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INTRODUCING DASEP: THE DOUBLY ASYMMETRIC SIMPLE EXCLUSION PROCESS

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Abstract. Research in combinatorics has often explored the asymmetric simple exclusion process (ASEP). The ASEP, inspired by examples from statistical mechanics, involves particles of various species moving around a lattice. With the traditional ASEP particles of a given species can move but do not change species. In this paper a new combinatorial formalism, the DASEP (doubly asymmetric simple exclusion process), is explored. The DASEP is inspired by biological processes where, unlike the ASEP, the particles can change from one species to another. The combinatorics of the DASEP on a one dimensional lattice are explored, including the associated generating function. The stationary probabilities of the DASEP are explored, and results are proven relating these stationary probabilities to those of the simpler ASEP.

Keywords: ASEP, DASEP, lattice, algebraic combinatorics, steady state probabilities, species, lattice paths.

1. Introduction

The ASEP (asymmetric simple exclusion process) is a structure that has frequently been referred to in the combinatorics literature. In its simplest form, the ASEP consists of a one dimensional infinite lattice, with each point on the lattice being populated with either a particle or a hole. At random intervals, each particle attempts to move either to the left or the right with different but fixed probabilities (hence the term 'asymmetric'). The ASEP can be thought of as a form of Markov process as noted in [4] by Corteel et al. Multiline queues [5] were introduced by Ferrari et al. as a combinatorial approach to the analysis of the ASEP. Originally the ASEP particles were thought of as all belonging to a single species. More recent work by Cantini et al. [3] generalized the concept to multiple species and uncovered a link with Macdonald polynomials. Although we focus on the homogeneous ASEP (transition probabilities do not depend on position in the lattice), several researchers (Lam et al. [7], Ayyer et al. [1], Cantini [2], Mandelshtam [9], and Kim et al. [6]) have explored the inhomogeneous ASEP in which transition probabilities do depend on lattice position.

2. Definitions

Following [4], a partition can be defined as follows:

Definition 2.1. A partition λ is a nonincreasing sequence of n nonnegative integers $\lambda = (\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n \geq 0)$.

We will start by working through a simple example of the ASEP before introducing the new concept of the DASEP. We will ordinarily write a partition as defined above as an n-tuple: $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$.

Definition 2.2. We write $S_n(\lambda)$ to mean the set of all permutations of λ.

Example 2.3. So, for $\lambda = (2,2,1)$, $S_3(\lambda) = \{(2,2,1),(2,1,2),(1,2,2)\}$.

The multispecies asymmetric simple exclusion process ASEP(λ) is then defined to be a Markov process on $S_n(\lambda)$ with certain specific transition probabilities:

Definition 2.4. For all partitions λ as defined in Definition 2.1, ASEP(λ) is a Markov process on $S_n(\lambda)$. We let t be a constant with $0 \le t \le 1$. The transition probability $P_{\mu,\nu}$ between two permutations $\mu \in S_n(\lambda)$ and $\nu \in S_n(\lambda)$ is given by:

- If $\mu = (\mu_1, \ldots, \mu_k, i, j, \mu_{k+2}, \ldots, \mu_n)$ and $\nu = (\mu_1, \ldots, \mu_k, j, i, \mu_{k+2}, \ldots, \mu_n)$, with $i \neq j$, then $P_{\mu,\nu} = \frac{t}{n}$ if $i > j$ and $P_{\mu,\nu} = \frac{1}{n}$ if $j > i$.
- If $\mu = (i, \mu_2, \mu_3, \ldots, \mu_{n-1}, j)$ and $\nu = (j, \mu_2, \mu_3, \ldots, \mu_{n-1}, i)$ with $i \neq j$, then $P_{\mu,\nu} = \frac{t}{n}$ if $j > i$ and $P_{\mu,\nu} = \frac{1}{n}$ if $i > j$.
- If neither of the above conditions apply but $\nu \neq \mu$ then $P_{\mu,\nu} = 0$. If $\nu = \mu$ then $P_{\mu,\mu} = 1 - \sum_{\nu \neq \mu} P_{\mu,\nu}$.

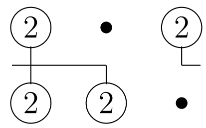
It is possible to compute steady state probabilities for ASEP(λ). For the purposes of the example that we will develop as we introduce DASEP, we are primarily interested in ASEP(λ) for $\lambda = (2,2,0)$, $\lambda = (2,1,0)$, and $\lambda = (1,1,0)$, so we will focus mostly on these three processes as we work through the computation of the steady state probabilities. Continuing to follow [4] as we develop this example, to compute these probabilities we need to define the concept of a multiline queue.

