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LEFT-TO-RIGHT MAXIMA IN DYCK PATHS

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Abstract. In a Dyck path a peak which is (weakly) higher than all the preceding peaks is called a strict (weak) left-to-right maximum. We obtain explicit generating functions for both weak and strict left-to-right maxima in Dyck paths. The proofs of the associated asymptotics make use of analytic techniques such as Mellin transforms, singularity analysis and formal residue calculus.

Keywords: Dyck paths, generating functions, asymptotics, left-to-right maxima.

1. General introduction

A Dyck path is a lattice path in the first quadrant, that starts at the origin (0,0) with an up step ($u = (1, 1)$) and thereafter only up and down ($d = (1, -1)$) steps are allowed under the conditions that it may not go below the x-axis and that it may terminate only if the end point is on the x-axis. A Dyck path with n up steps must end at the point $(2n, 0)$; see the definition in [16]. Such a Dyck path is said to have length $2n$. For a detailed study of properties of Dyck paths see [7]. For further recent work on Dyck paths; see [1–4, 6, 9, 15].

Given an arbitrary Dyck path, we mean by a *strict left-to-right maximum*, any peak (successive pair of the form ud) in the Dyck path which is greater than the height all peaks to its left. A *weak left-to-right maximum* is a peak which is greater than or equal to the height of all peaks to its left. From here on, by left-to-right maxima we mean strict left-to-right maxima unless otherwise stated.

A standard combinatorial problem is the accounting for the number of left-to-right maxima in combinatorial structures such as permutations and words over a fixed alphabet. In this paper we focus on obtaining a generating function for the number of left-to-right maxima in Dyck paths. This is a bivariate generating function which tracks the number of up steps by z and the number of left-to-right maxima by x. We also obtain a generating function for the total number of left-to-right maxima in Dyck paths with n up steps.

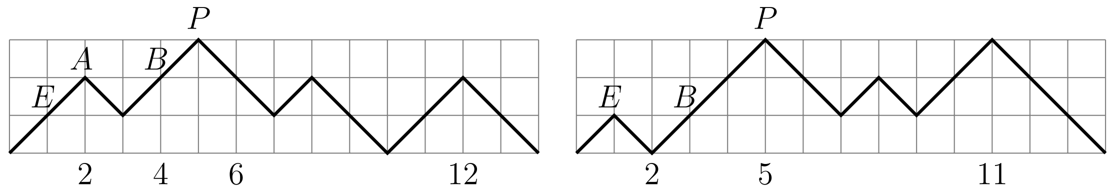

Figure 1. Two Dyck paths of length 14 and height 3

As an introduction to the method we will use for the construction of the first generating function above, here follows a sketch (Figure 1) of two Dyck paths of height 3. The left-to-right maxima are marked in the left case by A and P and in the right case by E and P. P also marks the first maximum height attained by the Dyck paths. We begin at the origin with a u step tracked in the generating function by z which leaves us at the point E. This single up step is followed by a possibly empty upside-down Dyck path of maximum height 1.

In the left example in Figure 1, this part is indeed empty (and therefore not requiring x) but not in the right example where the path between E and B is an upside-down Dyck path of height 1 which gives rise to a left-to-right maximum thus requiring an x tracker. Then we have another single u step and we proceed recursively in this way leaving us eventually at the next left-to-right maximum which is point A in the left example and P in the right. In the left example, right of A is again a possibly empty upside-down Dyck path, this time of maximum height 2 where the non empty case is tracked again by x. We are referring to the path between A and B which is actually of height 1. Once P is reached, it is followed by the rest of the path which is conceived as a *right to left portion* of a Dyck path. In the section dealing with this, the generating function for these latter Dyck paths ending at height r will be given and used, as will the generating function for Dyck paths of a fixed height h, which is used as indicated above for the possibly empty upside-down Dyck paths that occur sequentially before the point P is attained.

