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DIFFERENTIAL ALGEBRAIC GENERATING FUNCTIONS OF WEIGHTED WALKS IN THE QUARTER PLANE

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Abstract. In the present paper we study the nature of the trivariate generating functions of weighted walks in the quarter plane. Combining the results of this paper to previous ones, we complete the proof of the following theorem. The series satisfies a nontrivial algebraic differential equation in one of its variables, if and only if it satisfies a nontrivial algebraic differential equation in each of its variables.

Keywords: Random walks in quarter plane, elliptic functions, transcendence.

1. Introduction

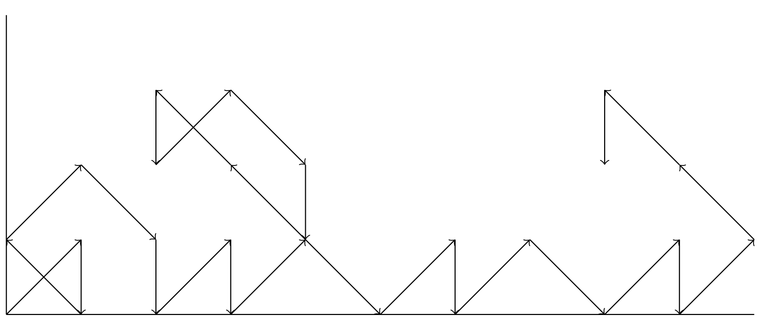
Framework. Consider a walk with small steps in the nonnegative quadrant $\mathbb{Z}^2_{\geq 0} = \{0, 1, 2, \ldots\}^2$ starting from $P_0 := (0, 0)$, that is a succession of points

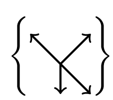
$$P_0, P_1, \ldots, P_k,$$

where each P_n lies in the quarter plane, where the moves (or steps) $P_{n+1} - P_n$ belong to $\{0, \pm 1\}^2$, and the probability to move in the direction $P_{n+1} - P_n = (i, j)$ may be interpreted as some weight parameter $d_{i,j} \in [0, 1]$, with $\sum_{(i,j)\in\{0,\pm1\}^2} d_{i,j} = 1$. The model of the walk (or model for short) is the data of the $d_{i,j}$ and the step set of the walk is the set of directions with nonzero weights, that is

$$\mathcal{S} = \{(i, j) \in \{0, \pm 1\}^2 \mid d_{i,j} \neq 0\}.$$

If $d_{0,0} = 0$ and if the nonzero $d_{i,j}$ all have the same value, we say that the model is unweighted. The following picture provides an example of a walk in the nonnegative quadrant:



$\mathcal{S} =$ 

Such objects are very natural both in combinatorics and probability theory: they are interesting for themselves and also because they are strongly related to other discrete structures; see [4, 6] and references therein.

The weight of the walk is defined to be the product of the weights of its component steps. For any $(i,j) \in \mathbb{Z}_{\geq 0}^2$ and any $k \in \mathbb{Z}_{\geq 0}$, we let $q_{i,j,k}$ be the sum of the weights of all walks reaching the position (i,j) from the initial position $(0,0)$ after k steps. We introduce the corresponding trivariate generating function

$$Q(x,y;t) := \sum_{i,j,k\geq 0} q_{i,j,k} x^i y^j t^k.$$

Being the generating function of probabilities, $Q(x,y;t)$ converges for all $(x,y,t) \in \mathbb{C}^3$ such that $|x|,|y| \leq 1$ and $|t| < 1$. Note that in several papers, as in [4], it is not assumed that $\sum d_{i,j} = 1$. However, after a rescaling of the t variable, we may always reduce to this case.

Statement of the main result. As we will see in the sequel, this paper takes part in a long history of articles that study the algebraic and differential relations satisfied by $Q(x,y;t)$. For any choice of a variable $\star$ among x,y,t, we say that $Q(x,y;t)$ is $\partial_\star$-algebraic if there exists $n \in \mathbb{Z}_{\geq 0}$, such that there exists a nonzero multivariate polynomial $P_\star \in \mathbb{C}(x,y,t)[X_0,\dots,X_n]$, such that

$$0 = P_\star(Q(x,y;t),\dots,\partial_\star^n Q(x,y;t)).$$

We stress that in the above definition, it is equivalent to require $0 \neq P_\star \in \mathbb{Q}[X_0,\dots,X_n]$; see Remark 3.1. Otherwise, we say that the series $Q(x,y;t)$ is $\partial_\star$-differentially transcendental.

Since the three variables x,y and t play a different role, we might expect the series to be of different nature with respect to the three derivatives. The main result of this paper, quite unexpected at first sight, shows that it is not the case. More precisely, using results of this paper and combining them to partial cases already known (see the discussion in the sequel), we complete the proof of the following main theorem.

Theorem 1.1. *The following facts are equivalent:*

- *The series $Q(x,y;t)$ is ∂_x-algebraic;*
- *The series $Q(x,y;t)$ is ∂_y-algebraic;*
- *The series $Q(x,y;t)$ is ∂_t-algebraic.*

Note that an algorithm is given in [16, Section 5] to decide whether the generating function is differentially algebraic in the x variable or not, but this does not provide the differential equation when it exists.

State of the art. More generally, the question of studying whether $Q(x,y;t)$ satisfies algebraic (resp. linear differential, resp. algebraic differential) equations attracted the attention of many authors in the last decade. In the unweighted case, the problem was first addressed in the seminal paper [4] and solved using several methods, such as combinatorics, computer algebra, complex analysis, and more recently, difference Galois theory; see [2, 3, 7–9, 19–21]. We refer to the introduction of [12] for a history of the cited results, from which it follows that Theorem 1.1 is valid for the unweighted models.

The main difficulty in generalizing those results to weighted models is that, contrary to the unweighted framework, there are infinitely many weighted models. However, certain unweighted results are still valid in the weighted cases, while some others are proved by a case-by-case argument, and therefore cannot be generalized straightforwardly. So beyond the generalization, we believe that replacing case-by-case proofs by systematic arguments has its own interest since it shows that the unweighted version of Theorem 1.1 has not appeared by accident in a finite number of cases, and illustrates a general phenomenon.

In many situations, the equivalence between the ∂_x-algebraicity and the ∂_y-algebraicity can be straightforwardly deduced in this weighted context from the proof of [8, Proposition 3.10]. In [7, Theorem 2] it was proved that the ∂_t-algebraicity implies the ∂_x-algebraicity. So it remains to show the converse. In [2], the authors show that all ∂_x-differentially algebraic unweighted models have a decoupling function. They use this property to prove the ∂_t-algebraicity in that case. In [8], using difference Galois theory, the authors show that such unweighted models admit a telescoping relation. We refer to [16] for precise definitions of the two latter notions. In [11], it is proved that the ∂_x-algebraic weighted models also have a telescoping relation. Finally in [16] the equivalence between the existence of a telescoping relation and the existence of decoupling functions is shown. This implies that a ∂_x-algebraic series admits a certain decomposition into elliptic functions.

The main difficulty is that the existence of such decompositions is proved for fixed values of t, so nothing is known about the dependence in t of the coefficients. For instance, the function $x\Gamma(t)$, seen as a function of x is simple for all fixed value of t (it is rational!) but it is differentially transcendental with respect to t, due to Hölder's result. We then have to make a careful study of the t-dependence of such elliptic relations, and use some results of ∂_t-algebraicity of the Weierstrass function in [2]. Finally, we are able to show that the ∂_x-algebraicity implies the ∂_t-algebraicity. The following diagram summarizes the various contributions toward the proof of Theorem 1.1.

$$\begin{array}{ccccc}
\partial_y\text{-algebraicity} & \underset{[8]}{\Longleftrightarrow} & \partial_x\text{-algebraicity} & \underset{[7]}{\Longleftarrow} & \partial_t\text{-algebraicity} \\
 & & \Big\Downarrow [11] & & \Big\Uparrow \text{This paper} + [2] \\
 & & \text{Telescoping relation} & \overset{[16]}{\Longleftrightarrow} & \text{Decoupling function}
\end{array}$$

Structure of the paper. The paper is organized as follows. In Section 2 we provide some reminders of objects appearing in the study of models of walks in the quarter plane. More precisely, we will study well-known properties of the kernel curve and explain how the generating function may be continued. We will also explain why Theorem 1.1 is correct in some degenerate cases that we may withdraw. In Section 3 we prove technical results on differential algebraicity. Some intermediate results stay valid in the framework of algebraic functions and/or solution of linear differential equations, but to simplify the exposition, we chose to present this section in a unified framework, making some intermediate results suboptimal. Finally Section 4 is devoted to the proof of Theorem 1.1. We split our study in two cases depending on whether the so-called group of the walk is finite or not.

2. Kernel of the walk

2.1. Functional equation. The kernel of the walk is the polynomial defined by

$$K(x,y;t) := xy(1 - tS(x,y)),$$

where $S(x,y)$ denotes the jump polynomial

$$\begin{aligned} S(x,y) &= \sum_{(i,j)\in\{0,\pm1\}^2} d_{i,j} x^i y^j \\ &= A_{-1}(x)\frac{1}{y} + A_0(x) + A_1(x)y \\ &= B_{-1}(y)\frac{1}{x} + B_0(y) + B_1(y)x, \end{aligned}$$

with $A_i(x) \in x^{-1}\mathbb{R}[x]$, $B_i(y) \in y^{-1}\mathbb{R}[y]$ (we recall that we consider weights $d_{i,j} \in [0,1]$). The kernel plays an important role in the so-called kernel method and the techniques we are going to apply will vary depending on its algebraic properties, that have been studied in [14] (when $t = 1$), and in [10,11] (when $t \in (0,1)$). The starting point is the following fundamental functional equation.

Lemma 2.1. *The generating function $Q(x,y;t)$ satisfies the functional equation*

$$K(x,y;t)Q(x,y;t) = xy + K(x,0;t)Q(x,0;t) + K(0,y;t)Q(0,y;t) - K(0,0;t)Q(0,0;t).$$

Proof. As a walk is either empty, or a smaller walk to which one added a step (removing the cases leaving the quarter-plane), one has the following combinatorial functional equation

$$Q(x,y;t) = 1 + tS(x,y)Q(x,y;t) - t\tfrac{B_{-1}(y)}{x}Q(0,y;t) - t\tfrac{A_{-1}(x)}{y}Q(x,0;t) + t\tfrac{d_{-1,-1}}{xy}Q(0,0;t),$$

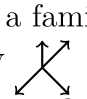
where the last summand removes the corresponding double counting. Multiplying by xy, we get Lemma 2.1. □

2.2. Degenerate cases. Like in [10], we will discard the following degenerate cases.

Definition 2.2 (Degenerate model)**.** Let us fix $t \in (0,1)$. A model of walk is called degenerate if one of the following holds:

- $K(x,y;t)$ factors in non-constant polynomials in $\mathbb{C}[x,y]$,
- $K(x,y;t)$ has x-degree (or y-degree) less than or equal to 1.