Definition 2.5. A ball system B is an $L \times n$ matrix each element of which is either 0 or 1. Moreover for all i the number of 1's in row $i+1$ is less than or equal to the number of 1's in row i.

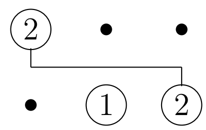
Definition 2.6. Given a ball system B a multiline queue Q is obtained by augmenting B with a labeling and matching system. Each cell in B will be labelled with a number from 0 to L inclusive, and each cell with a 1 element in row $i+1$, for $i \ge 1$, will be matched to a cell with a 1 element in row i. Such a matching must be obtained through an application of the following algorithm:

- Step 1: Find the highest numbered row with unlabelled 1 elements. Label each of those elements with the number of the row. If this is row 1, or there are no remaining unlabelled 1 elements in the matrix, exit.
- Step 2: Find the row with labelled but unmatched elements. If this is row 1, go back to step 1. If it is row $i+1$, for $i \ge 1$, first match each labelled but unmatched element that can be matched to an unlabelled element directly below it to that element. This is considered a *trivial* match. Then proceed from right to left (highest to lowest numbered columns) matching each remaining labelled but unmatched element to an unlabelled element in the row below–these are the *nontrivial* matches. Give all newly matched elements in row i the same label as the element it has just been matched to. Repeat step 2.

A multiline queue is often visualized as a ball system with an element with a 1 value being shown as a ball and a 0 value by the absence of a ball. Matches between elements (balls) are drawn by lines between the matched balls. The following shows a multiline queue associated with ASEP(λ) where $\lambda = (2, 2, 0)$. Note that the line matching the ball at upper right to the one at the lower middle wraps around to the right.



The labels in the bottom row determine the partition of the associated ASEP. The above multiline queue has $\lambda = (2, 2, 0)$ since the bottom row includes two 2's and a 0–by convention an element without a ball is assumed to be labeled with a 0. Likewise the following would be a multiline queue with $\lambda = (2, 1, 0)$:



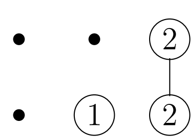
Each multiline queue is also associated with a permutation $\alpha \in S_n(\lambda)$ corresponding to the labels of its bottom row in *unsorted* order. For example, for the above multiline queue, $\lambda = (2, 1, 0)$ but $\alpha = (0, 1, 2)$. We will write $\lambda(Q) = \lambda$ and $\alpha(Q) = \alpha$.

3. Steady state probabilities with example

To determine steady state probabilities–and continue with the example started in the introduction–we next assign to each nontrivial matching p in Q two values $f(p)$ and $s(p)$. $f(p)$ is the number of choices that were available for the match when the match was made. $s(p)$ is the number of legal matches that were skipped, if we imagine ourselves considering possible matches from left to right and wrapping around the end if needed, before the actual choice was made. We can then define a weight on p as $\mathrm{wt}(p) = \frac{(1-t)t^{s(p)}}{1-t^{f(p)}}$. Here we are proceeding from [4] but with the simplifying assumption that $q = 1$, since in the sequel we will rely on a theorem that requires $q = 1$. Next we can define a weight on the entire multiline queue $\mathrm{wt}(Q) = \prod_{p \in Q} \mathrm{wt}(p)$ where the product is taken over all *nontrivial* matches p in Q. A theorem due to Martin [10] then gives the required steady state probabilities:

$$\Pr(\alpha) = \frac{\sum_{\alpha(Q)=\alpha} \mathrm{wt}(Q)}{\sum_{\lambda(Q)=\lambda} \mathrm{wt}(Q)}.$$

Before moving on to the DASEP, we need to evaluate the steady state probabilities for the examples that we will ultimately use to develop the DASEP. For the above multiline queue, there is exactly one nontrivial pair p. When this pair is matched, there are two available options so $f(p) = 2$. As we picked the second available option, $s(p) = 1$. So $\mathrm{wt}(Q) = \frac{(1-t)t}{1-t^2}$. As noted above, $\alpha = (0, 1, 2)$ and the only other multiline queue with $\alpha = (0, 1, 2)$ is as follows:



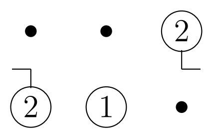
Here there is no nontrivial matching pair, so $\mathrm{wt}(Q) = 1$. Hence:

$$\sum_{\alpha(Q)=(0,1,2)} \mathrm{wt}(Q) = 1 + \frac{(1-t)t}{1-t^2} = \frac{1+2t}{1+t}.$$

