential rate  $\rho$  in (1.2) will be determined:

- the Laplace transform  $L$  of the increment distribution  $\mu$ :

$$L(t) = \mathbb{E}\left(e^{\langle t, X_k \rangle}\right) = \int_{\mathbb{R}^d} e^{\langle t, y \rangle} \mu(dy),$$

- the dual cone  $K^*$  associated with  $K$  (see Figure 1 for an example):

$$K^* = \{x \in \mathbb{R}^d : \langle x, y \rangle \geq 0 \text{ for all } y \in K\}. \quad (1.5)$$

Obviously,  $K^*$  is a closed convex cone.

Throughout this paper, we make the following assumptions on the cone  $K$  and on the distribution  $\mu$  of the random walk increments:

- (A1) The cone  $K$  is closed, convex, with non-empty interior.
- (A2) The random walk is truly  $d$ -dimensional, i.e., there is no  $u \neq 0$  such that  $\langle u, X_1 \rangle = 0$  almost surely. Moreover, the random walk started at zero can reach the interior  $K^\circ$  of the cone: there exists  $k > 0$  such that  $\mathbb{P}^0(\tau > k, S_k \in K^\circ) > 0$ .
- (A3) The random walk increments are  $L^1$ . We call  $m = \mathbb{E}X_1 = \int y \mu(dy)$  the drift.
- (A4) There exists a point  $t_0 \in K^*$  and a neighborhood  $V$  of  $t_0$  such that the Laplace transform  $L$  of  $\mu$  is finite in  $V$  and  $t_0$  is a minimum point of  $L$  restricted to  $K^* \cap V$ .

We would like to give some intuition on the hypothesis (A4), which is designed to perform an exponential change of measure adapted to the geometry of the problem (see Subsection 2.1). First, (A4) implies the existence of *some* exponential moments, but not all; it is even not necessary that the increments have a moment of order one.

For example, consider a random variable  $X$  with probability density function

$$f(x) = c_\gamma \left( \frac{e^{-x^2}}{1+x^2} 1_{\{x>0\}} + \frac{e^{\gamma^2}}{1+x^2} 1_{\{x\leq 0\}} \right),$$

where  $\gamma > 0$  is some fixed parameter and  $c_\gamma$  is the normalizing constant. Since the negative part of  $X$  has a half-Cauchy distribution, the variable  $X$  is not integrable. However, its Laplace transform

$$L(t) = \mathbb{E}(e^{tX}) = c_\gamma \int_0^\infty \frac{e^{-tx}e^{\gamma^2} + e^{tx}e^{-x^2}}{1+x^2} dx$$

is finite and differentiable for all  $t > 0$ , and its derivative

$$L'(t) = c_\gamma \int_0^\infty (e^{2tx-x^2} - e^{\gamma^2}) \frac{xe^{-tx}}{1+x^2} dx$$

is negative for  $t \in (0, \gamma)$ . Thus  $L$  reaches a minimum at some  $t_0 > 0$  (clearly  $L(t) \rightarrow \infty$  as  $t \rightarrow \infty$ ). Now taking a random vector  $Z = (X, Y)$  with density  $f(x)f(y)$  gives a two-dimensional example where  $Z$  is not integrable but its Laplace transform is finite in the quadrant  $[0, \infty)^2$  and reaches a global minimum inside the quadrant.

Regarding the existence of a local minimum in  $K^*$ , hypothesis (A4) is discussed in [11] (see its subsection 2.3; the condition is called (H2) there), where an equivalent geometric condition is given: in the presence of all exponential moments, condition (A4) is satisfied if and only if the support of the distribution  $\mu$  is not included in any ‘bad’ half-space  $\{x \in \mathbb{R}^d : \langle x, u \rangle \leq 0\}$ , with  $u$  in the dual cone  $K^* \setminus \{0\}$ .

Under assumptions (A1)–(A4), we proved in [11] that the exponential rate  $\rho$  of the survival probability is equal to  $L(t_0)$ , meaning that for all  $x \in K$ ,

$$\lim_{n \rightarrow \infty} \mathbb{P}^x(\tau > n)^{1/n} = L(t_0).$$

Furthermore,  $L(t_0) < 1$  if and only if the drift  $m$  does not belong to the closed cone  $K$ . Here, we shall prove a little bit more:

**Theorem 1.3.** *Assume hypotheses (A1)–(A4) above. If  $m \notin K^\circ$ , then*

$$\mathbb{P}^x(\tau > n) = \rho^n B_n,$$

where  $\rho = L(t_0) \in (0, 1]$ ,  $\sqrt[n]{B_n} \rightarrow 1$  and  $B_n \rightarrow 0$ .

Regarding the interior drift case, we shall prove the following estimate, in the setting of small step walks in orthants:

**Theorem 1.4.** *For all  $d$ -dimensional weighted small step walks with a drift interior to the orthant  $K = [0, \infty)^d$ , not trapped and satisfying (A2), we have for all  $x \in \mathbb{N}^d$*

$$\mathbb{P}^x(\tau > n) - \mathbb{P}^x(\tau = \infty) = \Theta(\rho^n B_n),$$

where  $\rho \in (0, 1)$  and  $B_n$  satisfies  $\sqrt[n]{B_n} \rightarrow 1$  and  $B_n \rightarrow 0$ .