2. Left-to-right maxima in Dyck paths

We start this section by referring to the paper [14] by Prodinger on the first sojourn in Dyck paths. Using the notation from [14], we let $C(h)$ be the number of paths of height $\leq h$ with steps which follow all rules of Dyck paths except that they terminate at height h, and we let $A(h)$ be the number of Dyck paths of height $\leq h$ (which by definition end at height zero). It is shown in [14] that

$$C(h) := \frac{z^h\sqrt{1-4z^2}}{{\lambda_1}^{h+2} - {\lambda_2}^{h+2}}$$

and

$$A(h) := \frac{{\lambda_1}^{h+1} - {\lambda_2}^{h+1}}{{\lambda_1}^{h+2} - {\lambda_2}^{h+2}}, \tag{2.1}$$

where λ_1 and λ_2, are given by

$$\lambda_1 = \frac{1+\sqrt{1-4z^2}}{2}; \; \lambda_2 = \frac{1-\sqrt{1-4z^2}}{2}.$$

As explained in the introductory section, we consider a sequence of possibly empty Dyck paths of height $\leq h$ for $h = 1, 2, \ldots$. At the end of each path in the sequence, we have a single up step that leads to the next left-to-right maximum and eventually to the first overall maximum of the entire Dyck path. We let x count the number of left-to-right maxima attained by the Dyck path. This leads to our first theorem:

Theorem 2.1. *The generating function for the number of left-to-right maxima tracked by x, for Dyck paths of maximum height r and length tracked by z is*

$$F(x,z,r) := z^r x C(r) \prod_{h=1}^{r-1} (1 + x(A(h) - 1)).$$

So, the total number of left-to-right maxima for Dyck paths of fixed height r is found by differentiating the above function with respect to x and setting $x = 1$. The derivative at this point is given by

$$\begin{aligned}
\frac{\partial}{\partial x} F(x,z,r)\Big|_{x=1} &= z^r\ C(r) \prod_{h=1}^{r-1} A(h) + z^r C(r) \prod_{h=1}^{r-1} A(h) \sum_{i=1}^{r-1} \frac{A(i)-1}{A(i)} \\
&= z^r\ C(r) \prod_{h=1}^{r-1} A(h) \left(1 + \sum_{i=1}^{r-1} \frac{A(i)-1}{A(i)}\right) \\
&= z^r C(r) \prod_{h=1}^{r-1} A(h) \left(r - \sum_{i=1}^{r-1} \frac{1}{A(i)}\right) \qquad (2.2)
\end{aligned}$$

Note that $z^r C(r) \prod_{h=1}^{r-1} A(h)$ telescopes to become

$$\frac{z^{2r}\left(1-4z^2\right)}{\left(-\lambda_2^{1+r} + \lambda_1^{1+r}\right)\left(-\lambda_2^{2+r} + \lambda_1^{2+r}\right)}$$

but the full generating function becomes very complicated as a function of z.

To simplify this generating function, we substitute

$$z^2 = \frac{u}{(1+u)^2}$$

in (2.2) which implies

$$\lambda_1 = \frac{1}{1+u};\ \lambda_2 = \frac{u}{1+u};\ C(r) = \frac{(1-u)(1+u)^{1+r} z^r}{1-u^{2+r}};\ A(r) = \frac{(1+u)\left(1-u^{1+r}\right)}{1-u^{2+r}}$$

and obtain

$$T(r) := \frac{\partial}{\partial x} F(x,z,r)\Big|_{x=1} = \frac{(1-u)^2 u^r (1+u)}{\left(1-u^{1+r}\right)\left(1-u^{2+r}\right)} \left(r - \sum_{i=1}^{r-1} \frac{1-u^{2+i}}{(1+u)\left(1-u^{1+i}\right)}\right).$$

The full generating function for the total number of left-to-right maxima in all Dyck paths of length n is

$$\mathrm{Tot}(u) := \sum_{r=1}^{\infty} T(r).$$

Consequently, we have the following proposition:

Proposition 2.2. *The generating function* $\operatorname{Tot}(u)$ *for the total number of left-to-right maxima in Dyck paths of length* n *tracked by* z *is given by (using* $z^2 = \frac{u}{(1+u)^2}$*):*

$$\operatorname{Tot}(u) = \sum_{r=1}^{\infty} \frac{(1-u)^2 u^r (1+u)}{(1-u^{1+r})(1-u^{2+r})} \left(r - \sum_{i=1}^{r-1} \frac{1-u^{2+i}}{(1+u)(1-u^{1+i})} \right). \tag{2.3}$$

In order to obtain the series expansion for this, we use the equivalent inverse substitution for u, namely

$$u = \frac{1-2z^2-\sqrt{1-4z^2}}{2z^2}, \tag{2.4}$$

and obtain in terms of z,

$$\operatorname{Tot}(u) = z^2 + 2z^4 + 6z^6 + \boldsymbol{19z^8} + 63z^{10} + 216z^{12} + 758z^{14} + 2705z^{16} + 9777z^{18} + 35698z^{20} + O(z^{21}).$$

We illustrate the bold term of the series by means of the black dots in Figure 2.