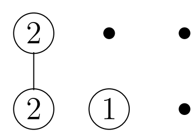
For reasons of symmetry:

$$\sum_{\alpha(Q)=(0,1,2)} \mathrm{wt}(Q) = \sum_{\alpha(Q)=(1,2,0)} \mathrm{wt}(Q) = \sum_{\alpha(Q)=(2,0,1)} \mathrm{wt}(Q) = \frac{1+2t}{1+t}.$$

Next we look at $\alpha = (0,1,2)$, for which there are also two multiline queues. The first of these is as follows:



Here there is one nontrivial matching pair p. When this pair is matched, there are two available options so $f(p) = 2$. As we picked the first available option, $s(p) = 0$. So $\mathrm{wt}(Q) = \frac{1-t}{1-t^2}$. The other multiline queue with $\alpha = (2,1,0)$ is as follows:



Again there is no nontrivial matching pair, so $\mathrm{wt}(Q) = 1$. Hence:

$$\sum_{\alpha(Q)=(2,1,0)} \mathrm{wt}(Q) = 1 + \frac{1-t}{1-t^2} = \frac{2+t}{1+t}.$$

For reasons of symmetry, one has

$$\sum_{\alpha(Q)=(2,1,0)} \mathrm{wt}(Q) = \sum_{\alpha(Q)=(1,0,2)} \mathrm{wt}(Q) = \sum_{\alpha(Q)=(0,2,1)} \mathrm{wt}(Q) = \frac{2+t}{1+t}.$$

So we get

$$\sum_{\lambda(Q)=(2,1,0)} \mathrm{wt}(Q) = 3(\frac{1+2t}{1+t}) + 3(\frac{2+t}{1+t}) = 9.$$

We are now ready to give the steady state probabilities

$$\Pr(0,1,2) = \Pr(1,2,0) = \Pr(2,0,1) = \frac{1+2t}{9(1+t)}$$

and

$$\Pr(2,1,0) = \Pr(1,0,2) = \Pr(0,2,1) = \frac{2+t}{9(1+t)}.$$

Trivial computations also give

$$\Pr(0,1,1) = \Pr(1,1,0) = \Pr(1,0,1) = \frac{1}{3}$$

and

$$\Pr(0,2,2) = \Pr(2,2,0) = \Pr(2,0,2) = \frac{1}{3}.$$

This concludes our computation for the steady state probabilities of this model; in the next section we introduce the DASEP model.