**A one-dimensional example.** We now illustrate our previous results on the example of the simple random walk on  $\mathbb{Z}$ , with jump probabilities  $q$  to the left ( $-1$ ) and  $p = 1 - q$  to the right ( $+1$ ) and let  $K = [0, \infty)$ . In this setting,

$$\tau = \inf\{n > 0 : S_n < 0\} = \inf\{n > 0 : S_n = -1\}. \quad (1.6)$$

It is well known that, for any positive starting point  $x \in \mathbb{N}$ , the series (1.1) equals

$$F(t) = \frac{1 - \phi(t)^{x+1}}{1 - t}, \quad \text{with } \phi(t) = \frac{1 - \sqrt{1 - 4pqt^2}}{2pt}. \quad (1.7)$$

We may directly observe on (1.7) that, as stated in our Theorems 1.1 and 1.2, the function  $F(t)$  is not rational. Indeed  $F(t) \in \mathbb{R}(t)$  if and only if  $(1 - \sqrt{1 - 4pqt^2})^{x+1} \in \mathbb{R}(t)$ . But an expression like  $(1 - \sqrt{P})^n$  where  $P$  is a polynomial belongs to  $\mathbb{R}(t)$  if and only if  $\sqrt{P}$  belongs to  $\mathbb{R}(t)$ , as seen by a binomial expansion, and  $\sqrt{P}$  is rational if and only if  $P$  is a square in  $\mathbb{R}[t]$ . This is not the case here with  $P(t) = 1 - 4pqt^2$ , unless  $p$  or  $q$  equals 0. However,  $F(t)$  defines an algebraic function (as usual for one-dimensional random walks; see [1]).

In the zero drift case (meaning that  $p = q = \frac{1}{2}$ ), expanding (1.7) at  $t = 1$  and using singularity analysis, one finds

$$\mathbb{P}^x(\tau > n) \sim (x+1) \sqrt{\frac{2}{\pi}} \frac{1}{n^{1/2}} \quad (\text{in particular } \rho = 1).$$

If the drift is negative ( $q > p$ ), the function  $F$  in (1.7) is analytic at 1 as  $\phi(1) = 1$ , and the singularities  $t = \pm \frac{1}{2\sqrt{pq}}$  will both contribute to the asymptotics, which reads

$$\mathbb{P}^x(\tau > n) \sim (x+1) \left(\frac{q}{p}\right)^{(x+1)/2} \left( \frac{1}{\frac{1}{2\sqrt{pq}} - 1} + \frac{(-1)^{x+n}}{\frac{1}{2\sqrt{pq}} + 1} \right) \frac{(2\sqrt{pq})^n}{\sqrt{2\pi}n^{3/2}}.$$

Finally, when the drift is positive ( $p > q$ ), the probability of survival admits the following two-term asymptotics (observe the similarity with the negative drift situation)

$$\mathbb{P}^x(\tau > n) = \left(1 - \left(\frac{q}{p}\right)^{x+1}\right) + (x+1) \left(\frac{q}{p}\right)^{(x+1)/2} \left( \frac{1}{\frac{1}{2\sqrt{pq}} - 1} + \frac{(-1)^{x+n}}{\frac{1}{2\sqrt{pq}} + 1} \right) \frac{(2\sqrt{pq})^n}{\sqrt{2\pi}n^{3/2}} + \dots$$

The three asymptotics above, which illustrate our Theorems 1.3 and 1.4, are obtained by studying the singularities of the generating function (1.7) and by using classical transfer theorems on the coefficients.

## 2. SURVIVAL PROBABILITY IN THE NON-INTERIOR DRIFT CASE:

### PROOF OF THEOREM 1.3

**2.1. Basics on the Laplace transform.** Let us first recall some basic properties. The Laplace transform of a random vector  $X = (X^{(1)}, \dots, X^{(d)}) \in \mathbb{R}^d$  with probability distribution  $\mu$  is the function  $L$  defined for  $t \in \mathbb{R}^d$  by

$$L(t) = \mathbb{E}\left(e^{(t,X)}\right) = \int_{\mathbb{R}^d} e^{(t,y)} \mu(dy).$$

It is finite in some neighborhood of the origin if and only if  $\mathbb{E}(e^{\alpha\|X\|})$  is finite for some  $\alpha > 0$ . If  $L$  is finite in some neighborhood of the origin, say  $\overline{B(0, r)}$ , then  $L$  is infinitely differentiable in  $B(0, r)$  and its partial derivatives are given there by

$$\frac{\partial L(t)}{\partial t_i} = \mathbb{E}(X^{(i)} e^{(t, X)}).$$

Therefore, the expectation  $\mathbb{E}X = (\mathbb{E}X^{(1)}, \dots, \mathbb{E}X^{(d)})$  of  $X$  is equal to the gradient of  $L$  at the origin  $\nabla L(0)$ . Notice that  $X$  is centered (i.e.,  $\mathbb{E}X = 0$ ) if and only if 0 is a critical point of  $L$ . Since  $L$  is a convex function, this means that 0 is a minimum point of  $L$  in  $\overline{B(0, r)}$ . Now suppose that  $L$  is finite in some ball  $\overline{B(t_0, r)}$ , and define a new probability measure  $\mu_*$  by

$$\mu_*(dy) = \frac{e^{(t_0, y)}}{L(t_0)} \mu(dy).$$

The Laplace transform  $L_*$  of  $\mu_*$  is linked to that of  $\mu$  by the relation  $L_*(t) = L(t_0 + t)/L(t_0)$ , and therefore  $L_*$  is finite in some neighborhood of the origin. As a consequence, applying the results above shows that any random vector  $X_*$  with distribution  $\mu_*$  satisfies:

- $\mathbb{E}(e^{\alpha\|X_*\|}) < \infty$  for some  $\alpha > 0$ ;
- $\mathbb{E}X_* = \nabla L(t_0)/L(t_0)$ .

As we shall see later, the relevant value of  $L$  for our problem is its minimum on the dual cone  $K^*$  defined by (1.5).