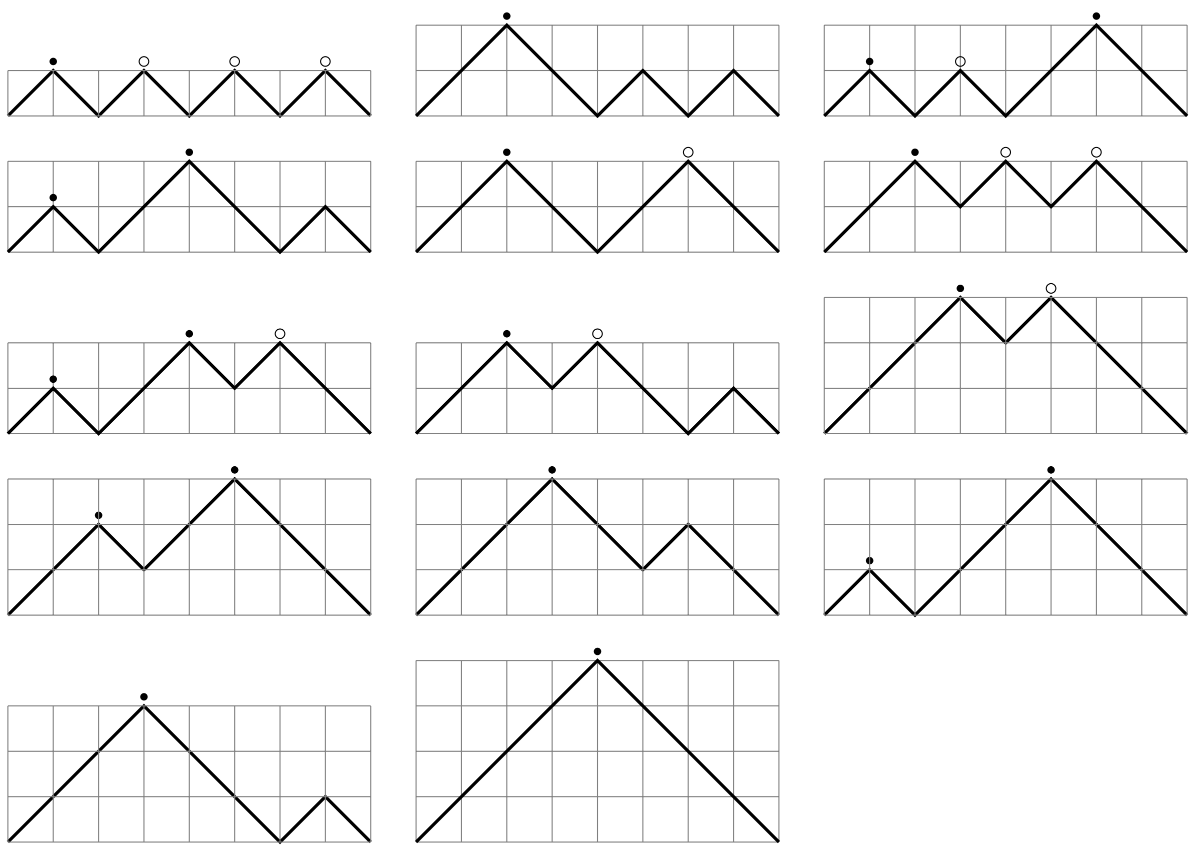

FIGURE 2. All 14 Dyck paths of length 8: they have 19 strict left-to-right maxima (indicated by black dots) and 10 weak left-to-right maxima (indicated by circles).

The type of series expansion for $\mathrm{Tot}(u)$ in Proposition 2.2 involves what is called Lambert series. There are currently no computer algebra packages that can automatically simplify expressions like Equation (2.3). Instead, it is therefore necessary to make the following lengthy calculations in order to derive Theorem 2.3.

