Séminaire Lotharingien de Combinatoire **95B** (2026)
Article #62, 12 pp.

Proceedings of the 38th Conference on Formal Power Series and Algebraic Combinatorics (Seattle)

Plabic tangles and cluster promotion maps

Chaim Even-Zohar[1], Matteo Parisi*[23], Melissa Sherman-Bennett[4], Ran Tessler[5], and Lauren Williams[6]

[1] *Faculty of Mathematics, Technion, Haifa, Israel;*
[2] *Max Planck Institute for Physics, Munich, Germany;*
[3] *Okinawa Institute for Science and Technology (OIST), Japan;*
[4] *Department of Mathematics, UC Davis, CA;*
[5] *Department of Mathematics, Weizmann Institute of Science, Israel;*
[6] *Department of Mathematics, Harvard University, Cambridge, MA.*

Abstract. Inspired by the BCFW recurrence for tilings of the amplituhedron, we introduce the general framework of *plabic tangles* that utilizes plabic graphs to define rational maps between products of Grassmannians called *promotions*. The central conjecture of the paper is that promotion maps are quasi-cluster homomorphisms, which we prove for several classes. In order to define promotion maps, we utilize m-vector-relation configurations (m-VRCs) on plabic graphs. We relate m-VRCs to the degree (a.k.a 'intersection number') of the amplituhedron map on positroid varieties and characterize all plabic trees with intersection number one, providing some generating functions. Promotion maps have important connections to the geometry and cluster structure of the amplituhedron and singularities of scattering amplitudes in planar $\mathcal{N} = 4$ super Yang–Mills theory.

Keywords: cluster algebras, vector-relation configuration, positroid, plabic graphs, amplituhedron, intersection numbers

1 Introduction

The *positive Grassmannian* $\mathrm{Gr}_{k,n}^{\geq 0}$ is the subset of the real Grassmannian in which all Plücker coordinates are nonnegative. It has a decomposition into *positroid cells* S_G [16] indexed by *plabic graphs* G of type (k, n), planar bicolored graphs which are embedded in a disk. The (tree) *amplituhedron* $\mathcal{A}_{n,k,m}(Z)$ is the image of the positive Grassmannian $\mathrm{Gr}_{k,n}^{\geq 0}$ under the *amplituhedron map* $\tilde{Z} : \mathrm{Gr}_{k,n}^{\geq 0} \to \mathrm{Gr}_{k,k+m}$. It was introduced by Arkani-Hamed and Trnka [3] in order to give a geometric interpretation of the *BCFW recurrence* [5] for *scattering amplitudes* in $\mathcal{N} = 4$ super Yang Mills theory (SYM). In particular, each way of iterating the BCFW recurrence gives rise to a collection of positroid cells of $\mathrm{Gr}_{k,n}^{\geq 0}$. Arkani-Hamed and Trnka conjectured that the amplituhedron map $\tilde{Z}$ is injective on these cells,

*matteo.parisi@oist.jp.

that the $\tilde{Z}$-images of these cells 'tile' the $m = 4$ amplituhedron $\mathcal{A}_{n,k,4}(Z)$, and that one can compute scattering amplitudes by summing the 'volumes' of the tiles.

In a previous work [6], which was joint with Lakrec, we proved the above *BCFW tiling conjecture*. We also proved the *cluster adjacency conjecture*, which says that facets of tiles are cut out by collections of *compatible cluster variables*. To prove these conjectures, we used a graphical recurrence [4, (3.3)] on plabic graphs that involves inserting two smaller plabic graphs into faces of a fixed "core" plabic graph (cf. [7], Figure 3). In particular, we showed that this graphical recurrence gives rise to a *quasi-cluster homomorphism* that is, a map that respects the *cluster algebra structure* on the Grassmannian, and in particular takes cluster variables to cluster variables.[1]

Main results. A goal of this work is to illustrate that the graphical recurrence and quasi-cluster homomorphism that we used to prove the BCFW tiling and cluster adjacency conjectures are part of a much vaster framework. In particular, in Section 3.1, we introduce *plabic tangles* – inspired by the notion of *planar tangle* [11] – which is the data of a plabic graph "core" G drawn inside an *outer disk*, together with ℓ *inner disks* or "blobs", each of which lies in a face of G (cf. Figure 2.a). The central construction of this paper is the association of a map between Grassmannians to each (sufficiently nice) plabic tangle.

We start in Section 2.5 by defining an *m-vector-relation configuration* (m-VRC) on a (bipartite) plabic graph G, which is an assignment of vectors $v_b \in \mathbb{C}^m$ and coefficients $r_e \in \mathbb{C}^*$ to each black vertex b and edge e of G, so that for each white vertex w, the linear relation $\sum_{e=\{b,w\}} r_e v_b = 0$ is satisfied. Then we consider plabic tangles with a core G which are *solvable*, that is, a generic configuration of n vectors in $\mathbb{C}^m$ on the outer boundary of G can be extended to a unique m-VRC (modulo gauge) on G. Each inner disk D is connected to some black vertices b of G, associated with vectors v_b in the m-VRC. Then we use the m-VRC to define a rational map (*geometric promotion*) from the Grassmannian $\mathrm{Gr}_{m,n}$ associated to the outer boundary of G, to the product of Grassmannians $\mathrm{Gr}_{m,D}$ associated to the inner disks of the plabic tangle (Definition 3.4). The pullback of this map is a map on the level of coordinate rings called *algebraic promotion*. We give a combinatorial characterization of when geometric promotion is a dominant map[2] (Theorem 3.8), in which case we call the plabic tangle *dominant*.