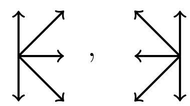
The notion of degeneracy thus seems to depend upon the parameter t. However, we will see in Proposition 2.3 below that the model is degenerate for a value of $t \in (0,1)$ if and only if it is degenerate for all values of $t \in (0,1)$. So, to lighten the terminology, we prefer not to stress this t-dependence and we say "degenerate" rather than "t-degenerate".

In what follows we will sometimes represent a family of models of walks with arrows. For instance, the family of models represented by  or, equivalently, $\{\nearrow, \searrow, \swarrow, \uparrow\}$ corresponds to models with $d_{1,0} = d_{0,-1} = d_{-1,1} = d_{-1,0} = 0$ and nothing is assumed on the other $d_{i,j}$. We stress the fact that the other $d_{i,j}$ (the weight of the arrows above) are allowed to be 0. In the following results, the behavior of the kernel curve *never* depends on $d_{0,0}$. This is the reason why, to reduce the amount of notations, we have decided not to associate an arrow to $d_{0,0}$.

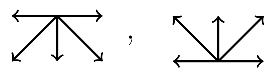
The following proposition has been proved in [14, Lemma 2.3.2] for $t = 1$, in [10, Proposition 1.2] for t is transcendental over $\mathbb{Q}(d_{i,j})$, and in [11, Proposition 1.3] for the other values of t in $(0, 1)$.

Proposition 2.3. *Let us fix $t \in (0, 1)$. A model of walk is degenerate if and only if at least one of the following holds:*

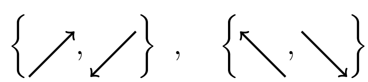
(a) *There exists $i \in \{-1, 1\}$ such that $d_{i,-1} = d_{i,0} = d_{i,1} = 0$. This corresponds to the following families of models*



(b) *There exists $j \in \{-1, 1\}$ such that $d_{-1,j} = d_{0,j} = d_{1,j} = 0$. This corresponds to the following families of models*



(c) *All the weights are 0 except maybe $\{d_{-1,-1}, d_{0,0}, d_{1,1}\}$ or $\{d_{-1,1}, d_{0,0}, d_{1,-1}\}$. This corresponds to the following families of models*



In virtue of the following lemma, Theorem 1.1 is valid for the degenerate models of walks. Therefore we will focus on models that are not degenerate.

Lemma 2.4. *Assume that the model of walk is degenerate. Then $Q(x, y; t)$ is algebraic over $\mathbb{C}(x, y, t)$ (and thus is differentially algebraic in its three variables).*

Proof. We use Proposition 2.3. Consider the cases (a), (b), and first configuration of the case (c). In the unweighted case it is proved in [4, Section 1.2] that $Q(x, y; t)$ is algebraic over $\mathbb{C}(x, y, t)$. The proof is the same in the weighted context but, to be self-contained, let us sketch the proof here. In the first configuration of case (a) the generating function is the same as the corresponding generating function of a model in the upper half-plane $\mathbb{Z}\times\mathbb{N}$. The latter is classically known to be algebraic over $\mathbb{C}(x, y, t)$, see for instance [5, Proposition 2]. In the second configuration of case (a), we have a unidimensional walk on the y-axis and such series is known to be rational, and therefore algebraic over $\mathbb{C}(x, y, t)$. The case (b) is similar. In the first configuration of case (c), we are considering a unidimensional walk on the half-line $\{(x, x), x \in \mathbb{N}\}$, and the generating function is algebraic. Since in all these cases, $Q(x, y; t)$ is algebraic over $\mathbb{C}(x, y, t)$, it is differentially algebraic in its three variables. In the last configuration of case (c), all the weights are 0 except maybe $\{d_{-1,1}, d_{0,0}, d_{1,-1}\}$, so the walk cannot leave $(0, 0)$ and we have

$$Q(x, y; t) = \sum_{k=0}^{\infty} d_{0,0}^k t^k = \frac{1}{1 - d_{0,0}t}.$$

Therefore the result holds in that case too. □

2.3. **Genus of the walk.** The kernel curve E_t is the complex affine algebraic curve defined by

$$E_t = \{(x, y) \in \mathbb{C} \times \mathbb{C} \mid K(x, y; t) = 0\}.$$

We shall now consider a compactification of this curve. We let $\mathbb{P}^1(\mathbb{C})$ be the complex projective line, that is the quotient of $(\mathbb{C} \times \mathbb{C}) \setminus \{(0, 0)\}$ by the equivalence relation $\sim$ defined by

$$(x_0, x_1) \sim (x_0', x_1') \Leftrightarrow \exists \lambda \in \mathbb{C}^*, (x_0', x_1') = \lambda(x_0, x_1).$$

The equivalence class of $(x_0, x_1) \in (\mathbb{C} \times \mathbb{C}) \setminus \{(0, 0)\}$ is denoted by $[x_0 : x_1] \in \mathbb{P}^1(\mathbb{C})$. The map $x \mapsto [x : 1]$ embeds $\mathbb{C}$ inside $\mathbb{P}^1(\mathbb{C})$. The latter map is not surjective: its image is $\mathbb{P}^1(\mathbb{C}) \setminus \{[1 : 0]\}$; the missing point $[1 : 0]$ is usually denoted by ∞. Now, we embed E_t inside $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ via $(x, y) \mapsto ([x : 1], [y : 1])$. The kernel curve $\overline{E}_t$ is the closure of this embedding of E_t. In other words, the kernel curve $\overline{E}_t$ is the algebraic curve defined by

$$\overline{E}_t = \{([x_0 : x_1], [y_0 : y_1]) \in \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \mid \overline{K}(x_0, x_1, y_0, y_1; t) = 0\}$$

where $\overline{K}(x_0, x_1, y_0, y_1; t)$ is the following degree two homogeneous polynomial in the four variables x_0, x_1, y_0, y_1

$$\overline{K}(x_0, x_1, y_0, y_1; t) = x_1^2 y_1^2 K\left(\frac{x_0}{x_1}, \frac{y_0}{y_1}; t\right) = x_0 x_1 y_0 y_1 - t \sum_{i,j=0}^{2} d_{i-1,j-1} x_0^i x_1^{2-i} y_0^j y_1^{2-j}.$$

Although it may seem more natural to take the closure of $\overline{E}_t$ in $\mathbb{P}^2(\mathbb{C})$, the above definition allows us to avoid unnecessary singularities.

The following proposition has been proved in [10, Proposition 2.1 and Corollary 2.6], when t is transcendental over $\mathbb{Q}(d_{i,j})$ and has been extended for a general $0 < t < 1$ in [11, Proposition 1.9].

Proposition 2.5. *Let us fix $t \in (0, 1)$ and assume that the model of the walk is not degenerate. The following facts are equivalent:*

(1) $\overline{E}_t$ is an elliptic curve;
(2) The set of authorized directions $\mathcal{S}$ is not included in any half-space with boundary passing through the origin.

Let us now discuss the case where for $t \in (0, 1)$ fixed, the model is not degenerate and $\overline{E}_t$ is not an elliptic curve. By Proposition 2.3 and Proposition 2.5, this corresponds to nondegenerate models that belong to one of the four families in Figure 1.

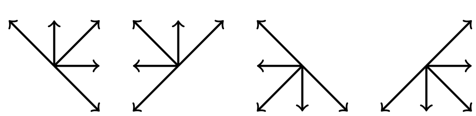


FIGURE 1. Our four nondegenerate models

Note that although the third configuration in Figure 1 is called nondegenerate, it leads to walks that never escape from $(0, 0)$ and thus their generating function is trivial.

The following lemma yields that Theorem 1.1 is valid for the families of models in Figure 1.

Lemma 2.6. *The following holds:*

(a) *Assume that the model of the walk is not degenerate and belongs to the first family in Figure 1. Then $Q(x,y;t)$ is differentially transcendental in its three variables.*
(b) *Assume that the model of the walk belongs to the second, third or the fourth family in Figure 1. Then $Q(x,y;t)$ is algebraic over $\mathbb{C}(x,y,t)$, and thus is differentially algebraic in its three variables.*

Proof. (a) This is [7, Corollary 2.2]; see also [9, Theorem 3.1].
(b) Consider the second family. We have $Q(x,0;t) = Q(0,0;t)$ and $K(x,0;t) = K(0,0;t)$. Then by Lemma 2.1,

$$K(x,y;t)Q(x,y;t) = K(0,y;t)Q(0,y;t) + xy. \tag{2.1}$$

Let us see that with the same arguments as for the walks in the half-plane, we deduce that $Q(x,y;t)$ is algebraic over $\mathbb{C}(x,y,t)$. The idea is to locally write $K(\phi(y;t),y;t) = 0$. Evaluating at $(\phi(y;t),y;t)$ we then have for convenient y and t, $0 = K(0,y;t)Q(0,y;t) + \phi(y;t)y$, proving that $Q(0,y;t)$ is algebraic over $\mathbb{C}(x,y,t)$. The functional equation (2.1) allows then to conclude that $Q(x,y;t)$ is algebraic over $\mathbb{C}(x,y,t)$. As in the proof of Lemma 2.4, we may deduce that $Q(x,y;t)$ is differentially algebraic in its three variables. The reasoning for the fourth family is similar. For the third family, the walk has to stay at $(0,0)$ and we have

$$Q(x,y;t) = \sum_{k=0}^{\infty} d_{0,0}^k t^k = \frac{1}{1-d_{0,0}t}.$$

Therefore the result holds in that case too. □

2.4. **Group of the walk.** From now on, we may focus on the case where $\overline{E}_t$ is an elliptic curve. Recall that we have seen in Proposition 2.3, that $K(x,y;t)$ has degree two in x and y, and nonzero coefficient of degree 0 in x and y. Hence, $A_1(x), A_{-1}(x), B_1(y), B_{-1}(y)$ are not identically zero.

Following [4, Section 3], [17, Section 3] or [14], and using the notations introduced in Section 2.3, we consider the rational involutions given by

$$i_1([x_0:x_1],[y_0:y_1]) = \left(\frac{x_0}{x_1}, \frac{A_1(\frac{x_0}{x_1})}{A_{-1}(\frac{x_0}{x_1})\frac{y_0}{y_1}}\right) \quad \text{and} \quad i_2([x_0:x_1],[y_0:y_1]) = \left(\frac{B_{-1}(\frac{y_0}{y_1})}{B_1(\frac{y_0}{y_1})\frac{x_0}{x_1}}, \frac{y_0}{y_1}\right).$$

Note that we have $i_1([x_0/x_1:1],[y_0/y_1:1]) = i_1([x_0:x_1],[y_0:y_1])$ and the same holds for i_2. Note also that i_1, i_2 are a priori not defined when the denominators vanish but we will see in the sequel that we may overcome this problem when we will restrict to $\overline{E}_t$.

For a fixed value of x, there are at most two possible values of y such that $(x,y) \in \overline{E}_t$. The involution i_1 corresponds to interchanging these values. A similar interpretation can be given for i_2. Therefore the kernel curve $\overline{E}_t$ is left invariant by the natural action of i_1, i_2.