To simplify Equation (2.3) we swap the order of the summations in the double sum, and thereafter use partial fractions on the r-indexed sum (which then telescopes as in line (2.5)) to obtain

$$\begin{aligned}
&\sum_{r=1}^{\infty}\frac{(1-u)^2u^r(1+u)}{(1-u^{1+r})\,(1-u^{2+r})}\sum_{i=1}^{r-1}\frac{1-u^{2+i}}{(1+u)\,(1-u^{1+i})}\\
&=(1-u)^2\sum_{i=1}^{\infty}\frac{1-u^{2+i}}{(1-u^{1+i})}\sum_{r=i+1}^{\infty}\frac{u^r}{(1-u^{1+r})\,(1-u^{2+r})}\\
&=(1-u)^2\sum_{i=1}^{\infty}\frac{1-u^{2+i}}{(1-u^{1+i})}\frac{u^{1+i}}{(1-u)\,(1-u^{2+i})}. \qquad (2.5)
\end{aligned}$$

Now changing the index of summation from i to r,

$$(1-u)\sum_{i=1}^{\infty}\frac{1-u^{2+i}}{(1-u^{1+i})}\frac{u^{1+i}}{(1-u^{2+i})}=(1-u)\sum_{r=1}^{\infty}\frac{u^{1+r}}{(1-u^{1+r})}.$$

Altogether,

$$\begin{aligned}
\mathrm{Tot}(u)&=\sum_{r=1}^{\infty}\frac{(1-u)^2u^r(1+u)r}{(1-u^{1+r})\,(1-u^{2+r})}-(1-u)\sum_{r=1}^{\infty}\frac{u^{1+r}}{(1-u^{1+r})}\\
&=\sum_{r=1}^{\infty}\frac{(1-u)u^r\left(r-u-ru^2+u^{3+r}\right)}{(1-u^{1+r})\,(1-u^{2+r})}\\
&=\sum_{r=1}^{\infty}\frac{ru^r-u^{1+r}-ru^{1+r}+u^{2+r}-ru^{2+r}+ru^{3+r}+u^{3+2r}-u^{4+2r}}{(1-u^{1+r})\,(1-u^{2+r})}.
\end{aligned}$$

Drop the first term ru^r in the numerator above and apply partial fractions to the rest of the summand which simplifies to

$$1-u+\frac{-1-r+2u-ru-u^2+ru^2}{(1-u)\,(1-u^{1+r})}+\frac{r+ru-ru^2}{(1-u)\,(1-u^{2+r})}.$$

The separated first term with numerator ru^r after partial fractions leads to

$$\frac{ru^r}{(1-u)\,(1-u^{1+r})}-\frac{ru^{r+1}}{(1-u)\,(1-u^{2+r})}.$$

Altogether,

$$\begin{aligned}
\mathrm{Tot}(u)&=\sum_{r=1}^{\infty}\left(1-u+\frac{-1-r+2u-ru-u^2+ru^2}{(1-u)\,(1-u^{1+r})}+\frac{r+ru-ru^2}{(1-u)\,(1-u^{2+r})}\right)\\
&\quad+\sum_{r=1}^{\infty}\frac{ru^r}{(1-u)\,(1-u^{1+r})}-\sum_{r=1}^{\infty}\frac{ru^{r+1}}{(1-u)\,(1-u^{2+r})}.
\end{aligned}$$

To facilitate the evaluation of the infinite sums, we define a new function (where ∞ is replaced temporarily by finite M in $\mathrm{Tot}(u)$), namely:

$$\mathrm{Tot2}(u) := \sum_{r=1}^{M}\left(1-u+\frac{-1-r+2u-ru-u^2+ru^2}{(1-u)\left(1-u^{1+r}\right)}+\frac{r+ru-ru^2}{(1-u)\left(1-u^{2+r}\right)}\right)$$
$$+\sum_{r=1}^{M}\frac{ru^r}{(1-u)\left(1-u^{1+r}\right)}-\sum_{r=1}^{M}\frac{ru^{r+1}}{(1-u)\left(1-u^{2+r}\right)}.$$

We now separate this into disjoint sums and shift the index of summation in the third and last sums:

$$\begin{aligned}
\mathrm{Tot2}(u) &= \sum_{r=1}^{M}(1-u)+\sum_{r=1}^{M}\frac{-1-r+2u-ru-u^2+ru^2}{(1-u)\left(1-u^{1+r}\right)}+\sum_{r=2}^{M+1}\frac{(r-1)\left(1+u-u^2\right)}{(1-u)(1-u^{1+r})}\\
&\quad+\sum_{r=1}^{M}\frac{ru^r}{(1-u)\left(1-u^{1+r}\right)}-\sum_{r=2}^{M+1}\frac{(r-1)u^r}{(1-u)\left(1-u^{1+r}\right)}\\
&=\sum_{r=1}^{M}(1-u)+\frac{-2+u}{(1-u)\left(1-u^2\right)}+\sum_{r=2}^{M}\frac{-1-r+2u-ru-u^2+ru^2}{(1-u)\left(1-u^{1+r}\right)}\\
&\quad+\sum_{r=2}^{M}\frac{(r-1)\left(1+u-u^2\right)}{(1-u)\left(1-u^{1+r}\right)}+\frac{M\left(1+u-u^2\right)}{(1-u)\left(1-u^{2+M}\right)}+\frac{u}{(1-u)\left(1-u^2\right)}\\
&\quad+\sum_{r=2}^{M}\frac{ru^r}{(1-u)\left(1-u^{1+r}\right)}-\sum_{r=2}^{M}\frac{(r-1)u^r}{(1-u)\left(1-u^{1+r}\right)}-\frac{Mu^{1+M}}{(1-u)\left(1-u^{2+M}\right)}.
\end{aligned}\tag{2.6}$$

We combine the terms in the sums from r equals 2 to M in (2.6) to get

$$\frac{-2+u+u^r}{(1-u)\left(1-u^{1+r}\right)}.$$

Then we simplify the rest to get

$$\begin{aligned}
\mathrm{Tot2}(u) &= \sum_{r=2}^{M}\frac{-2+u+u^r}{(1-u)\left(1-u^{1+r}\right)}-\frac{2}{1-u^2}\\
&\quad-\frac{M\left(-2+u+u^{1+M}+u^{2+M}-2u^{3+M}+u^{4+M}\right)}{(1-u)\left(1-u^{2+M}\right)}.
\end{aligned}$$

Note that $\mathrm{Tot2}(u)$ and $\mathrm{Tot}(u)$ match at least for terms up to $\left[u^M\right]$. Since for the present we are only interested in the terms up to $\left[u^M\right]$, we may set all higher power terms equal to zero, to produce

$$\mathrm{Tot2b}(u) = \sum_{r=2}^{M}\frac{-2+u+u^r}{(1-u)\left(1-u^{1+r}\right)}-\frac{2}{1-u^2}+M\frac{(2-u)}{(1-u)}.$$

Noting that $M = 1 + \sum_{r=2}^{M} 1$,

$$\begin{aligned}\mathrm{Tot2b}(u) &= \sum_{r=2}^{M} \frac{-2+u+u^r}{(1-u)\,(1-u^{1+r})} - \frac{2}{1-u^2} + \frac{(2-u)}{(1-u)} + \sum_{r=2}^{M} \frac{(2-u)}{(1-u)} \\ &= \sum_{r=2}^{M} \frac{-2+u+u^r}{(1-u)\,(1-u^{1+r})} + \frac{u}{1+u} + \sum_{r=2}^{M} \frac{(2-u)}{(1-u)}.\end{aligned}$$

Combine the summands in $\sum_{r=2}^{M}$. We may now allow $M \to \infty$ to finally obtain the simplified generating function as per the next theorem:

Theorem 2.3. *The simplified generating function for the total number of left-to-right maxima in Dyck paths is*

$$\mathrm{Tot}(u) = \sum_{r=1}^{\infty} \frac{(1-u)u^r}{1-u^{1+r}}. \tag{2.7}$$

2.1. **Formula for total number of left-to-right maxima.** In this section, we will obtain an exact formula for the total number of left-to-right maxima in terms of a well-known arithmetic function, namely the divisor function $d(r)$. Compare with [5]. Note that

$$\sum_{r=1}^{\infty} \frac{u^r}{1-u^r} = \sum_{r=1}^{\infty} d(r)\, u^r.$$

To read off coefficients from equation (2.7), we observe that for any formal power series $f(z)$

$$[z^{2n}]f(z) = [u^n](1-u)(1+u)^{2n-1} f(z(u)).$$

This can be justified by using formal residue calculus; see for example [12]. Therefore

$$\begin{aligned}[z^{2n}]\,\mathrm{Tot}(z) &= [u^n](1-u)(1+u)^{2n-1} \sum_{r=1}^{\infty} \frac{(1-u)u^r}{1-u^{1+r}} \\ &= [u^n](1-u)(1+u)^{2n-1} \sum_{r=1}^{\infty} (d(r+1)-d(r))u^r \\ &= \sum_{r=1}^{n} (d(r+1)-d(r)) \left(\binom{2n-1}{n-r} - \binom{2n-1}{n-r-1} \right).\end{aligned}$$