The main conjecture of our work (Conjecture 3.9) is that: *algebraic promotion maps by solvable dominant plabic tangles give rise to quasi-cluster homomorphisms on Grassmannians*.

In Section 3.2, we relate m-VRCs to the degree (a.k.a *intersection number*) of the amplituhedron map on positroid varieties (Definition 2.10). We prove (Theorem 3.13) that the number of m-VRCs (modulo gauge) on G that have a given generic configuration of n vectors in $\mathbb{C}^m$ on the outer boundary of G equals the m-intersection number of a positroid variety Π_G. It follows that: *a plabic tangle with core G is solvable if and only if the positroid variety Π_G has m-intersection number 1* (Corollary 3.14). Furthermore, we give a

[1] up to a Laurent monomial in frozen variables.

[2] i.e. its image is Zariski-dense in the codomain.

combinatorial characterization of when this happens and G is a tree (Theorem 3.17); in this case, we call G an *amplitree*, of which we provide some generating functions (Theorem 3.19). Finally, using plabic tangles that have amplitrees as cores, we prove our main Conjecture 3.9 for several interesting infinite families of promotion maps (Theorem 3.20).

Further motivation. Cluster promotions maps should play an important role in describing the geometry and the cluster structure of amplituhedron tiles and, more generally, images of positroid cells under the amplituhedron map (cf. Remarks 2.12, 3.15). On the cluster algebra side, cluster promotions maps can be used e.g. to find new classes of cluster variables for $\mathrm{Gr}_{m,n}$ or compatibility criteria among them. On the physics side, we expect singularities of scattering amplitudes in planar $\mathcal{N} = 4$ SYM to be images of cluster promotion maps, generalizing what we have seen with BCFW promotions. Our framework should thus shed light on the conjectural cluster structures and 'positivity phenomena' that arise in the study of scattering amplitudes, see [7], Remark 10.14.

2 Background

2.1 The (Positive) Grassmannian

The *Grassmannian* $\mathrm{Gr}_{k,n}(\mathbb{F})$ is the space of all k-dimensional subspaces of an n-dimensional vector space $\mathbb{F}^n$. Let $[n]$ denote $\{1,\ldots,n\}$, and $\binom{[n]}{k}$ denote the set of all k-element subsets of $[n]$. We can represent a point $V \in \mathrm{Gr}_{k,n}(\mathbb{F})$ as the row-span of a full-rank $k \times n$ matrix C with entries in $\mathbb{F}$. Then for $I = \{i_1 < \cdots < i_k\} \in \binom{[n]}{k}$, we let $\langle I\rangle_V = \langle i_1\, i_2\, \ldots\, i_k\rangle_V$ be the $k \times k$ minor of C using the columns I. The $\langle I\rangle_V$ are called the *Plücker coordinates* of V, and are independent of the choice of matrix representative C (up to common rescaling). The *Plücker embedding* $V \mapsto \{\langle I\rangle_V\}$ embeds $\mathrm{Gr}_{k,n}(\mathbb{F})$ into projective space[3] . If D is a totally ordered set, we use $\mathrm{Gr}_{k,D}$ to denote the Grassmannian $\mathrm{Gr}_{k,|D|}$, where the columns of a representing matrix are indexed by D.

Definition 2.1 (Positive Grassmannian and positroids)**.** [15, 16] We say that $V \in \mathrm{Gr}_{k,n}(\mathbb{R})$ is *totally nonnegative* if (up to a global change of sign) $\langle I\rangle_V \geq 0$ for all $I \in \binom{[n]}{k}$. Similarly, V is *totally positive* if $\langle I\rangle_V > 0$ for all $I \in \binom{[n]}{k}$. We let $\mathrm{Gr}_{k,n}^{\geq 0}$ and $\mathrm{Gr}_{k,n}^{>0}$ denote the set of totally nonnegative and totally positive elements of $\mathrm{Gr}_{k,n}(\mathbb{R})$, respectively. $\mathrm{Gr}_{k,n}^{\geq 0}$ is called the *totally nonnegative Grassmannian*, or sometimes just the *positive Grassmannian*.

If we partition $\mathrm{Gr}_{k,n}^{\geq 0}$ into strata based on which Plücker coordinates are strictly positive and which are 0, we obtain a cell decomposition of $\mathrm{Gr}_{k,n}^{\geq 0}$ into *positroid cells* [16]. Each positroid cell S gives rise to a matroid $\mathcal{M}$, whose bases are precisely the k-element subsets I such that the Plücker coordinate $\langle I\rangle$ does not vanish on S; $\mathcal{M}$ is called a *positroid*.

[3] We will identify C with its row-span and also drop the subscript V on Plücker coordinates when it does not cause confusion. We consider Plückers to be alternating in the indices e.g. $\langle i_1\, i_2\, \ldots\, i_k\rangle = -\langle i_2\, i_1\, \ldots\, i_k\rangle$.

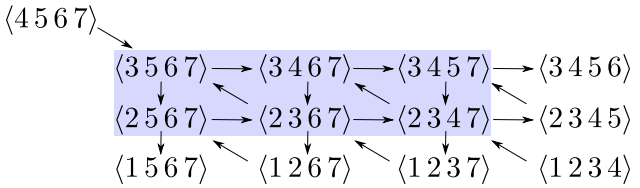


Figure 1: The rectangle seed $\Sigma_{4,7}$. Mutable variables are in the colored box.