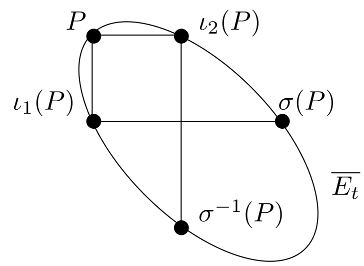


FIGURE 2. The maps i_1 and i_2 restricted to the kernel curve $\overline{E}_t$ are denoted by ι_1 and ι_2, respectively.

We denote by ι_1, ι_2 the restriction of i_1, i_2 to $\overline{E}_t$; see Figure 2. Again, these functions are a priori not defined where the denominators vanish. However, by [10, Proposition 4.1], this is only an "apparent problem". To be precise, the authors proved this for t transcendental over $\mathbb{Q}(d_{i,j})$ but the proof is still valid when $\overline{E}_t$ is an elliptic curve. We then obtain that ι_1 and ι_2 can be extended to morphisms of $\overline{E}_t$. We recall that a rational map f from $\overline{E}_t$ to $\overline{E}_t$ is a morphism if it is regular at any $P \in \overline{E}_t$, i.e. if f can be represented in suitable affine charts containing P and $f(P)$ by a rational function with nonvanishing denominator at P.

Let us finally define $\sigma = \iota_2 \circ \iota_1$. Note that such a map is called a QRT-map and has been widely studied; see [13].

Definition 2.7 (Group of the walk)**.** We call G the group generated by ι_1 and ι_2 and we call G_t the specialization of this group for a fixed value of $0 < t < 1$.

In the unweighted case, the algebraic nature of the generating series depends on whether σ has finite or infinite order. More precisely, G is finite if and only if the generating function is holonomic, i.e. satisfies a nontrivial linear differential equation with coefficients in $\mathbb{C}(x, y, t)$ in each of its three variables. On the other hand, when G is infinite, G_t can be either finite or infinite; see [15] for concrete examples. However, in that situation, the set of values of t such that G_t is finite is countable, see [8, Proposition 2.6].

2.5. **Uniformization of the curve.** In this section, we consider the uniformization problem in the genus one context, that has been solved in [14] for the case $t = 1$, and [11] for the case $0 < t < 1$. Let us consider a nondegenerate model of walk and assume that for all $t \in (0, 1)$, $\overline{E}_t$ is an elliptic curve. By Proposition 2.5, this corresponds to the situation where the step set is not included in any half-plane whose boundary passes through $(0, 0)$. By [11, Proposition 2.1], the elliptic curve $\overline{E}_t$ is biholomorphic to $\mathbb{C}/(\omega_1(t)\mathbb{Z} + \omega_2(t)\mathbb{Z})$ for some lattice $\omega_1(t)\mathbb{Z} + \omega_2(t)\mathbb{Z}$ of $\mathbb{C}$ via some $(\omega_1(t)\mathbb{Z} + \omega_2(t)\mathbb{Z})$-periodic holomorphic map Λ

$$\begin{array}{rccc} \Lambda : & \mathbb{C} & \to & \overline{E}_t \\ & \Lambda(\omega) & := & (x(\omega; t), y(\omega; t)), \end{array}$$

where x, y are rational functions of $\wp$ and its derivative $\partial_\omega \wp$, and $\wp$ is the Weierstrass function associated with the lattice $\omega_1(t)\mathbb{Z} + \omega_2(t)\mathbb{Z}$:

$$\wp(\omega; t) = \frac{1}{\omega^2} + \sum_{(\ell_1, \ell_2) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \left(\frac{1}{(\omega + \ell_1\omega_1(t) + \ell_2\omega_2(t))^2} - \frac{1}{(\ell_1\omega_1(t) + \ell_2\omega_2(t))^2} \right).$$

Then, the field of meromorphic functions on $\overline{E}_t$ is isomorphic to the field of meromorphic functions on $\mathbb{C}/(\omega_1(t)\mathbb{Z}+\omega_2(t)\mathbb{Z})$, that is itself isomorphic to the field of meromorphic functions on $\mathbb{C}$ that are $(\omega_1(t),\omega_2(t))$-periodic. For $t\in(0,1)$ fixed, the latter field is equal to $\mathbb{C}(\wp,\partial_\omega\wp)$; see [23, Chapter 9, Theorem 1.8].

The maps ι_1, ι_2, and σ may be analytically lifted to the ω-plane $\mathbb{C}$ via the map Λ^{-1}. We denote these lifts by $\widetilde{\iota}_1$, $\widetilde{\iota}_2$, and $\widetilde{\sigma}$ respectively. So we have the commutative diagrams

$$\begin{array}{ccc} \overline{E}_t & \xrightarrow{\iota_k} & \overline{E}_t \\ \Lambda\uparrow & & \uparrow\Lambda \\ \mathbb{C} & \xrightarrow[\widetilde{\iota}_k]{} & \mathbb{C} \end{array} \qquad\qquad \begin{array}{ccc} \overline{E}_t & \xrightarrow{\sigma} & \overline{E}_t \\ \Lambda\uparrow & & \uparrow\Lambda \\ \mathbb{C} & \xrightarrow[\widetilde{\sigma}]{} & \mathbb{C} \end{array}$$

For any $[x_0:x_1]$ in $\mathbb{P}^1(\mathbb{C})$, we denote by $\Delta_1([x_0:x_1];t)$ the discriminant of the degree two homogeneous polynomial given by $y\mapsto \overline{K}(x_0,x_1,y;t)$. Let us write

$$\Delta_1([x_0:x_1];t)=\sum_{i=0}^{4}\alpha_i(t)x_0^i x_1^{4-i}.$$

By [11, Theorem 1.11], the discriminant $\Delta_1([x_0:x_1];t)$ admits four distinct continuous real roots $a_1(t),\dots,a_4(t)$. They are numbered such that the cycle of $\mathbb{P}^1(\mathbb{R})$ starting from -1 to ∞ and from $-\infty$ to -1 crosses the a_i in the order $a_1(t),\dots,a_4(t)$. Furthermore, $[1:0]$ is a root if and only if $\alpha_4(t)=0$. In [11, Section 1.4], we see that $\alpha_4(t)=t^2(d_{1,0}^2-4d_{1,-1}d_{1,1})$. It follows that $[1:0]$ is a root of $\Delta_1([x_0:x_1];t)$ for one value of $t\in(0,1)$, if and only if $[1:0]$ is a root of $\Delta_1([x_0:x_1];t)$ for all $t\in(0,1)$.

Similarly, we denote by $b_1(t),\dots,b_4(t)$ the continuous real roots of the discriminant $x\mapsto \overline{K}(x,y_0,y_1;t)$, numbered in the same way, and we write $\Delta_2([y_0:y_1];t)=\sum_{i=0}^{4}\beta_i(t)y_0^i y_1^{4-i}$.

The following formulas have been proved

- in [14, Section 3.3] when $t=1$,
- in [22] in the unweighted case,
- in [11, Proposition 2.1 and (2.16)], in the weighted case.

Proposition 2.8 ([11], Proposition 2.1, Lemma 2.6, and (2.16))**.** *For $i=1,2$, let us set $D_i(\star;t):=\Delta_i([\star:1];t)$. An explicit expression of the periods is given by the elliptic integrals*

$$\omega_1(t)=\mathbf{i}\int_{a_3(t)}^{a_4(t)}\frac{dx}{\sqrt{|D_1(x;t)|}}\in\mathbf{i}\mathbb{R}_{>0}\quad \textit{and}\quad \omega_2(t)=\int_{a_4(t)}^{a_1(t)}\frac{dx}{\sqrt{D_1(x;t)}}\in\mathbb{R}_{>0}.$$

An explicit expression of the holomorphic map $\Lambda(\omega;t)=(x(\omega;t),y(\omega;t))$ is given by

- *If $a_4(t)\neq[1:0]$, then* $x(\omega;t)=\left[a_4(t)+\frac{D_1'(a_4(t);t)}{\wp(\omega;t)-\frac{1}{6}D_1''(a_4(t);t)}:1\right]$;
- *If $a_4(t)=[1:0]$, then* $x(\omega;t)=[\wp(\omega;t)-\alpha_2(t)/3:\alpha_3(t)]$;
- *If $b_4(t)\neq[1:0]$, then* $y(\omega;t)=\left[b_4(t)+\frac{D_2'(b_4(t);t)}{\wp(\omega-\omega_3(t)/2;t)-\frac{1}{6}D_2''(b_4(t);t)}:1\right]$;
- *If $b_4(t)=[1:0]$, then* $y(\omega;t)=[\wp(\omega-\omega_3(t)/2;t)-\beta_2(t)/3:\beta_3(t)]$.

An explicit expression of the involutions is given by

$$\widetilde{\iota}_1(\omega) = -\omega, \quad \widetilde{\iota}_2(\omega) = -\omega + \omega_3 \quad \textit{and} \quad \widetilde{\sigma}(\omega) = \omega + \omega_3,$$

where

$$\omega_3(t) = \int_{a_4(t)}^{X_\pm(b_4(t);t)} \frac{dx}{\sqrt{D_1(x;t)}} \in (0, \omega_2(t)), \tag{2.2}$$

and $X_\pm(y;t)$ *are the two roots of* $\overline{K}(X_\pm(y;t), y; t) = 0$.

2.6. **Meromorphic continuation of the generating function.** Let us summarize here the results of [11, Section 2.3]. Let us fix $t \in (0,1)$. The generating function $Q(x,y;t)$ converges for $|x|, |y| < 1$. The projection of this set inside $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ has a nonempty intersection with the kernel curve $\overline{E}_t$. In virtue of Lemma 2.1, we then find for $|x|, |y| < 1$ and $(x,y) \in \overline{E}_t$,

$$0 = K(x,0;t)Q(x,0;t) + K(0,y;t)Q(0,y;t) - K(0,0;t)Q(0,0;t) + xy.$$

To shorten several expressions hereafter, it is convenient to rewrite this equation introducing new auxiliary series F_1 and F_2:

$$0 = F_1(x;t) + F_2(y;t) - K(0,0;t)Q(0,0;t) + xy. \tag{2.3}$$

Since the series $F_1(x;t)$ and $F_2(y;t)$ converge for $|x|$ and $|y| < 1$ respectively, we then continue $F_1(x;t)$ for $(x,y) \in \overline{E}_t$ and $|y| < 1$ with the formula:

$$F_1(x;t) = -F_2(y;t) + K(0,0;t)Q(0,0;t) - xy.$$

We continue $F_2(y;t)$ for $(x,y) \in \overline{E}_t$ and $|x| < 1$ similarly. There exists a connected set $\mathcal{O} \subset \mathbb{C}$ such that

- $\Lambda(\mathcal{O}) = \{(x,y) \in \overline{E}_t \text{ such that } |x| < 1 \text{ or } |y| < 1\}$;
- $\widetilde{\sigma}^{-1}(\mathcal{O}) \cap \mathcal{O} \neq \varnothing$;
- $\bigcup_{\ell \in \mathbb{Z}} \widetilde{\sigma}^{\ell}(\mathcal{O}) = \mathbb{C}$.

There also exist meromorphic functions on $\mathcal{O}$, $r_x(\omega;t)$ and $r_y(\omega;t)$, such that $r_x(\omega;t) = F_1(x(\omega;t);t)$ and $r_y(\omega;t) = F_2(y(\omega;t);t)$.