Thus we have shown:

Theorem 2.4. *The total number of left-to-right maxima in Dyck paths of semi-length n is given by*

$$\sum_{r=1}^{n} (d(r+1)-d(r)) \left(\binom{2n-1}{n-r} - \binom{2n-1}{n-r-1} \right).$$

3. Asymptotics for strict left-to-right maxima

In this section we find the asymptotic expression for the total number of strict left-to-right maxima in Dyck paths. We will follow the approach used to study the height of planted plane trees by Prodinger in [12]. For related asymptotic calculations concerning the height of trees and lattice paths; see [10, 11, 13] and the seminal article by de Bruijn, Knuth and Rice [5].

First, we extract coefficients of z^n in $\mathrm{Tot}(u)$. That is we find

$$[z^n]\frac{1-u}{u}\sum_{r=2}^{\infty}\frac{u^r}{1-u^r}.$$

When u is in terms of z^2, by (2.4) the function $\mathrm{Tot}(u)$ has its dominant singularity at $z = 1/2$ which is mapped to $u = 1$. To study this further we set $u = e^{-t}$ and let $t \to 0$. Thus

$$\frac{1-u}{u} = e^t(1-e^{-t}) = t + \frac{t^2}{2} + \frac{t^3}{6} + \cdots . \tag{3.1}$$

To estimate the harmonic sum $f_1(t) := \sum_{r=2}^{\infty}\frac{e^{-rt}}{1-e^{-rt}}$ as $t \to 0$, we take the Mellin transform of $f_1(t)$, see [8], which is $f_1^*(s) := \int_0^{\infty} f_1(t)t^{s-1}\,dt$. Thus

$$f_1^*(s) = \Gamma(s)\zeta(s)(\zeta(s)-1), \text{ for } \Re(s) > 1.$$

By using the Mellin inversion formula, we have $f_1(t) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} f_1^*(s)\,t^{-s}\,ds$ (again see [8]). By computing residues this yields

$$f_1(t) \sim \frac{-1+\gamma-\log(t)}{t} + \frac{3}{4} - \frac{13t}{144} + \cdots , \tag{3.2}$$

where γ is Euler's constant.

Let

$$g_1(t) := e^t(1-e^{-t})\,f_1(t).$$

From (3.1) and (3.2)

$$g_1(t) \sim -\log(t) - 1 + \gamma + \left(\frac{3}{4} + \frac{1}{2}(-1+\gamma-\log(t))\right)t + \cdots .$$

Let $y = \sqrt{1-4z^2}$ and writing $e^{-t} = u = \frac{1-y}{1+y}$, we find $t = -\log\frac{1-y}{1+y} = 2y + \frac{2y^3}{3} + \cdots$. In terms of the y variable, we therefore need to compute $g_1(2y + \frac{2y^3}{3} + \cdots)$.

$$g_1\left(2y + \frac{2y^3}{3} + \cdots\right) \sim (-1+\gamma-\log(2)-\log(y)) + \frac{1}{2}(1+2\gamma-2\log(2)-2\log(y))y - \frac{y^2}{3} + \cdots .$$

Replacing y by $\sqrt{1-4z^2}$ gives

$$-1+\gamma-\log(2)-\frac{1}{2}\log\left(1-4z^2\right) + \frac{1}{2}\left(1+2\gamma-2\log(2)-\log\left(1-4z^2\right)\right)\sqrt{1-4z^2} + \cdots .$$

To use singularity analysis, see [8], it is convenient to put $z^2 = x$, then we find the coefficient of x^n in the above expression as $n \to \infty$. It is asymptotically equal to

$$2^{2n}\left(\frac{1}{2n} - \frac{\log(n)}{4\sqrt{\pi}n^{3/2}} + \frac{1-3\gamma}{4n^{3/2}\sqrt{\pi}} + \cdots\right). \tag{3.3}$$

To obtain the mean value we must divide by the total number of Dyck paths of semi-length n, i.e., as $n \to \infty$

$$\frac{1}{n+1}\binom{2n}{n} = 2^{2n}\left(\frac{1}{n^{3/2}\sqrt{\pi}} - \frac{9}{8\ n^{5/2}\sqrt{\pi}} + \frac{145}{128\ n^{7/2}\sqrt{\pi}}\right) + \cdots . \tag{3.4}$$

Hence, dividing (3.3) by (3.4) yields

Theorem 3.1. *The average number of strong left-to-right maxima in Dyck paths of semi-length n, as $n \to \infty$ is*

$$\frac{\sqrt{\pi n}}{2} - \frac{\log(n)}{4} + \frac{1}{4}(1-3\gamma) + O(n^{-1/2}).$$

Remark 3.2. The asymptotic formula of Theorem 3.1 when $n = 200$ yields 11.0257 for the average number of strong left-to-right maxima. Using the exact formula of Theorem 2.4 divided by the Catalan number for $n = 200$ yields 11.0503 which is indeed a very good match.