We now investigate further properties of  $\mathbb{E}X_*$  when  $t_0$  satisfies the assumption (A4), i.e.,  $t_0$  is a local minimum point of  $L$  restricted to  $K^*$ . By convexity of  $L$ , the point  $t_0$  is necessarily a global minimum on  $K^*$ ; we don't assume  $t_0$  to be a global minimum on  $\mathbb{R}^d$ . Define the two sets

$$S = \{u \in \mathbb{R}^d : \exists \varepsilon > 0, \forall s \in [-\varepsilon, \varepsilon], t_0 + su \in K^*\},$$

$$S^+ = \{u \in \mathbb{R}^d : \exists \varepsilon > 0, \forall s \in [0, \varepsilon], t_0 + su \in K^*\}.$$

Of course  $S \subset S^+$ . Since  $K^*$  is a convex cone, the set  $S$  contains at least  $t_0$ , while the set  $S^+$  contains at least  $K^*$ . Assuming (A4), we observe the following:

- if  $u$  belongs to  $S^+$ , then the function  $\phi(s) = L(t_0 + su)$  defined on some small interval  $[0, \varepsilon]$  reaches a minimum at  $s = 0$ , hence  $\phi'(0) = \langle \nabla L(t_0), u \rangle \geq 0$ . This holds for all  $u \in K^*$  since  $K^* \subset S^+$ , therefore  $\nabla L(t_0)$  belongs to the dual cone  $(K^*)^*$  associated with  $K^*$ ;
- if  $u$  belongs to  $S$ , the function  $\phi(s)$  defined on some small interval  $[-\varepsilon, \varepsilon]$  reaches its minimum at  $s = 0$ , hence  $\phi'(0) = 0$ . Therefore  $\nabla L(t_0)$  is orthogonal to  $S$  (and so at least to  $t_0$  itself).

Translating these observations in terms of the expectation of  $X_*$ , we obtain:

**Lemma 2.1.** *Assume (A1) and (A4). The expectation  $\mathbb{E}X_*$  of any random vector with distribution  $\mu_*$  belongs to the cone  $K$  and is orthogonal to  $t_0$ .*

*Proof.* Since  $K$  is a closed convex cone, it is well known that  $(K^*)^* = K$  (see Exercise 2.31 in [4] for example). Everything now follows from the relation  $\mathbb{E}X_* = \nabla L(t_0)/L(t_0)$ .  $\square$

**2.2. Proof of Theorem 1.3.** We shall use the preceding  $t_0$  and  $\mu_*$  in order to perform an exponential change of measure. For any non-negative and measurable function  $f : \mathbb{R}^n \rightarrow [0, \infty)$ , elementary algebraic manipulations give:

$$\begin{aligned} \mathbb{E}^x(f(S_1, \dots, S_n)) &= \int_{\mathbb{R}^n} f\left(x + x_1, x + \sum_{i=1}^2 x_i, \dots, x + \sum_{i=1}^n x_i\right) \prod_{i=1}^n \mu(dx_i) \\ &= \rho^n \int_{\mathbb{R}^n} f\left(x + x_1, x + \sum_{i=1}^2 x_i, \dots, x + \sum_{i=1}^n x_i\right) e^{-\langle t_0, \sum_{i=1}^n x_i \rangle} \prod_{i=1}^n \mu_*(dx_i) \\ &= \rho^n e^{\langle t_0, x \rangle} \mathbb{E}_*^x\left(f(S_1, \dots, S_n) e^{-\langle t_0, S_n \rangle}\right), \end{aligned}$$

where

- $\rho = L(t_0)$ ,
- $\mathbb{E}_*^x$  is the expectation with respect to  $\mathbb{P}_*^x$ , a probability distribution under which  $(S_n)_{n \geq 0}$  is a random walk with increment distribution  $\mu_*$  and started at  $S_0 = x$ .

Taking  $f(s_1, \dots, s_n) = \prod_{i=1}^n 1_K(s_i)$  leads to

$$\mathbb{P}^x(\tau > n) = \rho^n e^{\langle t_0, x \rangle} \mathbb{E}_*^x\left(e^{-\langle t_0, S_n \rangle}, \tau > n\right),$$

so that Theorem 1.3 will follow from the two lemmas below:

**Lemma 2.2.** *Assume (A1)–(A4). Then, for all  $x \in K$ ,*

$$\lim_{n \rightarrow \infty} \sqrt[n]{\mathbb{E}_*^x\left(e^{-\langle t_0, S_n \rangle}, \tau > n\right)} = 1.$$

**Lemma 2.3.** *Assume (A1)–(A4). If the drift  $m = \mathbb{E}X_1$  does not belong to  $K^o$ , then for all  $x \in K$ ,*

$$\lim_{n \rightarrow \infty} \mathbb{E}_*^x\left(e^{-\langle t_0, S_n \rangle}, \tau > n\right) = 0.$$

Lemma 2.2 is fully proved in [11]. However, to make our paper self-contained, we propose here a short proof of it in a simplified setting. Instead of (A2) we will work under the following hypothesis: there exist  $k > 0$  and  $z \in K^o$  such that  $\mathbb{P}(\tau > k, S_k = z) > 0$ . In the majority of classical lattice random walks, the previous hypothesis is satisfied, as for instance for all 74 non-singular small step random walks considered in [3].

*Proof of Lemma 2.2.* First observe that on the event  $\{\tau > n\}$ , we have  $S_n \in K$ , hence  $\langle t_0, S_n \rangle \geq 0$  since  $t_0 \in K^*$ . As a consequence  $\mathbb{E}_*^x\left(e^{-\langle t_0, S_n \rangle}, \tau > n\right) \leq \mathbb{P}_*^x(\tau > n) \leq 1$ , and what remains to prove is that

$$\liminf_{n \rightarrow \infty} \sqrt[n]{\mathbb{E}_*^x\left(e^{-\langle t_0, S_n \rangle}, \tau > n\right)} \geq 1.$$

By inclusion of events and basic properties of the  $n$ -th root limit, it suffices to prove the result for  $x = 0$ , in which case we get rid of the  $x$  superscript on  $\mathbb{E}_*$  and  $\mathbb{P}_*$ . We compute a lower bound of the expectation as follows:

$$\mathbb{E}_*\left(e^{-\langle t_0, S_n \rangle}, \tau > n\right) \geq e^{-a_n} \mathbb{P}_*\left(|\langle t_0, S_n \rangle| \leq a_n, \tau > n\right),$$

with  $a_n = n^{3/4}$ . The  $e^{-a_n}$  term goes to 1 in the  $n$ -th root limit, thus we focus on the probability in the right-hand side.