Lemma 2.9 (Inclusion of poles)**.** *The set of poles of* $r_x(\omega;t)$ *inside* $\mathcal{O}$ *are contained in the set of poles of* $x(\omega;t)$ *with* $|y(\omega;t)| < 1$. *The set of poles of* $r_y(\omega;t)$ *inside* $\mathcal{O}$ *are contained in the set of poles of* $y(\omega;t)$ *with* $|x(\omega;t)| < 1$.

Proof. Let us use (2.3). On $\mathcal{O}$, we have

$$0 = r_x(\omega;t) + r_x(\omega;t) - K(0,0;t)Q(0,0;t) + x(\omega;t)y(\omega;t).$$

Let us focus on $r_x(\omega;t)$, the proof for $r_y(\omega;t)$ is similar. Recall that $F_1(x;t)$ has no poles for $|x| < 1$. Since $r_x(\omega;t) = F_1(x(\omega;t);t)$, we find that $r_x(\omega;t)$ has no poles when $|x(\omega;t)| < 1$. With $\Lambda(\mathcal{O}) = \{(x,y) \in \overline{E}_t || x| < 1 \text{ or } |y| < 1\}$, we deduce that a pole of $r_x(\omega;t)$ inside $\mathcal{O}$ satisfies $|y(\omega;t)| < 1$. We use $r_y(\omega;t) = F_2(y(\omega;t);t)$, and the fact that $F_2(y;t)$ has no poles for $|y| < 1$ to deduce that $r_y(\omega;t)$ has no poles when $|y(\omega;t)| < 1$. Therefore, the poles of $r_x(\omega;t)$ inside $\mathcal{O}$ corresponds to the poles of $x(\omega;t)y(\omega;t)$ with $|y(\omega;t)| < 1$. The result follows. ∎

With $\bigcup_{\ell\in\mathbb{Z}} \widetilde{\sigma}^{\ell}(\mathcal{O}) = \mathbb{C}$ and $\widetilde{\sigma}^{-1}(\mathcal{O}) \cap \mathcal{O} \neq \varnothing$, we then extend $r_x(\omega;t)$ and $r_y(\omega;t)$ as meromorphic functions on $\mathbb{C}$ where they satisfy the functional equations

$$r_x(\omega + \omega_3(t);t) = r_x(\omega;t) + b_x(\omega;t), \tag{2.4}$$

$$r_x(\omega + \omega_1(t);t) = r_x(\omega;t), \tag{2.5}$$

$$\begin{aligned} r_y(\omega + \omega_3(t);t) &= r_y(\omega;t) + b_y(\omega;t), \\ r_y(\omega + \omega_1(t);t) &= r_y(\omega;t), \end{aligned} \tag{2.6}$$

where $b_x(\omega;t) = y(-\omega;t)(x(\omega;t) - x(\omega+\omega_3(t);t))$ and $b_y(\omega;t) = x(\omega;t)(y(\omega;t) - y(-\omega;t))$.

From the functional equations (2.5) and (2.6), the set of poles of $\omega \mapsto r_x(\omega;t)$ and $\omega \mapsto r_y(\omega;t)$ are $\omega_1(t)$ periodic. With the other functional equations and $\bigcup_{\ell\in\mathbb{Z}} \widetilde{\sigma}^{\ell}(\mathcal{O}) = \mathbb{C}$, we may deduce the expression of a discrete set containing the poles of r_x and r_y.

Lemma 2.10. *Let $\mathcal{P}_x$ be the poles of r_x in $\mathcal{O}$ and $\mathcal{P}_{b,x}$ be the poles of b_x in $\mathbb{C}$. The set of poles of $\omega \mapsto r_x(\omega;t)$ is included in $(\mathcal{P}_x + \omega_3(t)\mathbb{Z}) \bigcup (\mathcal{P}_{b,x} + \omega_3(t)\mathbb{Z})$. A similar statement holds for $r_y(\omega;t)$.*

3. Preliminary results on differential algebraicity

In this section, we prove some results on differential algebraicity, and more specifically on ∂_t-algebraicity of the functions that appear in Section 2.

Let us begin by definitions. Let $f(x_1,\dots,x_n)$ be a multivalued Puiseux series. For $i = 1,\dots,n$, we say that f is ∂_{x_i}-algebraic if and only if it satisfies a nontrivial algebraic differential equation in the variable x_i, with coefficients in $\mathbb{Q}$. We say that f is differentially algebraic in all its variables (or differentially algebraic for short) if and only if for all $1 \leq i \leq n$, f is ∂_{x_i}-algebraic.

The following remark, proved e.g. in [18, Proposition 8, page 101], will be used several times in the sequel.

Remark 3.1. Let $f_1,\dots,f_n$ be differentially algebraic functions meromorphic on a common domain. A function satisfies a nontrivial algebraic differential equation with coefficients in $\mathbb{C}(f_1,\dots,f_n)$ if and only if it satisfies a nontrivial algebraic differential equation with coefficients in $\mathbb{Q}$.

The following lemma shows that the set of differentially algebraic functions is stable under many operations.

Lemma 3.2 (Closure properties)**.** *The set of differentially algebraic functions meromorphic on a domain is a field stable under derivations. If f and g are differentially algebraic and $f \circ g$ is well-defined then $f \circ g$ is differentially algebraic as well. If f is differentially algebraic and admits an inverse f^{-1}, then f^{-1} is also differentially algebraic.*

Proof. See [8, Lemma 6.4] for the inverse property in the univariate case. The proof extends straightforwardly to the multivariate case. The rest of the statements follows from [2, Corollary 6.4 and Proposition 6.5]. □

In what follows, we might also consider functions of t defined only on some intervals of $(0,1)$. Let $\mathfrak{D}$ be the field of multivalued functions that admit an expansion as convergent Puiseux series for all $t \in (0,1)$, and that are differentially algebraic. In the sequel, when we will say that a function of t defined (a priori) only of some intervals of $(0,1)$ is differentially algebraic, it will be implicit that it may be continued as an element of $\mathfrak{D}$.

The goal of the following results is to prove that various functions that appear in the uniformization of the elliptic curve are ∂_t-algebraic.

Lemma 3.3 ([2, Lemma 6.10])**.** *The functions $\omega_1(t), \omega_2(t), \omega_3(t)$ belong to $\mathfrak{D}$.*[1] *Moreover, they are analytic on a complex neighborhood of $(0,1)$.*

Proposition 3.4. *Functions of $\mathfrak{D}(\wp(\omega;t), \partial_\omega \wp(\omega;t))$ are differentially algebraic in t and ω.*

Proof. Since the differentially algebraic functions form a field stable under the derivations (see Lemma 3.2), it suffices to show that $\wp(\omega;t)$ is differentially algebraic. It is well known that for $t \in (0,1)$ fixed, $\wp(\omega;t)$ is ∂_ω-algebraic. More precisely, it satisfies an equation of the form $(\partial_\omega \wp)^2 = 4\wp^3 - g_2(t)\wp - g_3(t)$, where $g_2(t), g_3(t)$ are the invariants of the elliptic curve. Differentiating with respect of ω allows us to eliminate the invariants, and obtain $\partial_\omega^3 \wp(\omega;t) = 12\wp(\omega;t)\partial_\omega \wp(\omega;t)$; see [1, (18.6.5)]. Hence $\wp(\omega;t)$ is ∂_ω-algebraic.

Let us prove the ∂_t-algebraicity. In virtue of [2, Proposition 6.7], $\wp$ satisfies a nontrivial ∂_t-algebraic equation with coefficients in $\mathbb{C}(\omega_1(t), \omega_2(t))$. By Lemma 3.3, the periods $\omega_1(t)$ and $\omega_2(t)$ of $\wp$ are differentially algebraic, so in virtue of Remark 3.1, $\wp(\omega;t)$ is ∂_t-algebraic. □

Remark 3.5. The same result holds with $\wp$ replaced by the Weierstrass function associated to the lattice $\omega_1(t)\mathbb{Z} + k\omega_2\mathbb{Z}$, or the lattice $\omega_1(t)\mathbb{Z} + \omega_3(t)\mathbb{Z}$.

Definition 3.6 (Principal part)**.** Let $f(\omega;t)$ be a meromorphic function at $\omega = a(t) \in \mathfrak{D}$, with Laurent series $f(\omega;t) = \sum_{\ell=\nu}^{\infty} a_\ell(t)(\omega - a(t))^\ell$. The principal part of f at $\omega = a(t)$ is the sum $\sum_{\ell=\nu}^{-1} a_\ell(t)(\omega - a(t))^\ell$ with the convention that it is 0 when $\nu \geq 0$. The coefficients of this principal part are $a_\nu(t), \dots, a_{-1}(t)$.

The following lemma will be used several times in the sequel.

Lemma 3.7. *The following statements hold:*

- *Let $d(t) \in \mathfrak{D}$ be an arbitrary function. We have $\wp(\omega;t) \in \mathfrak{D}((\omega + d(t)))$;*
- *$\wp(\omega;t) \in \omega^{-2}\mathfrak{D}[[\omega]]$;*
- *The coefficients of the principal parts of $\omega \mapsto \wp(\omega;t)$ belong to $\mathfrak{D}$.*

Proof. The last two assertions are straightforward consequences of the first one. Let us prove the first point. The function $d(t)$ and the poles of $\omega \mapsto \wp(\omega;t)$ are analytic on a convenient domain. So either $-d(t)$ is a pole of $\omega \mapsto \wp(\omega;t)$ with constant order with respect to t, or the set of t such that $-d(t)$ is a pole of $\omega \mapsto \wp(\omega;t)$ is discrete. It follows that the order of the pole of $\omega \mapsto \wp(\omega;t)$ at $-d(t)$ is constant except on a discrete set. Since for t fixed, $\omega \mapsto \wp(\omega;t)$ has pole of order at most two, we may write $\wp(\omega;t) = \sum_{\ell=k}^{\infty} c_\ell(t)(\omega + d(t))^\ell$.

[1] They even are solutions of linear differential equations.