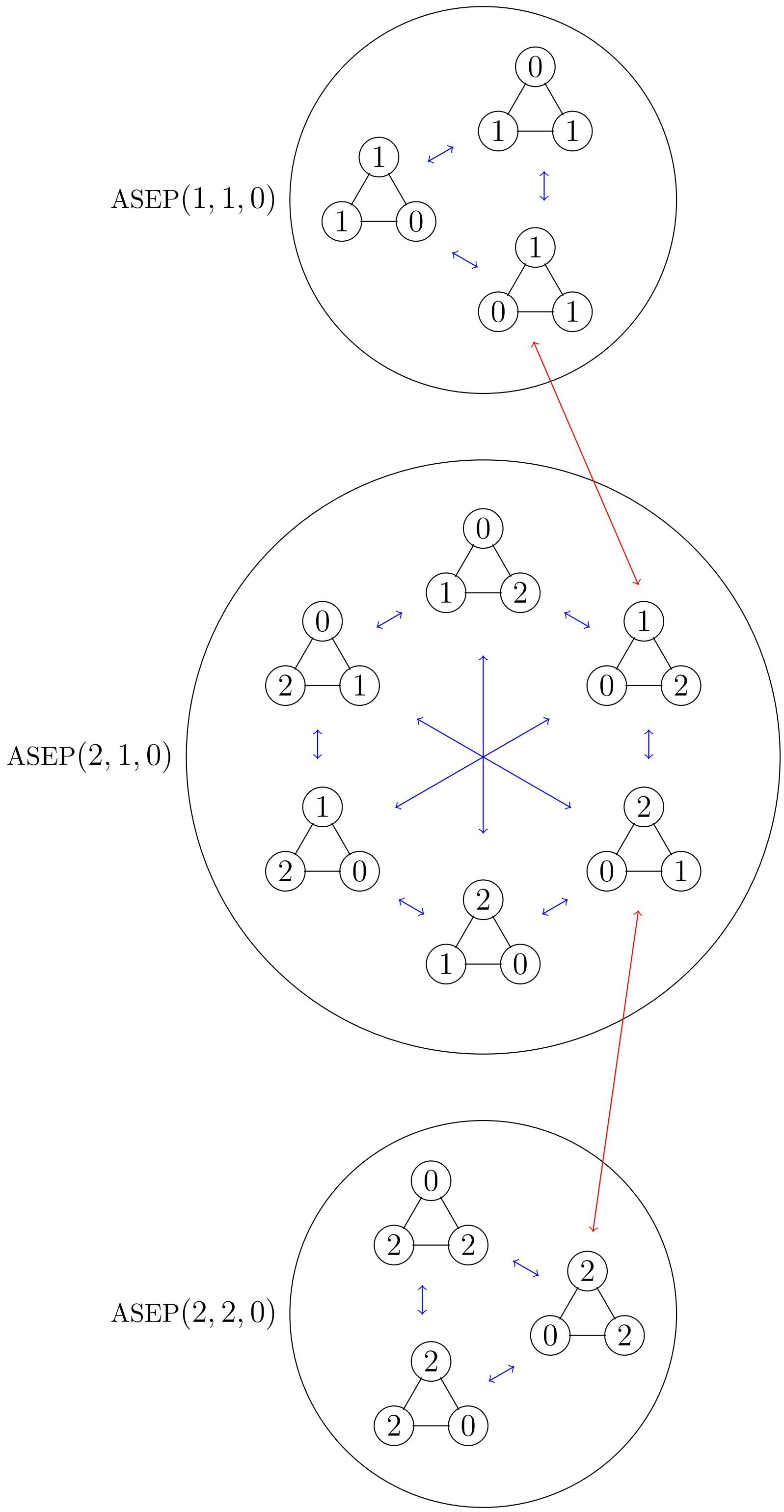


FIGURE 1. An example of DASEP(n, p, q): DASEP(3,2,2).

Each of the 12 small triangles represents a state of the DASEP in the circular lattice with $n = 3$ sites with $q = 2$ balls (i.e., the nonzero labels), each nonzero label is $\leq p = 2$ (i.e., one has $p = 2$ species). Each state corresponds to a permutation of the partition $(2, 2, 0)$, $(2, 1, 0)$, or $(1, 1, 0)$. The transitions are explained in Definition 4.1 hereafter.

4. Doubly asymmetric simple exclusion process

We are now ready to introduce the DASEP (doubly asymmetric simple exclusion process). While the ASEP is inspired by statistical mechanics where particles do not change species, the DASEP, by contrast, is inspired by biological processes where particles can change species, which we denote by DASEP(n,p,q) where n is the number of positions on the lattice, p is the number of types of species, and q is the number of particles.

Definition 4.1. For all positive integers n, p, and q with $n > q$, DASEP(n,p,q) is a Markov process on the set $\bigcup_{\lambda_1 \leq p,\ \lambda'_1 = q} S_n(\lambda)$, where one uses the notation of Definition 2.2, and where $\lambda'_1 = q$ refers to the dual partition [8] of λ, namely λ', and uses the fact that λ'_1 gives the number of nonzero terms in the original partition λ. The transition probability $P_{\mu,\nu}$ on two permutations μ and ν is as follows:

- If $\mu = (\mu_1, \ldots, \mu_k, i, j, \mu_{k+2}, \ldots, \mu_n)$ and $\nu = (\mu_1, \mu_2, ..., \mu_k, j, i, \mu_{k+2}, \ldots, \mu_n)$ with $i \neq j$, then $P_{\mu,\nu} = \frac{t}{3n}$ if $i > j$ and $P_{\mu,\nu} = \frac{1}{3n}$ if $j > i$.
- If $\mu = (i, \mu_2, \ldots, \mu_{n-1}, j)$ and $\nu = (j, \mu_2, \mu_3, ..., \mu_{n-1}, i)$ with $i \neq j$, then $P_{\mu,\nu} = \frac{t}{3n}$ if $j > i$ and $P_{\mu,\nu} = \frac{1}{3n}$ if $i > j$.
- If $\mu = (\mu_1, \ldots, \mu_k, i, \mu_{k+2}, \ldots, \mu_n)$ and $\nu = (\mu_1, \ldots, \mu_k, i+1, \mu_{k+2}, \ldots, \mu_n)$ with $i \geq 1$, then $P_{\mu,\nu} = \frac{u}{3n}$.
- If $\mu = (\mu_1, \ldots, \mu_k, i+1, \mu_{k+2}, \ldots, \mu_n)$ and $\nu = (\mu_1, \ldots, \mu_k, i, \mu_{k+2}, \ldots, \mu_n)$ with $i \geq 1$, then $P_{\mu,\nu} = \frac{1}{3n}$.
- If none of the above conditions apply but $\nu \neq \mu$ then $P_{\mu,\nu} = 0$. If $\nu = \mu$ then $P_{\mu,\mu} = 1 - \sum_{\nu \neq \mu} P_{\mu,\nu}$.