Using our hypothesis, we can use the first  $k\lfloor\sqrt{n}\rfloor$  steps to push the walk  $\lfloor\sqrt{n}\rfloor$  times in the direction  $z$  without leaving the cone: by inclusion of events and the Markov property, we have

$$\mathbb{P}_* \left( |\langle t_0, S_n \rangle| \leq a_n, \tau > n \right) \geq \alpha^{b_n} \mathbb{P}_*^{b_n z} \left( |\langle t_0, S_{n-kb_n} \rangle| \leq a_n, \tau > n - kb_n \right),$$

where  $\alpha = \mathbb{P}(\tau > k, S_k = z) > 0$  and  $b_n = \lfloor\sqrt{n}\rfloor$ . Here again, the  $\alpha^{b_n}$  term will disappear in the  $n$ -th root limit, and the  $-kb_n$  does not play any significant role in  $n - kb_n$ , so we are left to consider the probability

$$\mathbb{P}_*^{b_n z} \left( |\langle t_0, S_n \rangle| \leq a_n, \tau > n \right).$$

At this point, we take into account the “new drift”  $d = \mathbb{E}_* X_1$  of the random walk under  $\mathbb{P}_*$ , and consider the centered random walk  $\tilde{S}_n = S_n - nd$ . Lemma 2.1 asserts that:

- $d$  is orthogonal to  $t_0$ , so that  $\langle t_0, S_n \rangle = \langle t_0, \tilde{S}_n \rangle$ ,
- $d$  belongs to  $K$ , hence

$$\{\tau(\tilde{S}_\ell) > n\} = \{\tilde{S}_1, \dots, \tilde{S}_n \in K\} \subset \{S_1, \dots, S_n \in K\} = \{\tau > n\}.$$

Due to these facts, our probability can be bounded from below by

$$\begin{aligned} \mathbb{P}_*^{b_n z} \left( |\langle t_0, \tilde{S}_n \rangle| \leq a_n, \tau(\tilde{S}_\ell) > n \right) &= \mathbb{P}_* \left( |\langle t_0, b_n z + \tilde{S}_n \rangle| \leq a_n, \tau(b_n z + \tilde{S}_\ell) > n \right) \\ &= \mathbb{P}_* \left( |\langle t_0, z + \tilde{S}_n b_n^{-1} \rangle| \leq a_n b_n^{-1}, \tau(z + \tilde{S}_\ell b_n^{-1}) > n \right) \\ &\geq \mathbb{P}_* \left( \|\tilde{S}_\ell b_n^{-1}\| < \varepsilon \text{ for all } \ell = 1, \dots, n \right), \end{aligned}$$

where we have used the homogeneity of the cone, namely  $K/b_n = K$  on the second line, and then chosen  $\varepsilon > 0$  so that the ball  $B(z, \varepsilon) \subset K$ . Now recall that, under  $\mathbb{P}_*$ , the increments  $X_n$  of the random walk  $S_n$  have a distribution  $\mu^*$  with some exponential moments, hence the  $X_n$ 's are in  $L^2$ , and so do the increments  $X_n - d$  of the centered random walk  $\tilde{S}_n$ . Therefore, the functional central limit theorem [2, Thm 8.2] is in force and, in conjunction with Portmanteau theorem [2, Thm 2.1], we obtain

$$\liminf_{n \rightarrow \infty} \mathbb{P}_*^{b_n z} \left( |\langle t_0, \tilde{S}_n \rangle| \leq a_n, \tau(\tilde{S}_\ell) > n \right) \geq \mathbb{P}_* \left( \|B_t\| < \varepsilon \text{ for all } t \in [0, 1] \right) > 0,$$

where  $(B_t)_{t \in [0, 1]}$  is the image of a standard Brownian motion started at 0 under a (possibly degenerate) linear transformation. This concludes the proof of Lemma 2.2.  $\square$

*Proof of Lemma 2.3.* The proof will be done separately, according to whether  $t_0$  is zero or not. First assume  $t_0 \neq 0$ . On the event  $\{\tau > n\}$ , for all  $k = 1, \dots, n$ , we have that  $S_k \in K$ , hence  $R_k = \langle t_0, S_k \rangle \geq 0$  since  $t_0 \in K^*$ . Therefore

$$\mathbb{E}_*^x \left( e^{-\langle t_0, S_n \rangle}, \tau > n \right) \leq \mathbb{P}_*^x \left( R_k \geq 0 \text{ for all } k = 1, \dots, n \right).$$

Now, under  $\mathbb{P}_*^x$ , the process  $R_k = \langle t_0, S_k \rangle$  is a random walk with increments  $Y_k = \langle t_0, X_k \rangle$  having mean  $\langle t_0, \mathbb{E}_* X_1 \rangle = 0$  (see Lemma 2.1). Since the initial distribution  $\mu$  is truly  $d$ -dimensional and  $\mu_*$  is absolutely continuous with respect to  $\mu$ , the new distribution  $\mu_*$  is also truly  $d$ -dimensional.

Thus, under  $\mathbb{P}_*^x$ , the increments  $Y_k$  are non-degenerate (i.e., it does not hold that  $Y_k = 0$  almost surely). It is well known (see [9, Thm 1 & 2 of XII,2]) that for such a

one-dimensional random walk, almost surely,

$$-\infty = \liminf R_n < \limsup R_n = +\infty.$$

Accordingly,

$$\lim_{n \rightarrow \infty} \mathbb{P}_*^x(R_k \geq 0 \text{ for all } k = 1, \dots, n) = \mathbb{P}_*^x(R_k \geq 0 \text{ for all } k \geq 1) = 0.$$

We now turn to the case  $t_0 = 0$ . This time  $\langle t_0, S_n \rangle = 0$ , so we don't learn anything by considering this specific one-dimensional random walk. The idea is to replace  $t_0$  with an appropriate  $\tilde{t}_0$  and apply the same argument as before. To do this, observe that we know from Lemma 2.1 that  $\mathbb{E}_* X_1$  belongs to the cone  $K$ , but when  $t_0 = 0$  the change of measure has no effect:  $\mu_* = \mu$ . Hence the original drift  $m = \mathbb{E} X_1$  belongs to  $K$ . Since we assumed  $m \notin K^o$ , we are left with a drift  $m$  on the boundary  $\partial K$  of the cone  $K$ .