There are many ways to index positroid cells in $\mathrm{Gr}_{k,n}^{\geq 0}$ [16], such as *plabic graphs*:

Definition 2.2. A *plabic graph* $G = ((B, W), E)$ is a planar bipartite graph[4] embedded in a disk, with black vertices $1, 2, \ldots, n$ on the boundary of the disk. Let B and W denote the set of black and white vertices and E the set of edges. A *perfect orientation* $\mathcal{O}$ of G is a choice of orientation of each edge so that each black (white) vertex has all outgoing (ingoing) edges but one[5]. The *sink set* $I_{\mathcal{O}} \subset [n]$ of $\mathcal{O}$ is the set of boundary vertices i that are sinks. The positroid associated to G is the collection $\mathcal{M}(G) := \{I_{\mathcal{O}} : \mathcal{O}$ a perfect orientation of $G\}$. If $k := |I_{\mathcal{O}}|$ for a perfect orientation $\mathcal{O}$, then we say that G *has type* (k, n) and we denote as S_G the corresponding positroid cell in $\mathrm{Gr}_{k,n}^{\geq 0}$.

The Zariski closure of S_G in $\mathrm{Gr}_{k,n}(\mathbb{C})$ is called the *positroid variety* Π_G. The positroid variety is cut out by setting to 0 all Plücker coordinates $\langle I \rangle$ that vanish on S_G [12].

2.2 Cluster algebras

Cluster algebras were introduced by Fomin and Zelevinsky in [9] motivated by the study of total positivity; see [8] for an introduction. We give a quick definition of cluster algebras from *quivers*, which are finite oriented graphs. For a quiver Q without oriented cycles of length 1 and 2, one can define a *quiver mutation* $\mu_k(Q)$ at each vertex k of Q. This operation, described in [9], is an involution: $\mu_k^2(Q) = Q$.

Definition 2.3 (*Seeds*)**.** Choose $s \geq r$ positive integers. Let $\mathcal{F}$ be an *ambient field* of rational functions in r independent variables over $\mathbb{C}(x_{r+1}, \ldots, x_s)$. A *seed* in $\mathcal{F}$ is a pair $(\widetilde{\mathbf{x}}, Q)$, where $\widetilde{\mathbf{x}} = (x_1, \ldots, x_s)$ forms a free generating set for $\mathcal{F}$ and Q is a quiver on vertices $1, 2, \ldots, r, r+1, \ldots, s$. We call $\widetilde{\mathbf{x}}$ the *cluster* of a seed $(\widetilde{\mathbf{x}}, Q)$, and its elements are called *cluster variables*. The variables $c = \{x_{r+1}, \ldots, x_s\}$ are called *frozen* (or *coefficient variables*). The *frozen group* $\mathbb{M}$ is the group of Laurent monomials in the frozen variables.

Definition 2.4 (*Seed mutations*)**.** Let $(\widetilde{\mathbf{x}}, Q)$ be a seed in $\mathcal{F}$, and let k be a mutable vertex of Q. The *seed mutation* μ_k in direction k transforms $(\widetilde{\mathbf{x}}, Q)$ into the seed $\mu_k(\widetilde{\mathbf{x}}, Q) = (\widetilde{\mathbf{x}}', \mu_k(Q))$, where the cluster $\widetilde{\mathbf{x}}' = (x_1', \ldots, x_s')$ is defined as follows: $x_j' = x_j$ for $j \neq k$, and $x_k' \in \mathcal{F}$ is determined by the *exchange relation* $x_k' \, x_k = \prod_{i \to k} x_i + \prod_{k \to i} x_i$, where $i \to k$ (or $i \leftarrow k$) denotes an edge oriented from vertex i to k (or k to i)[6].

[4] We will always assume that plabic graphs are *reduced* and *leafless* [16].

[5] This is what is called a *reverse perfect orientation* in the conventions of [16]

[6] Note that one omits arrows between two frozen vertices as they do not affect seed mutation.

Definition 2.5 (*Cluster algebra*)**.** Given an initial labeled seed $(\widetilde{\mathbf{x}}, Q)$, we let $\mathcal{X}$ denote the union of all cluster variables obtained performing all possible mutation sequences from the initial seed. Let $\mathbb{C}[c^{\pm 1}]$ be the *ground ring* consisting of Laurent polynomials in the frozen variables. The *cluster algebra* $\mathcal{A} = \mathcal{A}(\widetilde{\mathbf{x}}, Q)$ is the $\mathbb{C}[c^{\pm 1}]$-subalgebra of the ambient field $\mathcal{F}$ generated by all cluster variables, with coefficients which are Laurent polynomials in the frozen variables: $\mathcal{A} = \mathbb{C}[c^{\pm 1}][\mathcal{X}]$. We say that $\mathcal{A}$ has *rank* r because each cluster contains r cluster variables. Cluster (or frozen) variables that belong to a common cluster are said to be *compatible*.

The Grassmannian $\mathrm{Gr}_{m,n}(\mathbb{C})$ has a natural cluster algebra structure, generated by the Plücker coordinates. A standard seed for this cluster structure is the *rectangles seed* $\Sigma_{m,n}$, with $m(n-m)+1$ variables, n of which are frozen (cf. Figure 1). The cluster algebra $\mathcal{A}(\Sigma_{m,n})$ is the homogeneous coordinate ring $\mathbb{C}[\mathrm{Gr}_{m,n}]$ of the Grassmannian [18].

2.3 Quasi-cluster homomorphisms of cluster algebras

Definition 2.6. Given a seed $\Sigma = ((x_1, \ldots, x_s), Q)$ for a cluster algebra of rank $r \leq s$, and a mutable variable x_i (so that $1 \leq i \leq r$), the *exchange ratio* of x_i (with respect to Σ) is

$$\hat{y}_\Sigma(x_i) \;=\; \prod_{j:i\to j} x_j^{\#\,\mathrm{arr}(i\to j)} \prod_{j:j\to i} (x_j^{-1})^{\#\,\mathrm{arr}(j\to i)}$$

where $\#\,\mathrm{arr}(i \to j)$ denotes the number of arrows from i to j in the quiver Q.