Figure 1 shows the simple example of the DASEP that we are working through. All possible transitions within a single ASEP (the first and second bullet points in the definition above) are shown with blue arrows on this diagram. To keep the diagram relatively clean in appearance, only selected transitions between different ASEPs (the third and fourth bullet points) are shown (with red arrows). Other ASEPs such as ASEP$(1,0,0)$ or ASEP$(2,0,0)$ are not shown since these are not part of DASEP$(3,2,2)$. This is because, per Definition 4.1, for DASEP$(3,2,2)$ we always have $\lambda'_1 = 2$, whereas for ASEP$(1,0,0)$ and ASEP$(2,0,0)$, we would have $\lambda'_1 = 1$.

Similar to with the ASEP, with the DASEP we wish to compute steady state probabilities for permutations α which we will call $\mathrm{Pd}(\alpha)$. We will focus on continuing to develop the example we have been working on which turns out to be DASEP$(3,2,2)$. Here $n = 3$ means that the particles move on the circular lattice with 3 sites, $p = 2$ means that each particle is allowed to take on the value 0, 1, or 2, and $q = 2$ means that each permutation α has exactly 2 nonzero values. We therefore find ourselves interested in the following 12 steady state probabilities:

$$\mathrm{Pd}(0,1,1), \mathrm{Pd}(0,1,2), \mathrm{Pd}(0,2,1), \mathrm{Pd}(0,2,2), \mathrm{Pd}(1,0,1), \mathrm{Pd}(1,0,2),$$
$$\mathrm{Pd}(1,1,0), \mathrm{Pd}(1,2,0), \mathrm{Pd}(2,0,1), \mathrm{Pd}(2,0,2), \mathrm{Pd}(2,1,0), \mathrm{Pd}(2,2,0).$$

Note here that particles in the DASEP are allowed to switch back and forth between species 1 and 2, but not back and forth from 0 to anything else. That is because a value of 0 is understood to not so much be a species but the absence of a species. Due to symmetries we can now focus on solving for the following four probabilities:

$$w = \mathrm{Pd}(0,1,1), \quad x = \mathrm{Pd}(0,1,2), \quad y = \mathrm{Pd}(0,2,1), \quad z = \mathrm{Pd}(0,2,2).$$

From the above transition probabilities, this reduces to solving the system

$$\begin{cases} 2uw = x + y \\ (2+t)x + x + ux = (1+2t)y + z + uw \\ (1+2t)y + y + uy = (2+t)x + uw + z \\ 2z = u(x+y) \end{cases}$$

which in turn implies the relation

$$(5 + 2t + u)x = (3 + 4t + u)y.$$

We can then ask ourselves the question of when the proportions of steady state probabilities for the DASEP are the same as for the previous ASEP. Noting that $\mathrm{Pr}(0,1,2) = \frac{1+2t}{9(1+t)}$ and $\mathrm{Pr}(2,1,0) = \frac{2+t}{9(1+t)}$ such equality will happen if $(5+2t+u)(1+2t) = (3+4t+u)(2+t)$, or $5 + 2t + u + 10t + 4t^2 + 2tu = 6 + 8t + 2u + 3t + 4t^2 + tu$, or $t(1+u) = 1+u$. So this will happen if and only if $t = 1$. We have therefore proven the following proposition.

Proposition 4.2. *If $D = \mathrm{DASEP}(3,2,2)$ is parameterized as described above by t and u, then the following two statements are equivalent:*

- *$t = 1$.*
- *For all partitions λ with $S_n(\lambda) \subseteq D$ and all permutations μ, $\nu \in S_n(\lambda)$, the following equality holds: $\frac{\mathrm{Pr}(\mu)}{\mathrm{Pr}(\nu)} = \frac{\mathrm{Pd}(\mu)}{\mathrm{Pd}(\nu)}$. That is, the ratio between steady state probabilities does not change in moving from the ASEP to the DASEP.*

In fact, we conjecture the following more general statement.

Conjecture 4.3. *If $D = \mathrm{DASEP}(n,p,q)$ is parameterized as described above by t and u, then the following two statements are equivalent:*

- *$t = 1$.*
- *For all partitions λ with $S_n(\lambda) \subseteq D$ and all permutations μ, $\nu \in S_n(\lambda)$, the following equality holds: $\frac{\mathrm{Pr}(\mu)}{\mathrm{Pr}(\nu)} = \frac{\mathrm{Pd}(\mu)}{\mathrm{Pd}(\nu)}$. That is, the ratio between steady state probabilities does not change in moving from the ASEP to the DASEP.*