If  $C$  is a closed cone, the interior of its dual cone has the following description:

$$(C^*)^o = \{x \in \mathbb{R}^d : \langle x, y \rangle > 0 \text{ for all } y \in C \setminus \{0\}\}$$

(see Exercise 2.31(d) in [4] for example). As a consequence, the boundary is given by

$$\partial C^* = \{x \in C^* : \langle x, y \rangle = 0 \text{ for some } y \in C \setminus \{0\}\},$$

and applying this to the closed convex cone  $C = K^*$  gives

$$\partial K = \{x \in K : \langle x, y \rangle = 0 \text{ for some } y \in K^* \setminus \{0\}\},$$

since  $(K^*)^* = K$ . Going back to our drift  $m \in \partial K$ , there exists some  $\tilde{t}_0 \in K^* \setminus \{0\}$  such that  $\langle \tilde{t}_0, m \rangle = 0$ . Setting  $\tilde{R}_k = \langle \tilde{t}_0, S_k \rangle$ , we obtain a centered and non-degenerate one-dimensional random walk such that  $S_k \in K$  implies  $\tilde{R}_k \geq 0$ . Therefore

$$\mathbb{E}_*^x(e^{-\langle t_0, S_n \rangle}, \tau > n) = \mathbb{P}^x(\tau > n) \leq \mathbb{P}_*^x(\tilde{R}_k \geq 0 \text{ for all } k = 1, \dots, n),$$

and the conclusion follows as in the first case.  $\square$

The proof of Theorem 1.3 is complete.

### 3. SURVIVAL PROBABILITY IN THE INTERIOR DRIFT CASE: PROOF OF THEOREM 1.4

In this section, we restrict our attention to the cone  $K = [0, \infty)^d$  and small step walks, i.e., random walks on  $\mathbb{Z}^d$  with increments  $X_k$  satisfying  $X_k \in \{-1, 0, 1\}^d$  almost surely. For such walks, we investigate the case of a drift  $m = \mathbb{E} X_k$  interior to the cone  $K$ , i.e., such that  $\langle m, e_i \rangle > 0$  for  $i = 1, \dots, d$ , where  $(e_1, \dots, e_d)$  denotes the standard basis of  $\mathbb{R}^d$ . We will use the notation  $X_k^{(i)} = \langle X_k, e_i \rangle$ . Since the drift is in the interior of  $K$ , we know that

$$\lim_{n \rightarrow \infty} \mathbb{P}^x(\tau > n) = \mathbb{P}^x(\tau = \infty) > 0$$

for all  $x \in K$ ; see Lemma 3.1 for a precise statement and a proof.

Here we wish to estimate the error term  $\delta_n = \mathbb{P}^x(\tau > n) - \mathbb{P}^x(\tau = \infty)$ . We exclude the case where  $\delta_n = 0$  for all  $n$  by assuming that the random walk is not trapped, i.e., the increments satisfy  $\mathbb{P}(X_k \notin K) > 0$ . Under this assumption we will prove Theorem 1.4, namely that

$$\mathbb{P}^x(\tau > n) - \mathbb{P}^x(\tau = \infty) = \Theta(\rho^n B_n).$$

Before going into the proof, we collect preliminary estimates on  $\mathbb{P}^x(\tau = \infty)$ .

**3.1. Exact formula for a one-dimensional small step walk.** First of all, we consider the one-dimensional setting with  $p = \mathbb{P}(X_k = 1)$ ,  $r = \mathbb{P}(X_k = 0)$ ,  $q = \mathbb{P}(X_k = -1)$ ,  $p + r + q = 1$ . Let  $\tau$  be as in (1.6) and assume  $m = p - q > 0$ . Then it is known that, for all  $x \in \mathbb{N}$ ,

$$\mathbb{P}^x(\tau = \infty) = 1 - \left(\frac{q}{p}\right)^{x+1}.$$

If  $q > 0$ , this can be rewritten as

$$\mathbb{P}^x(\tau = \infty) = 1 - \gamma e^{-sx}, \quad (3.1)$$

where  $\gamma = q/p$  and  $s > 0$  is the unique solution to  $e^{-s} = q/p$ .

One way to obtain the formula above is to use the discrete harmonicity of the function  $u_x = \mathbb{P}^x(\tau = \infty)$ : by the Markov property, we have  $u_x = qu_{x-1} + ru_x + pu_{x+1}$  for all  $x \geq 1$ , which is solved in  $u_x = a + b\left(\frac{q}{p}\right)^x$ . Then  $a$  and  $b$  are determined through initial and limit behaviors of  $u_x$ .

For future use, we notice the following fact: let

$$L(t) = \mathbb{E}\left(e^{tX_k}\right) = pe^t + r + qe^{-t}$$

be the Laplace transform associated with the random walk increments. Its derivative is given by  $L'(t) = pe^t - qe^{-t}$ . Evaluating at  $t = -s$ , where  $s$  is as above the solution to  $e^{-s} = q/p$ , leads to

$$L(-s) = 1 \quad \text{and} \quad L'(-s) = q - p = -m < 0. \quad (3.2)$$

The last value is exactly the opposite of the drift.