Let $\mathcal{A}$ be a cluster algebra and $\mathbb{M}$ its frozen group. For elements $x, y \in \mathcal{A}$, we say that x *is proportional to* y, writing $x \propto y$, if $x = My$ for some Laurent monomial $M \in \mathbb{M}$.

Definition 2.7 (Quasi-cluster homomorphism, [10])**.** Let $\mathcal{A}$ and $\overline{\mathcal{A}}$ be two cluster algebras of the same rank r, and with respective frozen groups $\mathbb{M}$ and $\overline{\mathbb{M}}$. Then an algebra homomorphism $f : \mathcal{A} \to \overline{\mathcal{A}}$ that satisfies $f(\mathbb{M}) \subseteq \overline{\mathbb{M}}$ is called a *quasi-cluster homomorphism* from $\mathcal{A}$ to $\overline{\mathcal{A}}$ if there are seeds $\Sigma = ((x_1, \ldots, x_s), Q)$ and $\overline{\Sigma} = ((\bar{x}_{\bar{1}}, \ldots, \bar{x}_{\bar{s}}), \bar{Q})$ for $\mathcal{A}$ and $\overline{\mathcal{A}}$, such that: $f(x_i) \propto \bar{x}_{\bar{\imath}}$ and $f(\hat{y}_\Sigma(x_i)) = \hat{y}_{\overline{\Sigma}}(\bar{x}_{\bar{\imath}})$ for $1 \leq i \leq r$, and we write $\Sigma \propto \overline{\Sigma}$.

Proposition 2.8. *[10, Proposition 3.2] If $\Sigma \propto \overline{\Sigma}$, then $\mu_k(\Sigma)$ is similar to $\mu_k(\overline{\Sigma})$. In particular, a quasi-cluster homomorphism f maps any collection of compatible cluster variables in $\mathcal{A}$ into a collection of compatible cluster variables in $\overline{\mathcal{A}}$, up to a Laurent monomial in the frozens in $\overline{\mathbb{M}}$.*

2.4 The amplituhedron map

Let $\mathrm{Mat}^{\circ}_{n,p}$ denote the set of full-rank $n \times p$ matrices, and $\mathrm{Mat}^{>0}_{n,p} \subset \mathrm{Mat}^{\circ}_{n,p}$ be the subset of matrices whose maximal minors are positive.

Definition 2.9. Let $Z \in \mathrm{Mat}^{\circ}_{n,k+m}$, where $k+m \leq n$. The *amplituhedron map* $\tilde{Z} : \mathrm{Gr}_{k,n} \dashrightarrow \mathrm{Gr}_{k,k+m}$ is the rational map defined by $\tilde{Z}(C) := CZ$, where C is a $k \times n$ matrix representing an element of $\mathrm{Gr}_{k,n}$, and CZ is a $k \times (k+m)$ matrix representing an element of $\mathrm{Gr}_{k,k+m}$. If $Z \in \mathrm{Mat}^{>0}_{n,k+m}$, $\tilde{Z}$ is well-defined when restricted to the positive Grassmannian $\mathrm{Gr}^{\geq 0}_{k,n}$, and the *tree amplituhedron* $\mathcal{A}_{n,k,m}(Z) \subset \mathrm{Gr}_{k,k+m}$ is the image $\tilde{Z}(\mathrm{Gr}^{\geq 0}_{k,n})$.

Definition 2.10 (Intersection number)**.** Let $\Pi_G \subset \mathrm{Gr}_{k,n}$ be a positroid variety of dimension km. The *m-intersection number* of Π_G, denoted $\mathrm{IN}_m(G)$, is d if the amplituhedron map $\tilde{Z} : \Pi_G \dashrightarrow \mathrm{Gr}_{k,k+m}$ restricted to Π_G is generically d-to-1, and it is zero otherwise.

Lam gave a Z-independent characterization of m-intersection number:

Proposition 2.11 ([13, Proposition 4.8])**.** *Suppose $\Pi_G \subset \mathrm{Gr}_{k,n}$ has dimension km. Then $\mathrm{IN}_m(G)$ is the coefficient of the Schur polynomial $s_{(n-k-m)^k}$ in the cohomology class $[\Pi_G] \in H^*(\mathrm{Gr}_{k,n})$. Equivalently, fix $X \in \mathrm{Gr}_{n-m,n}$ and let $\mathrm{Gr}_k(X) := \{V \in \mathrm{Gr}_{k,n} : V \subset X\}$. Then*

$$[\mathrm{Gr}_k(X)] \cap [\Pi_G] = \mathrm{IN}_m(G) \cdot [\mathrm{pt}] \quad \text{in } H^*(\mathrm{Gr}_{k,n}).$$

Remark 2.12. One often considers a related semi-algebraic version of the story. For a positroid cell $S_G = \Pi_G \cap \mathrm{Gr}^{\geq 0}_{k,n}$, one can consider $Z_G := \overline{\tilde{Z}(S_G)}$, the closure of the image of the positroid cell. We call Z_G a *tile* if $\tilde{Z}$ is 1-to-1 on S_G and $\dim \Pi_G = km$. All known tiles Z_G happen to have intersection number one, however this does not follow in general from the definitions (cf. [7], Remark 2.27).