Partial proof. We will prove this only in the $\Longrightarrow$ direction. If $t = 1$ we can replace λ with a similar partition but with species of the same type being replaced by similar distinct species. For example, if $\lambda = (3,3,3,2,1,0,...)$ we would map this to $\hat{\lambda} = (3_1, 3_2, 3_3, 2, 1, 0, ...)$ and allow adjacent species originally of the same type to be exchanged with the same transition probability. This will create a completely symmetric situation, so all steady state probabilities are equal. As an equal number of $\hat{\lambda}$'s can be derived from each λ this means all original steady state probabilities are equal as well, so $\frac{\mathrm{Pr}(\mu)}{\mathrm{Pr}(\nu)} = \frac{\mathrm{Pd}(\mu)}{\mathrm{Pd}(\nu)} = 1$. This completes the proof in the $\Longrightarrow$ direction. $\square$

Let us motivate Conjecture 4.3 by showing that it holds on one example. Following are the nine values (of the nine steady state probabilities) we must solve for to prove this conjecture for DASEP$(3,3,2)$:

$$\begin{aligned} a_1 &= \mathrm{Pd}(0,1,1), & a_2 &= \mathrm{Pd}(0,2,2), & a_3 &= \mathrm{Pd}(0,3,3),\\ b_1 &= \mathrm{Pd}(0,2,3), & b_2 &= \mathrm{Pd}(0,1,3), & b_3 &= \mathrm{Pd}(0,1,2),\\ c_1 &= \mathrm{Pd}(0,3,2), & c_2 &= \mathrm{Pd}(0,3,1), & c_3 &= \mathrm{Pd}(0,2,1). \end{aligned}$$

These values can be obtained by solving the following set of nine equations:

$$\begin{cases} 2ua_1 = b_3 + c_3 \\ (2+t)b_3 + ub_3 + ub_3 + b_3 = (1+2t)c_3 + b_2 + a_2 + ua_1 \\ (1+2t)c_3 + uc_3 + uc_3 + c_3 = (2+t)b_3 + c_2 + a_2 + ua_1 \\ a_2 + a_2 + ua_2 + ua_2 = b_1 + c_1 + ub_3 + uc_3 \\ (2+t)b_2 + ub_2 + b_2 = (1+2t)c_2 + b_1 + ub_3 \\ (1+2t)c_2 + uc_2 + c_2 = (2+t)b_2 + c_1 + uc_3 \\ 2a_3 = ub_1 + uc_1 \\ (2+t)b_1 + ub_1 + b_1 + b_1 = (1+2t)c_1 + a_3 + ub_2 + ua_2 \\ (1+2t)c_1 + uc_1 + c_1 + c_1 = (2+t)b_1 + a_3 + ua_2 + uc_2. \end{cases}$$

Without working through all the details, this can be solved to give

$$\begin{aligned} &(4u^3 + 36u^2t + 90ut^2 + 72t^3 + 32u^2 + 206ut + 270t^2 + 108u + 322t + 120)c_3 \\ &= (4u^3 + 24u^2t + 54ut^2 + 36t^3 + 44u^2 + 190ut + 198t^2 + 160u + 350t + 200)b_3. \end{aligned}$$

As previously discussed, $\Pr(0,1,2) = \frac{1+2t}{9(1+t)}$ and $\Pr(2,1,0) = \frac{2+t}{9(1+t)}$, so for $b_3 = \mathrm{Pd}(0,1,2)$ and $c_3 = \mathrm{Pd}(2,1,0)$ to be in the same ratio we would require $b_3 = k(1+2t)$ and $c_3 = k(2+t)$ for some k. It follows, after also dividing through by 2, that

$$\begin{aligned} &(2u^3 + 18u^2t + 45ut^2 + 36t^3 + 16u^2 + 103ut + 135t^2 + 54u + 161t + 60)(t+2) \\ &= (2u^3 + 12u^2t + 27ut^2 + 18t^3 + 22u^2 + 95ut + 99t^2 + 80u + 175t + 100)(2t+1). \end{aligned}$$

This can be expanded to

$$\begin{aligned} &2u^3t + 18u^2t^2 + 45ut^3 + 36t^4 + 4u^3 + 52u^2t + 193ut^2 \\ &\; + 207t^3 + 32u^2 + 260ut + 431t^2 + 108u + 382t + 120 \\ = \; &4u^3t + 24u^2t^2 + 54ut^3 + 36t^4 + 2u^3 + 56u^2t + \\ &217ut^2 + 216t^3 + 22u^2 + 255ut + 449t^2 + 80u + 375t + 100. \end{aligned}$$

This can be reduced to

$$2u^3t + 6u^2t^2 + 9ut^3 - 2u^3 + 4u^2t + 24ut^2 + 9t^3 - 10u^2 - 5ut + 18t^2 - 28u - 7t - 20 = 0.$$

This can be factored as

$$(t-1)(2u^3 + 6u^2t + 9ut^2 + 10u^2 + 33ut + 9t^2 + 28u + 27t + 20) = 0.$$

Since $u \geq 0$ and $t \geq 0$, it follows that $t = 1$. This completes the proof in the $\Longleftarrow$ direction for the DASEP$(3,3,2)$ case.