Note that the coefficients $c_\ell(t)$ may have a pole when the order of the pole of $\omega \mapsto \wp(\omega;t)$ at $d(t)$ increases. In virtue of the field property of Lemma 3.2, combined with Proposition 3.4, we find that $(\omega+d(t))^{-k}\wp(\omega;t)$ is differentially algebraic. Note that $c_k(t)$ is the value at $-d(t)$ of the ∂_t-algebraic function $(\omega+d(t))^{-k}\wp(\omega;t)$. By the field property of Lemma 3.2 and Proposition 3.4, $(\omega+d(t))^{-k}\wp(\omega;t)$ is differentially algebraic in its two variables. By the composition property of Lemma 3.2 it follows that $c_k(t)\in\mathfrak{D}$. Let us fix $k\leq n$ and assume that for $\ell=k,\dots,n$, $c_\ell(t)\in\mathfrak{D}$. Let us show that $c_{n+1}(t)\in\mathfrak{D}$. This will prove the result by induction. Let us define $h_n(\omega;t)=\wp(\omega,t)-\sum_{\ell=k}^{n} c_\ell(t)(\omega+d(t))^\ell$. By Proposition 3.4, the field property of Lemma 3.2, and the induction hypothesis, the function $t\mapsto h_n(\omega;t)$ is differentially algebraic in its two variables. Note that $c_{n+1}(t)$ is the value at $-d(t)$ of $(\omega+d(t))^{-(n+1)}h_n(\omega;t)$. By the composition property of Lemma 3.2 it follows that $c_k(t)\in\mathfrak{D}$. □

As a consequence of what precedes, we deduce:

Corollary 3.8. *The following holds:*

- *The functions $x(\omega;t)$ and $y(\omega;t)$ are differentially algebraic in their two variables;*
- *For $d(t)\in\mathfrak{D}$, we have $x(\omega;t),y(\omega;t)\in\mathfrak{D}((\omega+d(t)))$;*
- *The poles and the coefficients of the principal parts of $\omega\mapsto x(\omega;t)$ and $\omega\mapsto y(\omega;t)$ belong to $\mathfrak{D}$.*

Proof. We use the expressions of $x(\omega;t)$ and $y(\omega;t)$ given in Proposition 2.8. The elements involved in the expression are meromorphic on some complex neighborhood of $(0,1)$ in the t-plane and are differentially algebraic by Proposition 3.4. Since the differentially algebraic elements form a field, see Lemma 3.2, the first point follows. Using Lemma 3.7, we deduce that $x(\omega;t),y(\omega;t)\in\mathfrak{D}((\omega+d(t)))$ for all $d(t)\in\mathfrak{D}$. Then the coefficients of the principal parts of $\omega\mapsto x(\omega;t)$ and $\omega\mapsto y(\omega;t)$ belong to $\mathfrak{D}$.

It remains to prove the differential algebraicity of the poles. Let $a(t)$ be a pole of $\omega\mapsto x(\omega;t)$ or $\omega\mapsto y(\omega;t)$. Then $a(t)$ is a continuous function solution of $\wp(a(t);t)=b(t)$, where $b(t)$ is ∂_t-algebraic.

Assume first that $\partial_\omega\wp(a(t);t)$ is identically zero or $a(t)$ is a pole of $\wp(\omega;t)$. By [23, page 270], this corresponds to the case where $a(t)\in\omega_1(t)\frac{\mathbb{Z}}{2}+\omega_2(t)\frac{\mathbb{Z}}{2}$. By Lemma 3.3, $a(t)$ is meromorphic on a complex neighborhood of $(0,1)$ and is ∂_t-algebraic. Then, it belongs to $\mathfrak{D}$.

Assume now that $\partial_\omega\wp(a(t);t)$ is not identically zero and $\wp(a(t);t)=b(t)$. Then, by the implicit function theorem, $a(t)$ admits an expansion as a meromorphic function on a complex neighborhood of any $t\in(0,1)$ with $\partial_\omega\wp(a(t);t)\neq 0$. On that domain $\wp$ is locally invertible and its inverse is differentially algebraic in its two variables by Lemma 3.2. So we may write $\wp^{-1}(b(t);t)=a(t)$, where $\wp^{-1}$ is the local inverse of $\wp$. With the composition and inverse properties of Lemma 3.2, we deduce that $a(t)$ is ∂_t-algebraic. Furthermore, by the implicit function theorem, it admits an expansion as a convergent series on a complex neighborhood of any $t\in(0,1)$. The set of t such that $\partial_\omega\wp(a(t);t)\neq 0$ being dense, we find that the differential equation holds everywhere. This concludes the proof. □

Recall, see Section 2.6, that

$$b_x(\omega;t) = y(-\omega;t)(x(\omega;t) - x(\omega+\omega_3(t);t)) \text{ and } b_y(\omega;t) = x(\omega;t)(y(\omega;t) - y(-\omega;t)).$$

Corollary 3.9. *The following holds:*

- *The functions $b_x(\omega;t)$ and $b_y(\omega;t)$ are differentially algebraic in their two variables;*
- *For $d(t) \in \mathfrak{D}$, we have $b_x(\omega;t), b_y(\omega;t) \in \mathfrak{D}((\omega + d(t)))$;*
- *The poles and the coefficients of the principal parts of $\omega \mapsto b_x(\omega;t)$ and $\omega \mapsto b_y(\omega;t)$ belong to $\mathfrak{D}$.*

Proof. By Lemma 3.3, $\omega_3(t)$ belongs to $\mathfrak{D}$. This is now a straightforward application of Corollary 3.8, combined with the field property of Lemma 3.2. □

Toward the proof of Theorem 1.1, we are going to face to many situations where the series is known to be ∂_x-algebraic (or ∂_y-algebraic) for all fixed values t. More precisely the differential algebraicity of the series will be proved to be equivalent to the existence of functions that are for all t fixed, elliptic functions. Unfortunately, few things are known about the t-dependence of the coefficients. The following result will be the main ingredient in the proof of Theorem 1.1 since it gives a framework where we can state that the elliptic functions are differentially algebraic in all their variables.

Theorem 3.10. *Let $\omega \mapsto f(\omega;t)$ be a function such that:*

- *For all $t \in (0,1)$, $\omega \mapsto f(\omega;t) \in \mathbb{C}(\wp(\omega;t), \partial_\omega \wp(\omega;t))$.*
- *There are countably many elements of $\mathfrak{D}$, whose union forms the set of poles of $\omega \mapsto f(\omega;t)$.*
- *The coefficients of the principal parts of $\omega \mapsto f(\omega;t)$ are in $\mathfrak{D}$.*
- *There exists $a(t) \in \mathfrak{D}$ such that $f(a(t);t) \in \mathfrak{D}$.*

Then, $f(\omega;t)$ is differentially algebraic in its two variables.

Remark 3.11. At first sight, nothing is explicitly assumed on the t-dependence of $t \mapsto f(\omega;t)$. However, the assumptions on the poles, on the principal parts, and on the special value $f(a(t);t)$, will imply that $t \mapsto f(\omega;t)$ is analytic on a convenient domain.

Proof. If f is constant in the ω variable, then the result is clear. Assume that $f(\omega;t)$ is not constant. Let $a \in \mathbb{C}$. By the field property in Lemma 3.2, $f(\omega+a;t)$ satisfies the assumptions of Theorem 3.10. By the composition property, $f(\omega;t)$ is differentially algebraic in its two variables if and only if $f(\omega+a;t)$ is differentially algebraic in its two variables. Then, without loss of generality, we may reduce to the case where for any pole $b(t)$ of $\omega \mapsto f(\omega;t)$, $\partial_\omega \wp(b(t);t)$ is not identically zero. We may also assume that $a(t)$ is not a pole of $\omega \mapsto \wp(\omega;t)$.

Let us begin with the case where $\omega \mapsto f(\omega;t)$ is an even function. As we can see in the proof of [23, Lemma 1.9], we may write

$$f(\omega;t) = c(t) \prod_{i=1}^{\kappa_z} f_i(\omega;t) \prod_{j=1}^{\kappa_p} g_j(\omega;t),$$

where

- $c(t)$ is a function that does not depend upon ω;
- $f_i(\omega;t)$ are of the form $\wp(\omega;t) - \wp(a(t);t)$, where $a(t)$ are zeros of $\omega \mapsto f(\omega;t)$;
- $g_j(\omega;t)$ are of the form $(\wp(\omega;t) - \wp(b(t);t))^{-1}$, where $b(t)$ are poles of $\omega \mapsto f(\omega;t)$.

Then, a partial fraction decomposition yields a sum of the form

$$f(\omega;t)=\widetilde{c}(t)+\sum_{i=1}^{n_\infty}a_{i,\infty}(t)\wp(\omega;t)^i+\sum_j\sum_{i=1}^{n_j}\frac{a_{i,j}(t)}{(\wp(\omega;t)-\wp(b_j(t);t))^i}. \tag{3.1}$$

By assumption, the $b_j(t)$ are differentially algebraic. Recall, see Lemma 3.7, that for all j, we have $\wp(\omega;t)\in\mathfrak{D}((\omega+b_j(t)))$ (resp. $\wp(\omega;t)\in\omega^{-2}\mathfrak{D}[[\omega]]$). Then, for every i,j,

$$\frac{a_{i,j}(t)}{\Big(\wp(\omega;t)-\wp(b_j(t);t)\Big)^i}=\frac{a_{i,j}(t)}{\Big(\partial_\omega\wp(b_j(t);t)(\omega-b_j(t))\Big)^i}+O((\omega-b_j(t))^{-i+1}).$$

By the composition property of Lemma 3.2 and Proposition 3.4, for all k,ℓ the function $\Big(\partial_\omega^k\wp(b_j(t);t)\Big)^\ell$ is differentially algebraic. Let us write the Taylor expansion of the function

$$f(\omega;t)=\sum_{i=-n_j}^{\infty}\widetilde{a}_i(t)(\omega-b_j(t))^i.$$

Then, for $i<0$, one has

$$\widetilde{a}_i(t)=\frac{a_{i,j}(t)}{\partial_\omega\wp(b_j(t);t)^i}+f_{i,j},\ \text{ where } f_{i,j}\in\mathfrak{D}(a_{i+1,j}(t),\dots,a_{n_j,j}(t)).$$

Since the coefficients of the principal part at $b_j(t)$ are differentially algebraic we have $\widetilde{a}_i(t)\in\mathfrak{D}$. By Lemma 3.2, $\mathfrak{D}$ is a field, and we find by a decreasing induction that for all $1\le i\le n_j$, $a_{i,j}(t)\in\mathfrak{D}$. Similarly, for all i, we have

$$a_{i,\infty}(t)\wp(\omega;t)^i=\omega^{-2i}a_{i,\infty}(t)+O(\omega^{-2i+1}).$$

Then the coefficient of the term in ω^{-2i} with $i>0$ is of the form $a_{i,\infty}(t)+f_i$, where $f_i\in\mathfrak{D}(a_{i+1,\infty}(t),\dots,a_{n_\infty,\infty}(t))$. Since the coefficients of the principal part at 0 are differentially algebraic, we find $a_{i,\infty}(t)+f_i\in\mathfrak{D}$. By Lemma 3.2, $\mathfrak{D}$ is a field, and we find by a decreasing induction that for all $1\le i\le n_\infty$, $a_{i,\infty}(t)\in\mathfrak{D}$. Recall that by assumption, $f(a(t);t)$ is ∂_t-algebraic. By Lemma 3.2 and Proposition 3.4, we find

$$\widetilde{d}(t):=\sum_{i=1}^{n_\infty}a_{i,\infty}(t)\wp(a(t);t)^i+\sum_j\sum_{i=1}^{n_j}\frac{a_{i,j}(t)}{(\wp(a(t);t)-\wp(b_j(t);t))^i}\in\mathfrak{D}.$$

By the subtraction property of Lemma 3.2 we deduce that $\widetilde{c}(t)=f(a(t);t)-\widetilde{d}(t)$ is ∂_t-algebraic. In virtue of Lemma 3.2 and Proposition 3.4, every term in the right-hand side of (3.1) is differentially algebraic. With the field property of Lemma 3.2, this concludes the proof in the even case.