Remark 3.3. The number of strong left-to-right maxima is bounded above by the height of the path, which is known to be $\sim \sqrt{\pi n}$ as $n \to \infty$, (see, e.g., [12]). We see that asymptotically the average number is half of the height.

4. Weak left-to-right maxima in Dyck paths

For this question we first need a generating function for Dyck paths of height $\le h$ which have only a single return to the x axis. So using the formula above from (2.1), we obtain the generating function for these where $h \ge 1$ as

$$D(h,z) = z^2 A(h-1).$$

Now in order to construct the generating function $E(h,x,z)$ for the number of times a Dyck path of height $\le h$ and length n tracked by z, returns to 0 where the latter is tracked by a variable x in the generating function, we construct a sequence of such Dyck paths where each term in the generating function for this sequence is multiplied by x. Thus we obtain

$$E(h,x,z) = \frac{1}{1 - xD(h,z)}.$$

We now reiterate the construction in Theorem 2.1 to obtain the following theorem.

Theorem 4.1. *The generating function for the number of weak left-to-right maxima, tracked by x, for Dyck paths of maximum height r and length tracked by z is*

$$F(x,z,r) := z^{r+1}xC(r-1)\prod_{h=1}^{r} E(h,x,z). \tag{4.1}$$

To obtain the generating function for the total number of weak left-to-right maxima, we once again differentiate (4.1) with respect to x and evaluate this at $x = 1$. We obtain

Theorem 4.2. *The generating function for the total number of weak left-to-right maxima for Dyck paths of length n tracked by z is*

$$\mathrm{WTot}(u) := \sum_{r=1}^{\infty} \frac{(1-u)u^r\left(1-u^2\right)}{\left(1-u^{1+r}\right)\left(1-u^{2+r}\right)}\left(1-r+(1+u)\sum_{i=1}^{r}\frac{1-u^{1+i}}{1-u^{2+i}}\right),$$

where $z^2 = \frac{u}{(1+u)^2}$.

Proof. The derivative of (4.1) is

$$\frac{\partial}{\partial x}F(x,z,r)\bigg|_{x=1} = z^{r+1}C(r-1)\prod_{h=1}^{r}E(h,1,z)\left(1+\sum_{i=1}^{r}\frac{D(i,z)}{1-D(i,z)}\right).$$

Putting $z^2 = \frac{u}{(1+u)^2}$ in the formula above we obtain

$$z^{r+1}C(r-1)\prod_{h=1}^{r}\frac{1}{1-z^2A(h-1)} = \frac{(1-u)u^r\left(1-u^2\right)}{\left(1-u^{1+r}\right)\left(1-u^{2+r}\right)},$$

while the remaining bracketed part becomes

$$1-r+(1+u)\sum_{i=1}^{r}\frac{1-u^{1+i}}{1-u^{2+i}}.$$

□

Now, we simplify Theorem 4.2. The double sum becomes

$$\left(1-u^2\right)^2\sum_{i=1}^{\infty}\frac{1-u^{1+i}}{1-u^{2+i}}\sum_{r=i}^{\infty}\frac{u^r}{\left(1-u^{1+r}\right)\left(1-u^{2+r}\right)}.$$

We use partial fractions on the r-sum and then the double sum telescopes to

$$\frac{\left(1-u^2\right)^2}{(1-u)u}\sum_{i=1}^{\infty}\frac{u^{i+1}}{1-u^{i+2}}.$$

This is then combined with the single sum which simplifies to

$$\sum_{r=1}^{\infty}\left(\frac{(1-u)u^r\left(1-u^2\right)(1-r)}{\left(1-u^{1+r}\right)\left(1-u^{2+r}\right)}+\frac{\left(1-u^2\right)^2u^{1+r}}{(1-u)u\left(1-u^{2+r}\right)}\right). \tag{4.2}$$