**3.2. Estimate for  $\mathbb{P}^x(\tau < \infty)$  in the  $d$ -dimensional small step case.** Let us go back to our  $d$ -dimensional small step walk  $(S_n)_n$  with drift  $m$  interior to the cone  $K = [0, \infty)^d$  and such that  $\mathbb{P}(X_k \notin K) > 0$ . The simple inclusion of events

$$\{\exists n > 0, \langle S_n, e_i \rangle < 0\} \subset \{\tau < \infty\} \subset \bigcup_{i=1}^d \{\exists n > 0, \langle S_n, e_i \rangle < 0\}$$

leads to the bounds

$$\frac{g(x)}{d} \leq \mathbb{P}^x(\tau < \infty) \leq g(x), \quad (3.3)$$

where  $g(x) = \sum_{i=1}^d \mathbb{P}^x(\exists n > 0, \langle S_n, e_i \rangle < 0)$ . Now, for each  $i$ , the one-dimensional small step walk  $(\langle S_n, e_i \rangle)_n$  with increments  $X_k^{(i)}$  has a drift  $\mathbb{E}X_k^{(i)} = \langle m, e_i \rangle > 0$ . Since  $\mathbb{P}(X_k \notin K) > 0$ , the set  $I$  of indices  $i$  for which  $\mathbb{P}(X_k^{(i)} = -1) > 0$  is non-empty, and applying the exact formula (3.1) of the preceding paragraph, we obtain:

$$g(x) = \sum_{i \in I} \gamma_i e^{-s_i \langle x, e_i \rangle},$$

where  $\gamma_i = \mathbb{P}(X_k^{(i)} = -1) / \mathbb{P}(X_k^{(i)} = 1) \in (0, 1)$  and  $s_i > 0$  is the unique solution to  $e^{-s_i} = \gamma_i$ .

**3.3. Proof of Theorem 1.4.** Fix  $x \in \mathbb{N}^d$  and set

$$\delta_n = \mathbb{P}^x(\tau > n) - \mathbb{P}^x(\tau = \infty) = \mathbb{P}^x(\tau > n, \text{ but } S_m \notin K \text{ for some } m > n).$$

By the Markov property of the random walk, we can express  $\delta_n$  as follows:

$$\delta_n = \mathbb{E}^x \left( \tau > n, \mathbb{P}^{S_n}(\tau < \infty) \right),$$

so that inequality (3.3) leads to  $\delta_n = \Theta(g_n)$ , where  $g_n = \mathbb{E}^x(\tau > n, g(S_n))$ . It remains to estimate

$$g_n = \sum_{i \in I} \gamma_i \mathbb{E}^x \left( \tau > n, e^{\langle S_n, -s_i e_i \rangle} \right).$$

To do this, we apply to each term in the sum a specific exponential change of measure. Set

$$\mu_{*i}(dy) = \frac{e^{\langle -s_i e_i, y \rangle}}{L(-s_i e_i)} \mu(dy),$$

where  $\mu$  is the common distribution of the increments  $X_k$  of the random walk, and  $L(t) = \mathbb{E}(e^{t \cdot X_k})$  is their Laplace transform. Then basic algebraic manipulations as in Section 2.2 lead to

$$\mathbb{E}^x \left( \tau > n, e^{\langle S_n, -s_i e_i \rangle} \right) = L(-s_i e_i)^n e^{\langle -s_i e_i, x \rangle} \mathbb{P}_{*i}^x(\tau > n).$$

Now observe that  $t \mapsto L(te_i) = \mathbb{E}(e^{t X_k^{(i)}})$  is the one-dimensional Laplace transform of the increments  $X_k^{(i)}$ . Since  $s_i$  is the solution to  $e^{-s_i} = \gamma_i = \frac{\mathbb{P}(X_k^{(i)} = -1)}{\mathbb{P}(X_k^{(i)} = 1)}$ , we are in the same situation as in (3.2), so that

$$L(-s_i e_i) = 1 \quad \text{and} \quad \frac{\partial L}{\partial t_i}(-s_i e_i) = -\langle m, e_i \rangle < 0.$$

Therefore, equation (3.3) reads

$$\mathbb{E}^x \left( \tau > n, e^{\langle S_n, -s_i e_i \rangle} \right) = e^{\langle -s_i e_i, x \rangle} \mathbb{P}_{*i}^x(\tau > n),$$

and the new drift under  $\mathbb{P}_{*i}^x$ , which is given by the gradient of  $L$  at the point  $-s_i e_i$ , has a strictly negative  $i$ -th coordinate. As a consequence, this drift does not belong to the cone  $K = [0, \infty)^d$ , and it follows from Theorem 1.3 that

$$\mathbb{P}_{*i}^x(\tau > n) = \rho_i^n B_{i,n},$$

where  $\rho_i \in (0, 1)$ ,  $\sqrt[n]{B_{i,n}} \rightarrow 1$  and  $B_{i,n} \rightarrow 0$  as  $n \rightarrow \infty$ . Finally, we get

$$g_n = \sum_{i \in I} \gamma_i \rho_i^n B_{i,n},$$

which can be rewritten in the form  $g_n = \rho^n B_n$ , by selecting

$$\rho = \max\{\rho_i : i \in I\} < 1.$$

It is then clear that  $\sqrt[n]{B_n} \rightarrow 1$  and  $B_n \rightarrow 0$ , and the proof is complete.

### 3.4. Positivity of the escape probability.

**Lemma 3.1.** *Assume (A1) and (A2). If the drift  $m = \mathbb{E}X_1$  belongs to  $K^\circ$ , then the function  $h(x) = \mathbb{P}^x(\tau = \infty)$  satisfies:*

(1)  *$h$  is harmonic for the killed random walk, i.e.,*

$$h(x) = \mathbb{E}^x(h(S_n), \tau > n).$$

(2)  *$h(x) > 0$  for all  $x \in K$ .*

(3)  *$\lim_{t \rightarrow \infty} h(tu) = 1$  for all  $u \in K^\circ$ .*

*Proof.* Item (1) is just the Markov property applied at time  $n$ . The relation is valid disregarding the position of the drift. We now prove (2).

*First step.* We begin with a simple geometric fact. For any  $z \in K^\circ$ , the non-decreasing sequence of sets  $K - kz$  will ultimately cover the whole space, i.e.,  $\cup_{k \geq 0} (K - kz) = \mathbb{R}^d$ . To see this, select  $\varepsilon > 0$  such that  $B(z, \varepsilon) \subset K$ . For any  $x \in \mathbb{R}^d$ , there exists  $k > 0$  such that  $\|x/k\| < \varepsilon$ , hence  $z + \frac{x}{k}$  belongs to  $K$ . By homogeneity of  $K$ , it follows that  $kz + x \in K$ , i.e.,  $x \in K - kz$ .