5. Proof of the conjecture for DASEP$(3, p, 2)$

It would be an endless game to prove the conjecture "case by case", with more and more cumbersome computations, so let us now prove it for an infinite family of models. More precisely, we now prove Conjecture 4.3 for DASEP$(3, p, 2)$ (our previous examples covered the cases $p = 2$ and $p = 3$). To solve this case we essentially need to solve for each of p^2 prior probabilities $p_{i,j} = \mathrm{Pd}(0, i, j)$ for $1 \leq i, j \leq p$. The steady state probabilities can be obtained by solving a set of p^2 linear equations each of which essentially demands equilibrium for each of the possible states of the process. The generic form of such an equation, for $i < j$, is given by

$$(4 + t + 2u)p_{i,j} = (1 + 2t)p_{j,i} + p_{i+1,j} + p_{i,j+1} + up_{i-1,j} + up_{i,j-1}. \tag{5.1}$$

For $i > j$ the equation is

$$(3 + 2t + 2u)p_{i,j} = (2 + t)p_{j,i} + p_{i+1,j} + p_{i,j+1} + up_{i-1,j} + up_{i,j-1}$$

For $i = j$ the equation simplifies to

$$(2 + 2u)p_{i,i} = p_{i+1,i} + p_{i,i+1} + up_{i-1,i} + up_{i,i-1}.$$

The equation may be similarly simplified for other edge cases such as $i = 1 < j$, $i < j = p$, $i = 1 < j = p$, $i > j = 1$, $i = p > j$, $i = p > j = 1$, $i = j = 1$, and $i = j = p$. For the sake of brevity we do not list all such cases in detail.

From the first above equation we can define a polynomial $A_{i,j}$ by gathering all terms on the left:

$$A_{i,j} := (4 + t + 2u)p_{i,j} - (1 + 2t)p_{j,i} - p_{i+1,j} - p_{i,j+1} - up_{i-1,j} - up_{i,j-1}.$$

We can similarly define $A_{i,j}$ under the conditions stated for the various edge cases. We next define a $p^2 \times p^2$ matrix B as follows:

$$B_{p(i_1-1)+j_1, p(i_2-1)+j_2} = [p_{i_1,j_1}]A_{i_2,j_2}.$$

The next step is to prove that the rank of B is $p^2 - 1$. To see this, we first observe that the sum of all rows of B is identically zero, meaning that the rank cannot be p^2. For the rank to then be $p^2 - 1$, we would then need to show that no nontrivial linear combination of a proper subset of the rows can be zero. If we let row i, j be $R_{i,j}$ and for some coefficients $c_{i,j}$ we have $\sum_{i,j} c_{i,j} R_{i,j} = 0$, then we need to show that if any $c_{i,j} = 0$, then all $c_{i,j} = 0$. The only rows with a t term in column i, j will be $R_{i,j}$ and $R_{j,i}$. Hence if $c_{i,j} = 0$, it follows that $c_{j,i}$=0.

We next show that if $c_{i,j} = 0$ it follows that $c_{i-1,j-1} = 0$. We can do this by first showing that $c_{i-1,j}$ and $c_{i,j-1}$ must be negations of one another. The only rows with a u term in column i, j will be $R_{i,j}$, $R_{i-1,j}$, and $R_{i,j-1}$, with the latter two having the same coefficient. Hence the following two statements are equivalent: $c_{i,j} = 0$ and $c_{i-1,j} + c_{i,j-1} = 0$. We can similarly show that $c_{i,j} = 0$ and $c_{i+1,j} + c_{i,j+1} = 0$ are equivalent. So from $c_{i,j} = 0$ we can derive $c_{i-1,j-1} = 0$. By repeated application of the same argument we will get $c_{k,1} = 0$ or $c_{1,k} = 0$ for some k.

Likewise, using the equations for the edge cases $i = 1 < j$ and $i > j = 1$, the only rows with a u term in column $1, k$ will be $R_{1,k}$ and $R_{1,k-1}$ and the only rows with a u term in column $k, 1$ will be $R_{k,1}$ and $R_{k-1,1}$. So from $c_{k,1} = 0$ we can derive $c_{k-1,1} = 0$ and from $c_{1,k} = 0$ we can derive $c_{1,k-1} = 0$. By repeated application of this we will get to $c_{1,1} = 0$. By reversing the above arguments it follows that $c_{i,j} = 0$ for any i, j and we have proven:

Lemma 5.1. *The rank of the matrix B as defined above is $p^2 - 1$.*

We next prove a result about the values of the $p_{i,j}$.