In order to further simplify (4.2) we replace ∞ by finite M and then apply partial fractions to the summand of the first term which splits up as

$$\frac{(-1+r)(1-u)(1+u)}{u\left(1-u^{1+r}\right)}-\frac{(-1+r+1)(1-u)(1+u)}{u\left(1-u^{2+r}\right)}-\frac{(1-u)(1+u)}{u\left(1-u^{2+r}\right)}.$$

This is telescoping and simplifies to

$$-\frac{(-1+M+1)(1-u)(1+u)}{u\left(1-u^{2+M}\right)}+\sum_{r=1}^{M}\left(\frac{\left(1-u^2\right)^2u^{1+r}}{(1-u)u\left(1-u^{2+r}\right)}-\frac{(1-u)(1+u)}{u\left(1-u^{2+r}\right)}\right).$$

Now, replace M by $\sum_{r=1}^{M}1$. Then, letting M tend to ∞, and finally combining all summands, we obtain

Theorem 4.3. *The simplified generating function for the total number of weak left-to-right maxima for Dyck paths of length n tracked by z is*

$$\mathrm{WTot}(u) = \sum_{r=1}^{\infty} \frac{(1-u^2)\,u^r}{1-u^{2+r}}.$$

This has series expansion

$$z^2+3z^4+9z^6+\mathbf{29z^8}+98z^{10}+341z^{12}+1210z^{14}+4356z^{16}+15860z^{18}+58276z^{20}+O\left(z^{21}\right).$$

This is illustrated in Figure 2, where the dots and circles mark all 29 of the weak left-to-right maxima in Dyck paths of length 8.

4.1. **Formula for total number of weak left-to-right maxima.** In this section, we again obtain an exact formula for the total number of left-to-right maxima in terms of the divisor function $d(r)$. To read off coefficients from Theorem 4.3, as before

$$[z^{2n}]f(z) = [u^n](1-u)(1+u)^{2n-1}f(z(u)).$$

Therefore

$$\begin{aligned}[z^{2n}]\,\mathrm{WTot}(z) &= [u^n](1-u)(1+u)^{2n-1}\sum_{r=1}^{\infty}\frac{(1-u^2)u^r}{1-u^{2+r}}\\ &= [u^n](1-u)(1+u)^{2n-1}\sum_{r=1}^{\infty}(d(r+2)-d(r))u^r.\end{aligned}$$

We thus get the following theorem.

Theorem 4.4. *The total number of weak left-to-right maxima in Dyck paths of semi-length n is given by*

$$\sum_{r=1}^{n}(d(r+2)-d(r))\left(\binom{2n-1}{n-r}-\binom{2n-1}{n-r-1}\right).$$

5. Asymptotics for weak left-to-right maxima

To find an asymptotic expression for $\mathrm{WTot}(u)$, we reiterate the approach in Section 3. This yields

Theorem 5.1. *The average number of weak left-to-right maxima in Dyck paths of semi-length n, as $n \to \infty$ is*

$$\sqrt{\pi n} - \log(n) + \frac{1}{2}(5-6\gamma) + O(n^{-1/2}).$$

Remark 5.2. The asymptotic formula of Theorem 5.1 when $n = 200$ yields 20.536 for the average number of weak left-to-right maxima. Using the exact formula of Theorem 4.4 divided by the Catalan number for $n = 200$ yields 20.368. Taking larger n improves the accuracy.

6. Open problems

Theorem 2.4 and Theorem 4.4 are very similar to each other with only a slight change in their respective summands; this suggests that there is an underlying combinatorial proof. Also from Theorems 2.3 and 4.3 we obtain

$$\mathrm{WTot}(u) = \frac{1+u}{u}\,\mathrm{Tot}(u) - 1.$$

We think that accounting for these similarities may be an interesting combinatorial problem which we leave to the reader. It might also be possible to derive Theorem 2.3 in a simpler and more direct way instead of using Proposition 2.2.

The statistic left-to-right maximum, 'lrmax' is quite important in permutations due to the fact that it is equidistributed with the 'cycle' statistic and is counted nicely by Stirling numbers of the first kind. One of the main reasons for studying lrmax in permutations is this equidistribution with the number of cycles. Hence in the current setting one should anticipate a counterpart statistic in Dyck path to be equidistributed with lrmax.

With respect to permutations, there are other statistics equidistributed with lrmax. In the Dyck path context one can also introduce such concepts. This may potentially lead to further interesting research.

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