Remark 2.13. Scattering amplitudes in planar $\mathcal{N} = 4$ SYM have fundamental building blocks called *Yangian invariants*, which are functions on $\mathrm{Gr}_{4,n}$ that encode *leading singularities*. Each Yangian invariant $\mathcal{Y}_G$ corresponds to a $4k$-dimensional positroid cell $S_G \subseteq \mathrm{Gr}^{\geq 0}_{k,n}$ [2]. If $\mathrm{IN}_4(G) = 1$, $\mathcal{Y}_G$ is a *rational* function whose poles correspond to facets of Z_G. If $\mathrm{IN}_4(G) > 1$, $\mathcal{Y}_G$ is as a sum of d functions which have *algebraic* singularities.

2.5 m-VRCs on Plabic Graphs

In this section we introduce *vector relation configurations* on plabic graphs (cf. Definition 2.2), essentially inspired[7] by [1]. We note that related frameworks had previously appeared in [2], [17] and [14]. We fix $m \geq 1$ to be a positive integer.

Definition 2.14. An *m-vector-relation configuration* (m-VRC) on a plabic graph G is an assignment of a vector $v_b \in \mathbb{C}^m$ to each black vertex and a coefficient $r_e \in \mathbb{C}^*$ to each edge such that the *boundary vectors* $v_1, \ldots, v_n$ on vertices $1, \ldots n$ span $\mathbb{C}^m$, and for each white vertex w, we have the linear relation $\sum_{e=\{b,w\} \in E} r_e v_b = 0$.

[7] our definitions and results are rather different than those of [1]; see [7], Remark 3.5 for a comparison.

We write $(\mathbf{v}, \mathbf{R}) := (\{v_b\}_{b\in B}, \{r_e\}_{e\in E})$ for an m-VRC. The *boundary* of $(\mathbf{v}, \mathbf{R})$ is the $m \times n$ matrix of boundary vectors $\partial(\mathbf{v}, \mathbf{R}) := \big[v_1 \cdots v_n\big]$.

For any m-VRC $(\mathbf{v}, \mathbf{R})$ on G, $\mathbf{v}$ is determined by the boundary $\partial(\mathbf{v}, \mathbf{R})$ and the coefficients $\mathbf{R}$, with explicit formulas (cf. [7], Lemma 3.6 and Corollary 3.7).

Definition 2.15. Let $(\mathbf{v}, \mathbf{R})$ be an m-VRC on a plabic graph G. For an internal vertex x of G and $t \in \mathbb{C}^*$, *gauge transformation by t at x* changes $r_e \mapsto tr_e$ for all edges e adjacent to x and, if $x = b$ is black, additionally changes $v_b \mapsto (1/t)v_b$.

We write $\mathcal{C}_m(G)$ for m-VRCs on G modulo gauge transformations and GL_m-action, which acts on $(\mathbf{v}, \mathbf{R})$ by left multiplication on all vectors. We denote $[\mathbf{v}, \mathbf{R}]$ an element of $\mathcal{C}_m(G)$ and $\partial[\mathbf{v}, \mathbf{R}] \in \mathrm{Gr}_{m,n}$ for its boundary. We write $\mathcal{C}_m^{\partial=\mathbf{z}}(G)$ for the subset of $\mathcal{C}_m(G)$ with fixed boundary $\mathbf{z} \in \mathrm{Gr}_{m,n}$.

Remark 2.16. m-VRCs are well-behaved under equivalence moves of plabic graphs ([7], Lemma 3.8).

Example 2.17. Let G_{ex} be the plabic graph in black without orientation in Figure 2.b. A 4-VRC on G_{ex} is obtained as follows: let $\mathbf{z} = \big[v_1 \cdots v_7\big] \in \mathrm{Gr}_{4,7}$ and pick $v_i \in \mathbb{C}^4$ as the boundary vectors. Let $\langle I_i\rangle$ be the Plücker coordinates of $\mathbf{z}$, where $I_i := \{3,4,5,6,7\} \setminus \{i\}$. Then we set: $w_1 = v_4\langle I_4\rangle - v_3\langle I_3\rangle$ and $w_2 = v_6\langle I_6\rangle - v_7\langle I_7\rangle$, and the coefficients to be: $r_e = 1$ ($r_e = -1$), if e is an internal edge oriented from black (white) to white (black); $r_e = (-1)^{\sigma_i}\langle I_i\rangle$, if e is adjacent to the boundary vertex v_i and $(\sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7) = (1, 0, 0, 0, 1)$.

3 Results

3.1 m-VRCs, Promotion Maps, and Cluster Algebras.

Definition 3.1 (Plabic tangles)**.** Let ℓ be a positive integer. A *plabic tangle* $(G, \mathbf{D})$ is the data of a plabic graph $G = ((B, W), E)$, drawn inside a disk called an *outer disk* (with boundary vertices $\{1, \ldots, n\}$), together with ℓ *inner disks*, each of which lies in a face of G. Each inner disk has boundary vertices D_i (for $i \in [\ell]$), and each vertex $u \in D_i$ is connected to one element b_u of B via a segment such that the resulting graph is planar. Each inner and outer disk contains one $\star$*-marked interval*, and the boundary vertices of each disk are labeled clockwise in increasing order starting just after the $\star$. We use the notation $\mathbf{D} := \{D_i\}_{i\in[\ell]}$ for the boundary vertices of the inner disks. We often refer to G as the *core* and the inner disks as *blobs*, see Figure 2.a.

Definition 3.2. Let $(G, \mathbf{D})$ be a plabic tangle and $(\mathbf{v}, \mathbf{R})$ an m-VRC on G, the *D-boundary* of $(\mathbf{v}, \mathbf{R})$ on the blob D is the $m \times |D|$ matrix of vectors $\partial_D(\mathbf{v}, \mathbf{R}) := \Big[v_{b_1} \cdots v_{b_{|D|}}\Big]$ obtained as follows. Let $u_1, \ldots, u_{|D|}$ be the boundary vertices of the blob D read in clockwise order starting just after the $\star$. Then each vertex u_j is adjacent to a unique black vertex $b_j := b_{u_j} \in B$, associated to the vector v_{b_j} in $(\mathbf{v}, \mathbf{R})$, see Figure 2.a.