*Second step.* Let us consider the random walk  $(S_n)$  with drift  $m \in K^\circ$  and select  $\varepsilon > 0$  such that  $B(m, \varepsilon) \subset K$ . By the strong law of large numbers  $S_n/n \rightarrow m$  almost surely, therefore, for almost all  $\omega$ , there exists  $n_0 = n_0(\omega)$  such that

$$n \geq n_0 \Rightarrow \left\| \frac{S_n(\omega)}{n} - m \right\| < \varepsilon \Rightarrow S_n(\omega) \in K.$$

Considering now the first positions  $S_1(\omega), S_2(\omega), \dots, S_{n_0-1}(\omega)$ , the first step of the proof ensures that there exists  $k \geq 0$  such that they all belong to  $K - kz$ , where  $z \in K^\circ$  is to be fixed in the last step of the proof. Since one has

$$K \subset K - kz \text{ (recall that } K + K \subset K),$$

all positions  $S_n(\omega), n \geq n_0$ , also belong to  $K - kz$  and we obtain the following:

$$\mathbb{P}\left(\cup_{k \geq 0} \{S_n \in K - kz \text{ for all } n \geq 0\}\right) = 1.$$

Since the events inside the probability above form a non-decreasing sequence, it follows that

$$\lim_{k \rightarrow \infty} \mathbb{P}\left(S_n \in K - kz \text{ for all } n \geq 0\right) = 1. \quad (3.4)$$

*Last step.* To conclude, we invoke hypothesis (A2), that claims the existence of an integer  $\ell \geq 1$  such that

$$\mathbb{P}(\tau > \ell, S_\ell \in K^\circ) > 0.$$

Fix some  $u \in K^\circ$ . Since  $K^\circ = \cup_{\lambda > 0} (K + \lambda u)$ , there is a  $z = \lambda u \in K^\circ$  such that

$$\mathbb{P}(\tau > \ell, S_\ell \in K + z) = p > 0.$$

By the Markov property, a concatenation of  $m$  such  $\ell$ -steps paths leads to

$$\mathbb{P}(\tau > m\ell, S_{m\ell} \in K + mz) \geq p^m > 0.$$

On the other hand, it follows from (3.4) that there exists  $k \geq 0$  such that

$$\mathbb{P}(S_n \in K - kz \text{ for all } n \geq 0) \geq 1/2.$$

Now choose  $m \geq k$ . Since  $S_{m\ell} \in K + mz$  and  $S_n - S_{m\ell} \in K - kz$  imply  $S_n \in K$ , we obtain

$$\mathbb{P}(\tau = \infty) \geq \mathbb{P}(\tau > m\ell, S_{m\ell} \in K + mz) \times \mathbb{P}(S_n \in K - kz \text{ for all } n \geq 0) > 0.$$

We have just proved that  $g(0) > 0$ . The result follows since  $g(x) \geq g(0)$  for all  $x \in K$ , by inclusion of events.

We conclude with the proof of (3). The limit (3.4) obtained in the *second step* of Item (2) can be recast as:

$$\lim_{k \rightarrow \infty} \mathbb{P}^{kz}(\tau = \infty) = 1,$$

where  $z$  is any vector in  $K^\circ$ . Since  $g(x) = \mathbb{P}^x(\tau = \infty)$  is non-decreasing in every direction, the proof is completed.  $\square$

#### 4. PROOF OF THEOREMS 1.1 AND 1.2

In this section, we show that our estimates on  $a_n = \mathbb{P}^x(\tau > n)$  given in Theorems 1.3 and 1.4 are not compatible with the generating function  $F(t) = \sum_{n \geq 0} a_n t^n$  being rational, using classical singularity analysis for rational functions. The starting point is Theorem IV.9 in [10], which asserts the following:

**Theorem 4.1.** *If  $F(z) = \sum_{n \geq 0} a_n z^n$  is a rational function that is analytic at 0 and has poles at points  $\alpha_1, \alpha_2, \dots, \alpha_k$ , then its coefficients are sums of exponential-polynomials: there exist  $k$  polynomials  $P_j$  such that, for  $n$  larger than some fixed  $n_0$ ,*

$$a_n = \sum_{j=1}^k P_j(n) \alpha_j^{-n}.$$

Both estimates in Theorems 1.3 and 1.4 have the following form:

$$a_n = a + \Theta(\rho^n B_n),$$

where  $a \geq 0$ ,  $\rho \in (0, 1]$ ,  $\sqrt[n]{B_n} \rightarrow 1$  and  $B_n \rightarrow 0$ . Therefore Theorems 1.1 and 1.2 asserting the non-rationality of  $F$  will follow in both cases from the following elementary lemma.

**Lemma 4.2.** *Let  $c_1, \dots, c_k$  be distinct non-zero complex numbers and  $P_1, \dots, P_k$  be non-zero complex polynomials. Set  $a_n = \sum_{j=1}^k P_j(n) c_j^n$ . If  $a_n = a + \Theta(\rho^n B_n)$  for some  $a \geq 0$ ,  $\rho > 0$  and  $B_n > 0$  such that  $\sqrt[n]{B_n} \rightarrow 1$ , then necessarily  $B_n \not\rightarrow 0$ .*

*Proof.* If  $a_n = \sum_{j=1}^k P_j(n) c_j^n$ , then  $a_n - a$  has the same form, thus, without loss of generality, we can assume  $a = 0$ . Write  $c_j = r_j z_j$  with  $r_j > 0$  and  $|z_j| = 1$ . Let  $r = \max\{r_j : j = 1, \dots, k\}$  and let  $J$  be the subset of indices  $j$  such that  $r_j = r$ . Then

$$a_n = \sum_{j=1}^k P_j(n) c_j^n = r^n \left( \sum_{j \in J} P_j(n) z_j^n + o(t^n) \right),$$

where  $0 < t < 1$ . For future use, note that the numbers  $z_j, j \in J$  are all distinct (this is so since we kept at most one  $c_j$  in any fixed “direction”  $z_j$ : the one with maximum modulus).