Assume that $\omega\mapsto f(\omega;t)$ is odd. The function $\partial_\omega\wp(\omega;t)^{-1}f(\omega;t)$ is even, and $\omega_1(t)\mathbb{Z}+\omega_2(t)\mathbb{Z}$, the poles of $\partial_\omega\wp(\omega;t)$, are ∂_t-algebraic; see Lemma 3.3. Then, we may apply the even case to deduce that $f(\omega;t)$ is of the form

$$\partial_\omega\wp(\omega;t)\widetilde{c}(t)+\sum_{i=1}^{n_\infty}a_{i,\infty}(t)\partial_\omega\wp(\omega;t)\wp(\omega;t)^i+\sum_j\sum_{i=1}^{n_j}\frac{a_{i,j}(t)\partial_\omega\wp(\omega;t)}{(\wp(\omega;t)-\wp(b_j(t);t))^i}.$$

Setting $a_{0,\infty}(t) := \widetilde{c}(t)$, we may rewrite the latter expression as

$$\sum_{i=0}^{n_\infty} a_{i,\infty}(t)\partial_\omega\wp(\omega;t)\wp(\omega;t)^i + \sum_j \sum_{i=1}^{n_j} \frac{a_{i,j}(t)\partial_\omega\wp(\omega;t)}{(\wp(\omega;t)-\wp(b_j(t);t))^i}.$$

By Proposition 3.4, for all j, we have $\partial_\omega\wp(\omega;t) \in \mathfrak{D}((\omega - b_j(t)))$ (resp. we have $\partial_\omega\wp(\omega;t) \in \mathfrak{D}((\omega)))$. The same reasoning as in the even case shows that for all $1 \leq i \leq n_j$, the functions $a_{i,j}(t)$ are differentially algebraic. Similarly, for all $0 \leq i \leq n_\infty$, the functions $a_{i,\infty}(t)$ are differentially algebraic. By Proposition 3.4 and Lemma 3.2, we find that $f(\omega;t)$ is differentially algebraic. This completes the proof in the odd case.

Let us consider the general case. Note that by Proposition 3.4, $\wp(\omega;t) - \wp(a(t);t)$ is differentially algebraic. So for all n, Lemma 3.2 ensures that $f(\omega;t)$ is differentially algebraic if and only if $(\wp(\omega;t) - \wp(a(t);t))^n f(\omega;t)$ is differentially algebraic. So without loss of generality, we may reduce to the case where $f(\pm a(t);t) = 0$. We write $f(\omega;t) = f_+(\omega;t) + f_-(\omega;t)$, where

$$f_+(\omega;t) := \frac{f(\omega;t) + f(-\omega;t)}{2},$$
$$f_-(\omega;t) := \frac{f(\omega;t) - f(-\omega;t)}{2}.$$

The poles of $\omega \mapsto f_\pm(\omega;t)$ are poles of f or opposite of the latter. By Lemma 3.2, they are ∂_t-algebraic and the coefficients of the principal parts are in $\mathfrak{D}$. Since $f(\pm a(t);t) = 0$ we find $f_\pm(a(t);t) = 0$. In particular it is differentially algebraic. From the even and the odd cases, $f_\pm(\omega;t)$ are differentially algebraic in their two variables. Since the sum of two differentially algebraic functions is differentially algebraic, see Lemma 3.2, we deduce that $f(\omega;t) = f_+(\omega;t) + f_-(\omega;t)$ is differentially algebraic. □

Remark 3.12.

- As in Remark 3.5, we may consider $\widetilde{\wp}(\omega;t)$, the Weierstrass functions associated to the lattice $\omega_1(t)\mathbb{Z} + \omega_3(t)\mathbb{Z}$, or the lattice $\omega_1(t)\mathbb{Z} + k\omega_2(t)\mathbb{Z}$, with $k \in \mathbb{N}^*$. Then, the proof of Theorem 3.10 can be straightforwardly adapted to this new lattice. We then deduce the following. If $\omega \mapsto f(\omega;t)$ is a function such that:
  (1) For all $t \in (0,1)$, $\omega \mapsto f(\omega;t) \in \mathbb{C}(\widetilde{\wp}(\omega;t), \partial_\omega\widetilde{\wp}(\omega;t))$.
  (2) There are countably many elements of $\mathfrak{D}$, whose union forms the set of poles of $\omega \mapsto f(\omega;t)$.
  (3) The coefficients of the principal parts of $\omega \mapsto f(\omega;t)$ are in $\mathfrak{D}$.
  (4) There exists $a(t) \in \mathfrak{D}$ such that $f(a(t);t) \in \mathfrak{D}$.

  Then, $f(\omega;t)$ is differentially algebraic in its two variables.
- Let us now just assume that $\omega \mapsto f(\omega;t)$ satisfies the above first three points and let $a(t) \in \mathfrak{D}$ that is not a pole. Then, $f(\omega;t) - f(a(t);t)$ satisfies the four points and is therefore differentially algebraic. By construction, the function $f(\omega;t) - f(a(t);t)$ has the same principal parts as $f(\omega;t)$.

Although r_x and r_y are not elliptic functions, we will see in the next section that it is sufficient to control the behavior of their poles and coefficients in order to apply Theorem 3.10.

Lemma 3.13. *The following holds:*

(A1) *The poles and coefficients of the principal parts of $\omega \mapsto r_x(\omega;t)$ belong to $\mathfrak{D}$.*
(A2) *There exists $a(t) \in \mathfrak{D}$ such that $r_x(a(t);t) \in \mathfrak{D}$.*

Similar statements hold for r_y.

Proof. Let us prove the result for r_x, the reasoning for r_y is similar. We refer to Section 2.6 for the notations used in this proof.

Recall that the series $Q(x,y;t)$ converges for $|x|,|y|,|t|<1$. Let us consider t in $(0,1)$. Take $\omega \in \mathcal{O}$ (note that $\mathcal{O}$ depends continuously on t), for each of the domains $|x(\omega;t)|<1$ and $|y(\omega;t)|<1$, one has the following equality of functions:

$$F_1(x(\omega;t);t) = r_x(\omega;t) \qquad \text{and} \qquad F_2(y(\omega;t);t) = r_y(\omega;t),$$

with no poles on these domains. Via the equality $0 = r_x(\omega;t)+r_y(\omega;t)-K(0,0;t)Q(0,0;t)+x(\omega;t)y(\omega;t)$, and Lemma 2.9 on the inclusion of poles, we deduce that the poles inside $\mathcal{O}$ of $\omega \mapsto r_x(\omega;t)$ are the poles inside $\mathcal{O}$ of $\omega \mapsto x(\omega;t)y(\omega;t)$ with $|y(\omega;t)|<1$. What is more, on that domain, $\omega \mapsto x(\omega;t)y(\omega;t)$ and $\omega \mapsto r_x(\omega;t)$ have the same principal parts. By Corollary 3.8, the poles of $\omega \mapsto r_x(\omega;t)$ inside $\mathcal{O}$ are differentially algebraic. Furthermore, the corresponding principal parts have differentially algebraic coefficients.

Recall, see (2.4), that $r_x(\omega+\omega_3(t);t) = r_x(\omega;t)+b_x(\omega;t)$. By Corollary 3.9, the poles and the coefficients of the principal parts of $\omega \mapsto b_x(\omega;t)$ are differentially algebraic. By Lemma 3.3, $\omega_3(t)$ is differentially algebraic. Recall that $\bigcup_{\ell\in\mathbb{Z}} \widetilde{\sigma}^{\ell}(\mathcal{O}) = \mathbb{C}$. With (2.4) and what precedes, we get assertion (A1).

It remains to prove assertion (A2). To lighten the notations we omit the dependence in t in what follows. Let us write $K(x,y;t) = \widetilde{B}_0(y)+x\widetilde{B}_1(y)+x^2\widetilde{B}_2(y)$. Let $\omega_0(t) \in \mathcal{O}$ such that

$$y(\omega_0)=0 \qquad \text{and} \qquad x(\omega_0) = \frac{-\widetilde{B}_1(y(\omega_0))+\sqrt{\widetilde{B}_1(y(\omega_0))^2-4\widetilde{B}_0(y(\omega_0))\widetilde{B}_2(y(\omega_0))}}{2\widetilde{B}_2(y(\omega_0))}.$$

The y-valuation of $\widetilde{B}_2(y)$ being at most two, we consider the following subcases.

- If it is 0 or 1, the valuation of the algebraic function $y \times \frac{-\widetilde{B}_1(y)+\sqrt{\widetilde{B}_1(y)^2-4\widetilde{B}_0(y)\widetilde{B}_2(y)}}{\widetilde{B}_2(y)}$ is nonnegative and then ω_0 is not a pole of $x(\omega;t)y(\omega;t)$.
- If it is 2, then $4\widetilde{B}_0(y)\widetilde{B}_2(y)$ converges to 0 when y goes to 0 and hence the same holds for $-\widetilde{B}_1(y)+\sqrt{\widetilde{B}_1(y)^2-4\widetilde{B}_0(y)\widetilde{B}_2(y)}$.

We further find that $y \times \left(-\widetilde{B}_1(y)+\sqrt{\widetilde{B}_1(y)^2-4\widetilde{B}_0(y)\widetilde{B}_2(y)}\right) \in O(y^2)$. In that case, we find that ω_0 is not a pole of $x(\omega;t)y(\omega;t)$ either. With $K(0,0;t)Q(0,0;t) = F_2(y(\omega_0;t);t) = r_y(\omega_0;t)$, and $0 = r_x(\omega;t)+r_y(\omega;t)-K(0,0;t)Q(0,0;t)+x(\omega;t)y(\omega;t)$, we then find $0 = r_x(\omega_0;t)+x(\omega_0;t)y(\omega_0;t)$. It then suffices to show that $x(\omega_0;t)y(\omega_0;t)$ is differentially algebraic. With the expression of $y(\omega_0;t)$ in Proposition 2.8, we find that ω_0 is solution of an equation of the form $\wp(\omega_0;t) = b(t)$ with $b(t) \in \mathfrak{D}$. With the same reasons as in the proof of Corollary 3.8, we find that ω_0 is differentially algebraic, and $x(\omega_0;t), y(\omega_0;t) \in \mathfrak{D}$. Then $r_x(\omega_0(t);t) \in \mathfrak{D}$. This concludes the proof. □

The following result relates the differential transcendence of $Q(x,y;t)$ and the differential transcendence of $r_x(\omega;t)$ and $r_y(\omega;t)$.

Proposition 3.14. *The following statements are equivalent.*

- *The generating function $Q(x,y;t)$ is differentially algebraic in its three variables.*
- *The series $F_1(x;t)$ and $F_2(y;t)$ are differentially algebraic in their two variables.*
- *The meromorphic continuations $r_x(\omega;t)$ and $r_y(\omega;t)$ are differentially algebraic in their two variables.*

Proof. If $Q(x,y;t)$ is differentially algebraic then $Q(x,0;t)$ is differentially algebraic. Since $K(x,0;t)$ is differentially algebraic, we use the ring property of Lemma 3.2 to deduce that $F_1(x;t) = K(x,0;t)Q(x,0;t)$ is differentially algebraic. (The reasoning is similar for the differential algebraicity of $F_2(y;t)$). Conversely, if $F_1(x;t)$ and $F_2(y;t)$ are differentially algebraic, then, by evaluation, so is $Q(0,0;t)$. As the right-hand side of the expression in Lemma 2.1 is a sum and product of elements that are differentially algebraic, it is differentially algebraic (by the field property in Lemma 3.2). Therefore, $K(x,y;t)Q(x,y;t)$ is differentially algebraic. Thus, $Q(x,y;t)$ is differentially algebraic. So the first two points are equivalent.