If $\partial_D(\mathbf{v}, \mathbf{R})$ has rank m, then, by the GL_m-action on the m-VRC, we may regard $\partial_D(\mathbf{v}, \mathbf{R})$ as an element of $\mathrm{Gr}_{m,D}$.

Definition 3.3 (Solvability)**.** A plabic tangle $(G, \mathbf{D})$ is *m-solvable* if for generic $\mathbf{z} \in \mathrm{Gr}_{m,n}$, $\mathcal{C}_m^{\partial=\mathbf{z}}(G)$ has a unique element and each blob $D \in \mathbf{D}$ has size at least m.

We denote this unique element in $\mathcal{C}_m^{\partial=\mathbf{z}}(G)$ by $[\mathbf{v}, \mathbf{R}]^{\partial=\mathbf{z}}$. A *pinning* $\mathbf{p}(\mathbf{z})$ of a solvable plabic tangle $(G, \mathbf{D})$, is a choice of a representative $\mathbf{p}(\mathbf{z}) = (\{v_b(\mathbf{z})\}_{b\in B}, \{r_e(\mathbf{z})\}_{e\in E})$ for $[\mathbf{v}, \mathbf{R}]^{\partial=\mathbf{z}}$, where the vectors v_b and coefficients r_e are rational functions[8] of $\mathbf{z}$.

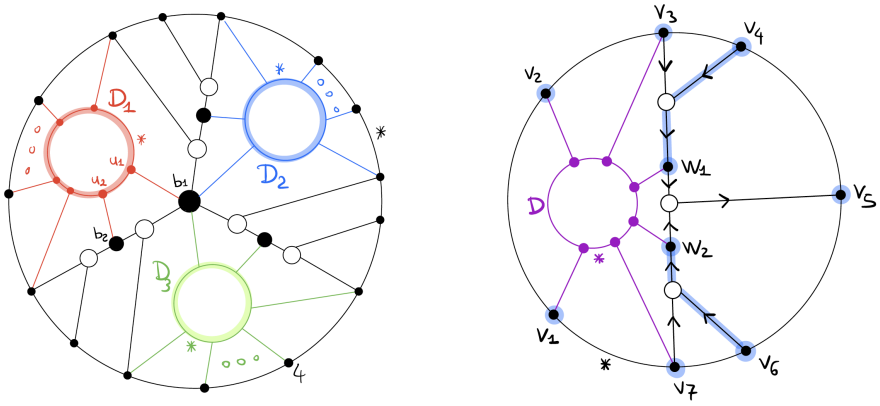


Figure 2: (a): a plabic tangle with 3 blobs; (b): plabic tangle with core G_{ex} (in black) displaying vectors of a 4-VRC of G_{ex} and a brushing (with orientation and paths).

Definition 3.4. Let $(G, \mathbf{D})$ be a solvable plabic tangle with a pinning $\mathbf{p}(\mathbf{z})$, we define[9]:

- the rational *geometric promotion* map:
$$\psi : \mathrm{Gr}_{m,n} \dashrightarrow \mathrm{Gr}_{m,D_1} \times \ldots \times \mathrm{Gr}_{m,D_\ell} \tag{3.1}$$
which maps a generic $\mathbf{z} \in \mathrm{Gr}_{m,n}$ to the D_i-boundary of $\mathbf{p}(\mathbf{z})$ on each Gr_{m,D_i}.
- the *(algebraic) promotion* map to be the pullback $\Psi := \psi^*$:
$$\Psi : \mathbb{C}(\widehat{\mathrm{Gr}}_{m,D_1}) \otimes \cdots \otimes \mathbb{C}(\widehat{\mathrm{Gr}}_{m,D_\ell}) \to \mathbb{C}(\widehat{\mathrm{Gr}}_{m,n}), \tag{3.2}$$
which acts on Plücker coordinates by substituting the column vector u_j in each blob D with the associated vector v_{b_j} in the D-boundary of $\mathbf{p}(\mathbf{z})$ (cf. Definition 3.2).

Example 3.5. The 4-VRC in Example 2.17 is a pinning for the plabic tangle in Fig. 2.b. Its geometric promotion $\psi : \mathrm{Gr}_{4,7} \dashrightarrow \mathrm{Gr}_{4,6}$, sends $\mathbf{z} = [v_1, \ldots, v_7]$ to the point $\psi(\mathbf{z}) = [v_1, v_2, v_3, w_1, w_2, v_7]$. The algebraic promotion map $\Psi : \mathbb{C}(\widehat{\mathrm{Gr}}_{4,6}) \to \mathbb{C}(\widehat{\mathrm{Gr}}_{4,7})$ sends e.g. the Plücker coordinate $\langle 1246\rangle$ into $\langle 1, 2, w_1, 7\rangle = \langle 1273\rangle\langle 4567\rangle - \langle 1274\rangle\langle 3567\rangle$.

[8] such a choice always exists, cf. [7], Proposition 6.15.

[9] for well-defined-ness we have to require that every D_i-boundary of $\mathbf{p}(\mathbf{z})$ has rank m for generic $\mathbf{z}$. In particular, this holds when we will consider ψ to be dominant (cf. Definition 3.6).

Definition 3.6. A solvable plabic tangle $(G, \mathbf{D})$ is *dominant* if ψ is a dominant map.

We give a combinatorial characterization of dominance in term of *brushing*.