We first show that  $r = \rho$ . Since  $a_n = \Theta(\rho^n B_n)$  and  $\sqrt[n]{B_n} \rightarrow 1$ , it follows that  $a_n/\rho^n$  goes to one in the  $n$ -th root limit. Thus, for any  $\varepsilon > 0$ ,

$$\left( (1 - \varepsilon)\rho \right)^n \leq a_n \leq \left( (1 + \varepsilon)\rho \right)^n$$

for  $n$  large enough. Therefore

$$\left( \frac{(1-\varepsilon)\rho}{r} \right)^n \leq \left| \sum_{j \in J} P_j(n) z_j^n + o(t^n) \right| \leq \left( \frac{(1+\varepsilon)\rho}{r} \right)^n \quad (4.1)$$

for  $n$  large enough. If  $\rho > r$  then we can choose  $\varepsilon > 0$  such that the lower bound is  $A^n$  for some  $A > 1$ .

But then we would have

$$A^n \leq \left| \sum_{j \in J} P_j(n) z_j^n + o(t^n) \right| \leq \sum_{j \in J} |P_j(n)| + |o(t^n)|$$

and this is impossible since  $\sum_{j \in J} |P_j(n)|$  grows polynomially. On the other hand, if  $\rho < r$  then we can choose  $\varepsilon > 0$  such that the upper bound in (4.1) is  $A^n$  for some  $A < 1$ . This implies

$$\sum_{j \in J} P_j(n) z_j^n \rightarrow 0.$$

Dividing this by  $n^p$ , where  $p$  stands for the maximum degree of polynomials  $P_j$ , leads to the convergence

$$\sum_{j \in J'} a_j z_j^n \rightarrow 0,$$

where  $J' \subset J$  is a non-empty subset of indices (those  $j$  for which  $P_j$  has degree  $p$ ) and the  $a_j$ 's are non-zero complex numbers. Since the numbers  $z_j$  are distinct complex numbers with modulus 1, this contradicts Lemma 4.3 below. The assertion  $r = \rho$  is now established, hence we have

$$\left| \sum_{j \in J} P_j(n) z_j^n + o(t^n) \right| = \frac{a_n}{\rho^n} = \Theta(B_n).$$

We have seen just before that this expression cannot go to zero as  $n \rightarrow \infty$ , thus  $B_n \not\rightarrow 0$ .  $\square$

**Lemma 4.3.** *Let  $z_1, \dots, z_k$  be distinct complex numbers with modulus  $\geq 1$ . If*

$$\lim_{n \rightarrow \infty} \sum_{j=1}^k a_j z_j^n = 0,$$

*then necessarily  $a_1 = \dots = a_k = 0$ .*

*Proof.* Denote by  $A_n$  the quantity  $\sum_{j=1}^k a_j z_j^n$ . Clearly, given any complex numbers  $\alpha_0, \dots, \alpha_{k-1}$ ,

$$\sum_{i=0}^{k-1} \alpha_i A_{n+i} = \sum_{j=1}^k a_j P(z_j) z_j^n, \quad (4.2)$$

where  $P(z) = \sum_{i=0}^{k-1} \alpha_i z^i$ . We can choose the polynomial  $P$  so as to have  $P(z_1) = 1$  and all other  $P(z_j) = 0$ . We then take the limit of (4.2) as  $n \rightarrow \infty$ , using the assumption of Lemma 4.3. We find that the term  $a_1 z_1^n$  should go to zero, which implies that  $a_1 = 0$ , since  $|z_1| \geq 1$ . A similar reasoning gives that all  $a_j = 0$ , and thus Lemma 4.3 is proved.  $\square$

## 5. THE EXCURSION GENERATING FUNCTION

In this section, we look at lattice random walks in convex cones. Besides the generating function of the survival probabilities (1.1), it is natural to ask whether the excursion generating function

$$E(t) = \sum_{n \geq 0} \mathbb{P}^x(\tau > n, S_n = y) t^n \quad (5.1)$$

can be rational, for given starting and ending points  $x, y \in K$ . When the cone  $K$  is an orthant  $\mathbb{N}^d$  and  $x = y = (0, \dots, 0)$ , the function  $E(t)$  reduces to the series  $F(0, \dots, 0; t)$  of (1.4). In order to state the result of this section, we introduce the following assumption:

(A4') There exists a point  $\tilde{t}_0 \in \mathbb{R}^d$  and a neighborhood  $V$  of  $\tilde{t}_0$  such that the Laplace transform  $L$  of  $\mu$  is finite in  $V$  and  $\tilde{t}_0$  is a minimum point of  $L$  restricted to  $V$ .

Since  $L$  is a convex function, the point  $\tilde{t}_0$  above is necessarily a global minimum. If  $\mu$  is truly  $d$ -dimensional (as assumed in (A2)), the function  $L$  is strictly convex and a necessary and sufficient condition for the existence of a global minimum is that the support of  $\mu$  is not included in any closed half-space.

**Theorem 5.1.** *For any distribution satisfying to (A1)–(A3), (A4') and such that the random walk takes its values on a lattice, the generating function  $E(t)$  in (5.1) is not a rational function.*

Contrary to our elementary and self-contained proof of Theorem 1.1, we don't have any elementary argument to prove Theorem 5.1. Instead, we may give a one-line proof based on earlier literature. Indeed, Denisov and Wachtel provide the following estimate in [5, Eq. (10)] (we use the generalization to convex cones as in [6, Cor. 1.3]):  $\mathbb{P}^x(\tau > n, S_n = y)$  is either 0 (for periodicity reasons) or asymptotic to

$$C(x, y) \tilde{\rho}^n n^{-p-d/2},$$

where  $\tilde{\rho} = L(\tilde{t}_0)$  with  $\tilde{t}_0$  as in (A4'),  $d$  is the dimension and  $p > 0$  is a geometric quantity related to the cone. One immediately concludes because the exponent of  $n$  is negative.

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