Assume that the series $F_1(x;t)$ is differentially algebraic in its two variables. Recall that $F_1(x(\omega;t);t) = r_x(\omega;t)$ where $x(\omega;t)$ is differentially algebraic; see Corollary 3.8. By composition of differentially algebraic functions, see Lemma 3.2, $r_x(\omega;t)$ is differentially algebraic. Conversely, on a domain where $x(\omega;t)$ is invertible, its inverse is also differentially algebraic; see Lemma 3.2. We conclude similarly that if $r_x(\omega;t)$ is differentially algebraic then $F_1(x;t)$ is differentially algebraic. A similar reasoning holds for the y variable and we find that $F_2(y;t)$ is differentially algebraic if and only if $r_y(\omega;t)$ is differentially algebraic. This proves the equivalence between the last two points. □

4. Differential algebraicity of the generating function

The goal of this section is to prove Theorem 1.1 (the ∂_x, ∂_y, and ∂_t differential algebraicity are equivalent). By Lemma 2.4, the result holds for all degenerate cases. By Lemma 2.6 and Proposition 2.5, it also holds when $\overline{E}_t$ *is not* an elliptic curve. So we now prove the case where $\overline{E}_t$ is an elliptic curve. Let G be the group of the walk (see Definition 2.7). Our proof handles separately the cases $|G| < \infty$ and $|G| = \infty$.

4.1. **Finite group case.**

Proposition 4.1. *Let us consider a nondegenerate model of walks, assume that $\overline{E}_t$ is an elliptic curve and $|G| < \infty$. Then, $Q(x,y;t)$ is ∂_x-algebraic, ∂_y-algebraic and ∂_t-algebraic.*

Proof. By Proposition 3.14, it suffices to show that $r_x(\omega;t)$ and $r_y(\omega;t)$ are differentially algebraic in their two variables. Let us only consider $r_x(\omega;t)$, the proof for $r_y(\omega;t)$ is similar. Recall that the $\omega_i(t)$ are continuous and that $\omega_3(t) \in (0, \omega_2(t))$ (see Equation (2.2)). Since $|G| < \infty$ and $\widetilde{\sigma}(\omega) = \omega + \omega_3(t)$, there exist $k, \ell \in \mathbb{N}^*$ with $\gcd(k,\ell) = 1$ such that $\omega_3(t)/\omega_2(t) = k/\ell$. By (2.4), we have $r_x(\omega + \omega_3(t);t) = r_x(\omega;t) + b_x(\omega;t)$, where $b_x(\omega;t) = y(-\omega;t)(x(\omega;t) - x(\omega + \omega_3(t);t))$. Let us recall some notations borrowed from the proof of [11, Theorem 4.1]. It is shown that we may write a decomposition of the form

$$r_x(\omega;t) = \psi(\omega;t) + \Phi(\omega;t)\phi(\omega;t). \tag{4.1}$$

More precisely,

- $\Phi(\omega;t)=\sum_{i=0}^{\ell-1} b_x(\omega+i\omega_3(t);t)$;
- $\phi(\omega;t)=\frac{\omega_1(t)}{2\mathbf{i}\pi}\zeta(\omega;t)-\frac{\omega}{\mathbf{i}\pi}\zeta(\omega_1(t)/2;t)$, where ζ is an opposite of the antiderivative of the Weierstrass function with periods $\omega_1(t)$ and $k\omega_2(t)$, that is

$$\zeta(\omega;t)=\frac{1}{\omega}+\sum_{(\ell_1,\ell_2)\in\mathbb{Z}^2\setminus\{(0,0)\}}\left(\frac{1}{\omega+\ell_1\omega_1(t)+\ell_2 k\omega_2(t)}-\frac{1}{\ell_1\omega_1(t)+\ell_2 k\omega_2(t)}+\frac{\omega}{(\ell_1\omega_1(t)+\ell_2 k\omega_2(t))^2}\right);$$

- for all $t\in(0,1)$, the function $\omega\mapsto\psi(\omega;t)$ is $(\omega_1(t),k\omega_2(t))$-periodic.

The idea is to prove successively that $\Phi(\omega;t)$, $\phi(\omega;t)$ and $\psi(\omega;t)$ are differentially algebraic. We will also see them as functions of ω and study their poles and principal parts.

Step 1: Study of $\Phi(\omega;t)$.

Lemma 4.2. *The following holds:*

- *There are countably many elements of $\mathfrak{D}$, whose union forms the set of poles of $\omega\mapsto\Phi(\omega;t)$.*
- *The coefficients of the principal parts of $\omega\mapsto\Phi(\omega;t)$ are in $\mathfrak{D}$.*
- *Φ is differentially algebraic in its two variables.*

Proof. Recall, see Lemma 3.3, that $\omega_3(t)\in\mathfrak{D}$. We may combine Corollary 3.9 and Lemma 3.2, to deduce that the poles and the coefficients of the principal parts of $\omega\mapsto\Phi(\omega;t)$ are ∂_t-algebraic. Furthermore by Lemma 3.2 and Proposition 3.4, Φ is differentially algebraic in its two variables. □

Step 2: Study of $\phi(\omega;t)$.

Before proving that $\phi(\omega;t)$ is differentially algebraic, let us study $\zeta(\omega;t)$.

Lemma 4.3. *The following holds:*

- *There are countably many elements of $\mathfrak{D}$, whose union forms the set of poles of $\omega\mapsto\zeta(\omega;t)$.*
- *The coefficients of the principal parts of $\omega\mapsto\zeta(\omega;t)$ are in $\mathfrak{D}$.*
- *ζ is differentially algebraic in its two variables.*

Proof. In virtue of Lemma 3.3, the periods $\omega_1(t),\omega_2(t)$ are differentially algebraic. Then, the poles and the coefficients of the principal parts of $\omega\mapsto\zeta(\omega;t)$ belong to $\mathfrak{D}$.

Let $\widetilde{\wp}$ be the Weierstrass function with periods $\omega_1(t),k\omega_2(t)$ and let us write the classical differential equation

$$\partial_\omega\widetilde{\wp}(\omega;t)^2=4\widetilde{\wp}(\omega;t)^3-\widetilde{g_2}(t)\widetilde{\wp}(\omega;t)-\widetilde{g_3}(t). \tag{4.2}$$

By Remark 3.5, $\widetilde{\wp}(\omega;t)=-\partial_\omega\zeta(\omega;t)$ is differentially algebraic in its two variables. Then, $\zeta(\omega;t)$ is ∂_ω-algebraic too. Let us prove the ∂_t-algebraicity. Let us differentiate (4.2) with respect to ∂_ω and simplify by $\partial_\omega\widetilde{\wp}(\omega;t)$, to find

$$2\partial_\omega^2\widetilde{\wp}(\omega;t)=12\widetilde{\wp}(\omega;t)^2-\widetilde{g_2}(t).$$

By Lemma 3.2, for all $k \geq 0$, the derivatives $\partial_\omega^k \widetilde{\wp}(\omega;t)$ are ∂_t-algebraic. Since the ∂_t-algebraic functions form a ring, see Lemma 3.2, we deduce that $\widetilde{g_2}(t)$ is ∂_t-algebraic. Using again the ring property of Lemma 3.2 in (4.2), we deduce that $\widetilde{g_3}(t)$ is ∂_t-algebraic too. We may see the elliptic functions as functions of ω and $\widetilde{g_2}, \widetilde{g_3}$. By [1, (18.6.19)],

$$(\widetilde{g_2}^3 - 27\widetilde{g_3}^2)\partial_{\widetilde{g_3}}\widetilde{\wp} = (3\widetilde{g_2}\zeta - \frac{9}{2}\widetilde{g_3}\omega)\partial_\omega\widetilde{\wp} + 6\widetilde{g_2}\widetilde{\wp}^2 - 9\widetilde{g_3}\widetilde{\wp} - \widetilde{g_2}^2. \tag{4.3}$$

We have $\partial_t\widetilde{\wp} = \partial_t\widetilde{g_3}\partial_{\widetilde{g_3}}\widetilde{\wp}$. If $\partial_t\widetilde{g_3} = 0$ then $\widetilde{\wp}$ does not depend on t. In particular, its poles are independent of t, which implies that the periods $\omega_1(t)$ and $k\omega_2(t)$ are independent of t. Then, $\zeta(\omega;t)$ is independent of t and therefore ∂_t-algebraic. We similarly deal with the case $\partial_t\widetilde{g_2} = 0$. So let us consider the situation where both functions $\partial_t\widetilde{g_2}$ and $\partial_t\widetilde{g_3}$ are not identically zero. By the derivation property of Lemma 3.2, we deduce that $\partial_t\widetilde{\wp}, \partial_t\widetilde{g_3}$ are ∂_t-algebraic. Since the ∂_t-algebraic functions form a field, see Lemma 3.2, we then find that $\partial_{\widetilde{g_3}}\widetilde{\wp} = \partial_t\widetilde{\wp}/\partial_t\widetilde{g_3}$ is ∂_t-algebraic. Since $\partial_t\widetilde{g_2} \neq 0$, we are in the situation where $\widetilde{g_2}$ is not identically zero. With (4.3), and the field property of Lemma 3.2, we deduce that $\zeta(\omega;t)$ is ∂_t-algebraic. This completes the proof of the lemma. □

Lemma 4.4. *The following holds:*

- *There are countably many elements of $\mathfrak{D}$, whose union forms the set of poles of $\omega \mapsto \phi(\omega;t)$.*
- *The coefficients of the principal parts of $\omega \mapsto \phi(\omega;t)$ are in $\mathfrak{D}$.*
- *ϕ is differentially algebraic in its two variables.*

Proof. In virtue of Lemma 3.3, the period $\omega_1(t)$ is differentially algebraic. By Lemma 3.2, and Lemma 4.3, we find that $\phi(\omega;t)$ is differentially algebraic in its two variables. Furthermore, the poles and the coefficients of the principal parts of $\omega \mapsto \phi(\omega;t)$ belong to $\mathfrak{D}$. □

Step 3: Study of $\psi(\omega;t)$.

Let us now study $\psi(\omega;t)$. By Lemma 3.13 there exists $a(t) \in \mathfrak{D}$ such that $r_x(a(t);t) \in \mathfrak{D}$. Furthermore, the poles and the coefficients of the principal parts of $\omega \mapsto r_x(\omega;t)$ are ∂_t-algebraic.