Definition 3.7 (Brushing)**.** A *brushing* $\mathcal{B} = (\mathcal{O}^D, \{P_u\}_{u\in D})$ for a plabic tangle $(G, \mathbf{D})$, is a choice for each blob $D \in \mathbf{D}$ of: an acyclic perfect orientation[10] $\mathcal{O}^D$ of G; a collection of oriented vertex-disjoint paths $\{P_u\}_{u\in D}$ in $\mathcal{O}^D$ such that P_u goes from some boundary vertex $i_u \in \{1, \ldots, n\}$ to the vertex b_u adjacent to $u \in D$, see Figure 2.b.

Theorem 3.8. *A solvable plabic tangle $(G, \mathbf{D})$ is dominant if and only if it admits a brushing.*

We propose that the class of solvable dominant plabic tangles has deep connection with cluster algebras.

Conjecture 3.9. *Let $(G, \mathbf{D})$ be a solvable dominant plabic tangle. Then there exists a pinning $\boldsymbol{p}$ of $(G, \mathbf{D})$ such that the following holds:*

1. *Geometric promotion ψ sends totally positive elements to totally positive elements.*
2. *Algebraic promotion Ψ gives a cluster quasi-homomorphism*

$$\mathcal{A}(\Sigma_{m,D_1} \sqcup \cdots \sqcup \Sigma_{m,D_\ell}) \to \mathcal{A}(\overline{\Sigma})$$

where $\overline{\Sigma}$ is obtained from a seed $\Sigma_{m,n}$ for $\widehat{\mathrm{Gr}}_{m,n}$ by freezing some variables.

In such a case, we call the algebraic promotion Ψ a cluster promotion map.

Example 3.10. The promotion maps in Example 3.5 satisfy Conjecture 3.9 as it is a special case of *star promotion* for $m = 4$ (Theorem 3.20). In particular, Ψ is a cluster promotion map. Indeed, e.g. $\Psi(\langle 1246\rangle) = \langle 1273\rangle\langle 4567\rangle - \langle 1274\rangle\langle 3567\rangle$, which is a quadratic cluster variable for $\mathrm{Gr}_{4,7}$, e.g. obtained by mutating $\langle 2367\rangle$ in $\Sigma_{4,7}$ (cf. Figure 1).

Conjecture 3.9 generalizes the BCFW quasi-cluster homomorphism [6], and asserts a deep connection between solvable plabic tangles, m-VRCs on plabic graphs and the cluster structure on the Grassmannian. Moreover, we prove it for several interesting infinite families of promotion maps (Theorem 3.20).

Remark 3.11. A brushing $\mathcal{B}$ can be used to define a distinguished pinning $\mathbf{p}_\mathcal{B}$ ([7], Definition 4.9). Then we can strengthen Conjecture 3.9 by restricting[11] to the pinning $\mathbf{p} = \mathbf{p}_\mathcal{B}$, for some brushing $\mathcal{B}$ of $(G, \mathbf{D})$.

Remark 3.12. The collection of dominant m-solvable plabic tangles $\mathcal{P}_m^{\mathrm{ds}}$ has an *operad* structure (cf. [7], Section 9). In particular, composing two plabic tangles $(G, \mathbf{D})$ and $(G', \mathbf{E})$ in $\mathcal{P}_m^{\mathrm{ds}}$ by inserting $(G', \mathbf{E})$ into one blob $D \in \mathbf{D}$ of G generates a new plabic tangle in $\mathcal{P}_m^{\mathrm{ds}}$. Moreover, algebraic promotion maps give *representations* of $\mathcal{P}_m^{\mathrm{ds}}$.

[10] every reduced plabic graph has an acyclic perfect orientation (cf. [7], Lemma A.6).

[11] up to also making a choice of signs σ, cf. [7], Definition 4.9.

3.2 m-VRCs, the Amplituhedron, and Intersection Numbers.

We relate m-VRCs and solvable plabic tangles to the (fiber of the) amplituhedron map and intersection numbers of positroid varieties (cf. Definition 2.10).

Theorem 3.13. *Let $\Pi_G \subset \mathrm{Gr}_{k,n}$ be of dimension km. Then for generic $\mathbf{z} \in \mathrm{Gr}_{m,n}$, the number $|\mathcal{C}_m^{\partial=\mathbf{z}}(G)|$ of m-VRC's with boundary $\mathbf{z}$ is equal to the intersection number $\mathrm{IN}_m(G)$ of Π_G.*

Corollary 3.14. *A plabic tangle $(G, \mathbf{D})$ with blobs of size at least m is m-solvable if and only if $\mathrm{IN}_m(G) = 1$.*

Remark 3.15. m-VRCs help understand fibers of the amplituhedron map. For example, when $\mathrm{IN}_m(G) = 1$, we can use the unique m-VRC with boundary $\mathbf{z}$ to invert the amplituhedron map $\tilde{Z}$ on the positroid variety Π_G. See [7], Remarks 6.16, 6.17.

We now give a combinatorial characterization of intersection number one trees.

Definition 3.16. Let G be a bipartite plabic tree of type[12] $(k, km+1)$. We say that G is *m-balanced* if for each edge e of G, if we write $G \setminus \{e\} = G_1 \sqcup G_2$ (giving "half" the edge e to each G_i), then for each $i = 1, 2$, we have: $m(k_i - 1) < \dim \Pi_{G_i} \leq mk_i$, where k_i is the k-statistic of the tree G_i. In this case, we say G is a *$(k, m)-$amplitree*.