Proposition 5.2. *One has*

$$p_{i,j} + p_{j,i} = \frac{2u^{i+j-2}}{\left(\sum_{k=0}^{n-1} u^k\right)^2} \qquad \text{and} \qquad p_{i,i} = \frac{u^{2i-2}}{\left(\sum_{k=0}^{n-1} u^k\right)^2}.$$

Proof. This can be proven by eliminating the variable t from the set of linear equations above. For example, if we add the equations for $i < j$ and $j < i$ we get the following:

$$\begin{aligned} &(4 + t + 2u)p_{i,j} + (3 + 2t + 2u)p_{j,i} \\ &= (2 + t)p_{i,j} + (1 + 2t)p_{j,i} + p_{i+1,j} + p_{j,i+1} + p_{i,j+1} + p_{j+1,i} \\ &\quad + up_{i-1,j} + up_{j,i-1} + up_{i,j-1} + up_{j-1,i}. \end{aligned}$$

If we let $q_{i,j} = p_{i,j} + p_{j,i}$ the above can be simplified to

$$(2 + 2u)q_{i,j} = q_{i+1,j} + q_{i,j+1} + uq_{i-1,j} + uq_{i,j-1}.$$

If we substitute in the values for $q_{i,j}$ from the theorem we are attempting to prove to the above equation, we see that it does satisfy the above equation. Therefore the values of $q_{i,j}$ given in the theorem represent one possible feasible solution to the set of equations. Moreover, via Lemma 5.1 about the rank of B, the solution must be unique. This completes the proof. □

To continue with the proof of Conjecture 4.3 in the $\Longleftarrow$ direction, we note that from $\frac{\Pr(\mu)}{\Pr(\nu)} = \frac{\mathrm{Pd}(\mu)}{\mathrm{Pd}(\nu)}$ it follows that $\frac{\Pr(0,2,1)}{\Pr(0,1,2)} = \frac{\mathrm{Pd}(0,2,1)}{\mathrm{Pd}(0,1,2)}$ or $\frac{2+t}{1+2t} = \frac{p_{2,1}}{p_{1,2}}$. This expands as $(2 + t)p_{1,2} = (1 + 2t)p_{2,1}$. From the above theorem we know that

$$p_{1,2} + p_{2,1} = \frac{2u}{\left(\sum_{k=0}^{n-1} u^k\right)^2}.$$

We can then solve for $p_{1,2}$ giving

$$p_{1,2} = \frac{2(1 + 2t)u}{3(1 + t)\left(\sum_{k=0}^{n-1} u^k\right)^2}.$$

From the equation (5.1) for $i = 1 < j$ we get

$$(3 + t + 2u)p_{1,2} = (1 + 2t)p_{2,1} + p_{2,2} + p_{1,3} + up_{1,1}.$$

Substitute in to get

$$\frac{2(3 + t + 2u)(1 + 2t)u}{3(1 + t)\left(\sum_{k=0}^{n-1} u^k\right)^2} = \frac{2(2 + t)(1 + 2t)u}{3(1 + t)\left(\sum_{k=0}^{n-1} u^k\right)^2} + \frac{3(1 + t)(1 + u)u}{3(1 + t)\left(\sum_{k=0}^{n-1} u^k\right)^2} + p_{1,3}.$$

This simplifies to

$$p_{1,3} = \frac{(5ut + u + t - 1)u}{3(1+t)\left(\sum_{k=0}^{n-1} u^k\right)^2}.$$

A similar argument to that used to produce the above equation for $p_{1,2}$ will give us

$$p_{1,3} = \frac{2(1+2t)u^2}{3(1+t)\left(\sum_{k=0}^{n-1} u^k\right)^2}.$$

Equating the last two equations and solving gives us $t = 1$. This completes the proof of

Theorem 5.3. *Conjecture 4.3 holds for* $D = \text{DASEP}(3, p, 2)$.

6. Future work

Three main potential directions for future work are indicated. One is that further results should be obtained with a view to eventually proving Conjecture 4.3. We proved it for DASEP$(3, p, 2)$ and the suggestion would be to prove it for DASEP$(n, 2, 2)$ and DASEP$(n, 2, q)$ before eventually proceeding to DASEP(n, p, q). Similarly considering the case where 0 represents a ball with species 0 rather than the absence of a species is a variant that should be explored. The other, and more ambitious, possible goal for future research would be to come up with a complete combinatorial characterization of the steady state probabilities for the DASEP. For the ASEP, this has been done in [4] and [10] leading to a deep relationship being discovered between the ASEP and Macdonald polynomials.

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