With (4.1), Lemma 4.2 and Lemma 4.4, we deduce that the poles of $\omega \mapsto \psi(\omega;t)$ are ∂_t-algebraic, and the coefficients of the principal parts are ∂_t-algebraic. Recall that for all t, $\omega \mapsto \psi(\omega;t)$ is $(\omega_1(t), k\omega_2(t))$-periodic. By Remark 3.12, we may build $\omega \mapsto \psi_0(\omega;t)$, that is differentially algebraic and $(\omega_1(t), k\omega_2(t))$-periodic, with same principal parts as $\omega \mapsto \psi(\omega;t)$. We have

$$r_x(\omega;t) = \psi(\omega;t) - \psi_0(\omega;t) + \Phi(\omega;t)\phi(\omega;t) + \psi_0(\omega;t). \tag{4.4}$$

Note that by construction $\omega \mapsto \psi(\omega;t) - \psi_0(\omega;t)$ has no poles. Since $\omega \mapsto r_x(\omega;t)$ has no poles at $a(t)$, we deduce with (4.4), that $\omega \mapsto \Phi(\omega;t)\phi(\omega;t) + \psi_0(\omega;t)$ has no poles at $a(t)$. Since $\Phi(\omega;t)\phi(\omega;t) + \psi_0(\omega;t)$ is differentially algebraic (as the sum of two differentially algebraic functions, see Lemma 3.2), with no poles at $a(t)$, we find that its evaluation at $a(t)$ is differentially algebraic. Since $r_x(a(t);t) \in \mathfrak{D}$ we use the ring property in Lemma 3.2 to deduce that $\psi(a(t);t) - \psi_0(a(t);t)$ is differentially algebraic.

Then, $\psi(\omega;t)-\psi_0(\omega;t)$ satisfies the assumptions of Theorem 3.10 (with $\omega_2(t)$ replaced by $k\omega_2(t)$) and we deduce that it is differentially algebraic by Remark 3.12. By Lemma 3.2, and the differential algebraicity of $\psi_0(\omega;t)$, we deduce that $\psi(\omega;t)$ is differentially algebraic.

Step 4: Study of $r_x(\omega;t)$.

Let us now finish the proof of Proposition 4.1. Since $\psi(\omega;t)$, $\Phi(\omega;t)$, and $\phi(\omega;t)$ are differentially algebraic in their two variables, we conclude that $r_x(\omega;t)=\psi(\omega;t)+\Phi(\omega;t)\phi(\omega;t)$ is differentially algebraic as the sum and product of differentially algebraic functions; see Lemma 3.2. □

4.2. **Infinite group case.** It now remains to handle the case where the group has infinite order. So let us consider a nondegenerate model of walks and assume that $\overline{E}_t$ is an elliptic curve and $|G|=\infty$. The equivalence between the ∂_x-algebraicity and the ∂_y-algebraicity can be straightforwardly deduced in this weighted context from the proof of [8, Proposition 3.10]. Let us see that the ∂_t-algebraicity implies the ∂_x-algebraicity. If $Q(x,y;t)$ is ∂_t-algebraic, then $Q(x,0;t)$ is ∂_t-algebraic. By [7, Theorem 3.12], if $Q(x,0;t)$ is ∂_t-algebraic, then it is ∂_x-algebraic. In virtue of Lemmas 2.1 and 3.2, we find that if $Q(x,0;t)$ is ∂_x-algebraic, then $Q(x,y;t)$ is ∂_x-algebraic. So to prove Theorem 1.1, it now suffices to show the following result.

Theorem 4.5. *Let us consider a nondegenerate model of walks, assume that $\overline{E}_t$ is an elliptic curve and $|G|=\infty$. If $Q(x,y;t)$ is ∂_x-algebraic, then it is ∂_t-algebraic.*

Proof. By Proposition 3.14, it suffices to show that $r_x(\omega;t)$ and $r_y(\omega;t)$ are differentially algebraic. Let us consider $r_x(\omega;t)$, the proof for $r_y(\omega;t)$ is similar. By Proposition 2.5, for all $t\in(0,1)$ fixed, $\overline{E}_t$ is an elliptic curve. Let G_t be the group G specialized at t. The order of the group G_t may depend upon t. However by [8, Proposition 2.6], see also [19, Proposition 14], which can be straightforwardly extended in the weighted framework, the set of $t\in(0,1)$ such that G_t has infinite order is dense. By assumption, for such t fixed, $x\mapsto F_1(x;t)$ is ∂_x-algebraic. By [16, Theorem 3.8], for all such t fixed there exists a $(\omega_1(t),\omega_2(t))$-periodic function $\widetilde{g}(\omega;t)$, such that

$$b_x(\omega;t)=\widetilde{g}(\omega+\omega_3(t);t)-\widetilde{g}(\omega;t). \tag{4.5}$$

By [16, Proposition 3.9], there exist $g(x;t)\in\mathbb{C}(x,t)$ and $h(y;t)\in\mathbb{C}(y,t)$ such that $g(x(\omega;t);t)=\widetilde{g}(\omega;t)$ and for all $(x,y)\in\overline{E}_t$,

$$xy=g(x;t)+h(y;t).$$

Since $g(x(\omega;t);t)=\widetilde{g}(\omega;t)$, we use Corollary 3.8 to deduce that we may continue $\widetilde{g}(\omega;t)$ in the t variable.

Step 1: Study of $\widetilde{g}(\omega;t)$.

Lemma 4.6. *The following holds:*

- *There are countably many elements of $\mathfrak{D}$, whose union forms the set of poles of $\omega\mapsto\widetilde{g}(\omega;t)$.*
- *The coefficients of the principal parts of $\omega\mapsto\widetilde{g}(\omega;t)$ are in $\mathfrak{D}$.*
- *$\widetilde{g}$ is differentially algebraic in its two variables.*

Proof. We claim that the poles of $\omega \mapsto \widetilde{g}(\omega;t)$ are of the form $\omega_0(t) + \ell\omega_3(t)$, where $\omega_0(t)$ is a pole of $\omega \mapsto b_x(\omega;t)$ and $\ell \in \mathbb{Z}$. To the contrary, assume that $a(t)$ is a pole that is not of this form. Then $a(t) - \omega_3(t)$ is a pole of $\omega \mapsto \widetilde{g}(\omega + \omega_3(t);t)$, that is not a pole of $\omega \mapsto b_x(\omega;t)$. With (4.5), we find that $a(t) - \omega_3(t)$ is a pole of $\omega \mapsto \widetilde{g}(\omega;t)$. We prove successively that for all $\ell \geq 0$, $a(t) - \ell\omega_3(t)$ is a pole of $\omega \mapsto \widetilde{g}(\omega;t)$. Since $\widetilde{g}(\omega;t)$ is $(\omega_1(t), \omega_2(t))$-periodic, $a(t) - \omega_3(t)\mathbb{N} + \omega_1(t)\mathbb{Z} + \omega_2(t)\mathbb{Z}$ are poles of $\omega \mapsto \widetilde{g}(\omega;t)$. Since $|G| = \infty$ and $\widetilde{\sigma}(\omega) = \omega + \omega_3(t)$, the sets $A_\ell := \{a(t) - \ell\omega_3(t) + \omega_1(t)\mathbb{Z} + \omega_2(t)\mathbb{Z}\}$, with $\ell \in \mathbb{N}$, are disjoint. Then, the set of poles of $\omega \mapsto \widetilde{g}(\omega;t)$ possesses an accumulation point which contradicts that the function is meromorphic. This proves the claim.

By Corollary 3.9, the poles of $\omega \mapsto b_x(\omega;t)$ are ∂_t-algebraic. By Lemma 3.3, $\omega_3(t)$ is ∂_t-algebraic too. With the claim, it follows that the poles of $\omega \mapsto \widetilde{g}(\omega;t)$ are ∂_t-algebraic. By Corollary 3.8, the coefficients of the principal parts of $\omega \mapsto x(\omega;t)$ are ∂_t-algebraic. With $g(x(\omega;t);t) = \widetilde{g}(\omega;t)$, and $g(x;t) \in \mathbb{C}(x,t)$, we deduce that the coefficients of the principal parts of $\omega \mapsto \widetilde{g}(\omega;t)$ are ∂_t-algebraic. Finally $\widetilde{g}(\omega;t)$ is differentially algebraic, as the composition of differentially algebraic functions; see Lemma 3.2. □

Step 2: Study of $\widetilde{f}(\omega;t) := r_x(\omega;t) - \widetilde{g}(\omega;t)$.

By (2.4) and (4.5), we find

$$\begin{aligned}\widetilde{f}(\omega + \omega_3(t);t) &= r_x(\omega + \omega_3(t);t) - \widetilde{g}(\omega + \omega_3(t);t)\\ &= r_x(\omega;t) + b_x(\omega;t) - (\widetilde{g}(\omega;t) + b_x(\omega;t)) = \widetilde{f}(\omega;t).\end{aligned}$$

Then, $\widetilde{f}(\omega;t)$ is $\omega_3(t)$-periodic. Recall that $\widetilde{g}(\omega;t)$ is $\omega_1(t)$-periodic. By (2.5), $r_x(\omega;t)$ is also $\omega_1(t)$-periodic. Therefore, $\omega \mapsto \widetilde{f}(\omega;t)$ is elliptic with periods $(\omega_1(t), \omega_3(t))$. Recall that the poles and the coefficients of the principal parts of $\omega \mapsto \widetilde{g}(\omega;t)$ are ∂_t-algebraic. By Lemma 3.13, the same holds for $r_x(\omega;t)$ and there exists $a(t) \in \mathfrak{D}$ such that $r_x(a(t);t)$ is differentially algebraic. By Remark 3.12, we may build $\omega \mapsto \widetilde{f}_0(\omega;t)$, that is differentially algebraic, $(\omega_1(t), \omega_3(t))$-periodic, and with same principal parts as $\omega \mapsto \widetilde{f}(\omega;t)$. Let us write

$$\widetilde{f}(\omega;t) - \widetilde{f}_0(\omega;t) = r_x(\omega;t) - \widetilde{g}(\omega;t) - \widetilde{f}_0(\omega;t).$$

The function $-\widetilde{g}(\omega;t) - \widetilde{f}_0(\omega;t)$ is differentially algebraic, as the sum of two differentially algebraic functions; see Lemma 3.2. Since $\widetilde{f}(\omega;t) - \widetilde{f}_0(\omega;t)$ has no poles and $a(t)$ is not a pole of $\omega \mapsto r_x(\omega;t)$, we deduce that $a(t)$ is not a pole of $\omega \mapsto -\widetilde{g}(\omega;t) - \widetilde{f}_0(\omega;t)$. Therefore, its evaluation at $a(t)$ is still differentially algebraic. Then, the same holds for $\omega \mapsto \widetilde{f}(\omega;t) - \widetilde{f}_0(\omega;t)$, which satisfies the assumptions of Theorem 3.10 (with $\omega_2(t)$ replaced with $\omega_3(t)$) and we deduce that it is differentially algebraic by Remark 3.12. Hence, $r_x(\omega;t) = \left(\widetilde{f}(\omega;t) - \widetilde{f}_0(\omega;t)\right) + \widetilde{g}(\omega;t) + \widetilde{f}_0(\omega;t)$ is differentially algebraic as the sum of differentially algebraic functions, see Lemma 3.2. This concludes the proof. □

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