Theorem 3.17. *Let G be a bipartite plabic tree of type $(k, km+1)$. Then $\mathrm{IN}_m(G) = 1$ if and only if G is m-balanced. If G is not m-balanced, then $\mathrm{IN}_m(G) = 0$.*

Example 3.18. Let G be the core of the plabic tangle in Figure 3.b and e.g. consider the edge e adjacent to the white vertex of degree 4 and the internal black vertex. Let $G \setminus \{e\} = G_1 \sqcup G_2$ as Definition 3.16. Then, $\dim \Pi_{G_1} = 3, k_1 = 1$ and $\dim \Pi_{G_2} = 6, k_2 = 2$, hence satisfying the necessary conditions for edge e in order for G to be 4-balanced.

We provide an explicit m-VRC on amplitrees with fixed boundary vectors in terms of the *Grassmann-Cayley algebra* (cf. [7], Proposition 7.21). Moreover, we find generating functions for the number $c_{k,m}$ of (k, m)-amplitrees for $k = 1, 2, 3$ ([7], Theorem 7.24, 7.25):

Theorem 3.19. *When $k = 1$, we have $c_{1,m} = 1$; for $k = 2, 3$ the generating functions are:*

$$\sum_{m \geq 0} c_{2,m} x^m = \frac{1+x}{(1-x)^4} \quad \text{and} \quad \sum_{m \geq 0} c_{3,m} x^m = \frac{1 + 28x + 56x^2 + 14x^3}{(1-x)^7},$$

$$\text{where } c_{2,m} = 1^2 + 2^2 + \cdots + m^2 = \frac{m(m+1)(2m+1)}{6}$$

is the sequence of square pyramidal numbers *(cf. A000330 in [19]).*

[12] if $\Pi_G \subset \mathrm{Gr}_{k,n}$ has dimension mk and G is a tree, then G is of type $(k, km+1)$.

In general, for fixed k, we expect $c_{k,m}$ to satisfy a generating function of the form $p(x)/(1-x)^{3k-2}$ where $p(x)$ is a polynomial of degree at most $3k-2$. This implies that $c_{k,m} = \Theta(m^{3k-3})$, as m tends to infinity. If $m \geq 2$ is fixed and k tends to infinity, then $c_{k,m}$ is exponential in k. It remains an intriguing problem to determine an explicit two-variable generating function keeping track of both k and m.

Finally, using Theorem 3.17, we can produce for each m infinite families of solvable plabic tangles whose cores are trees[13], and hence infinite families of promotion maps:

Theorem 3.20 ([7], Section 8)**.** *Conjecture 3.9 is true for:* star promotion *($k = 1$, any m, n);* spurion promotion *($k = 2$, $m = 4$, any n);* chain-tree promotion *($k = 3$, $m = 4$, any n); and* forest promotion *($k = 2$, $m = 3$, any n). For each promotion, the plabic tangles are in Figure 3.*

Remark 3.21. We also investigate plabic tangles $(G, \mathbf{D})$ with $\mathrm{IN}_m(G) > 1$. In [7], Section 10, we consider G to be the 4-*mass box*, for which $\mathrm{IN}_4(G) = 2$ and it has important connections to physics. We obtain two "promotion" maps, which cannot be quasi-cluster homomorphisms, because they involve a square root. Nevertheless, these maps still have intriguing positivity properties ([7], Corollary 10.12), namely (1) of Conjecture 3.9. This could point to a new algebraic structure beyond the framework of cluster algebras.

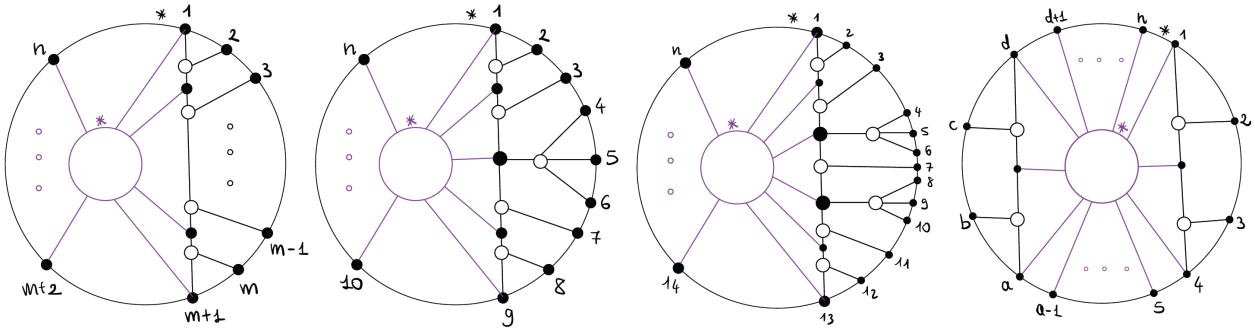


Figure 3: Plabic tangles giving: (a) star, (b) spurion, (c) chain-tree, (d) forest promotions

References

[1] N. Affolter, M. Glick, P. Pylyavskyy, and S. Ramassamy. "Vector-relation configurations and plabic graphs". *Selecta Math. (N.S.)* **30**.1 (2024), Paper No. 9, 55 pp. DOI.

[2] N. Arkani-Hamed, J. Bourjaily, F. Cachazo, A. Goncharov, A. Postnikov, and J. Trnka. *Grassmannian geometry of scattering amplitudes*. Cambridge University Press, Cambridge, 2016, pp. ix+194. DOI.

[13] as well as forests of amplitrees

[3] N. Arkani-Hamed and J. Trnka. "The amplituhedron". *J. High Energy Phys.* **10** (2014), p. 33.

[4] Y. Bai and S. He. "The amplituhedron from momentum twistor diagrams". *J. High Energy Phys.* 2 (2015), 065, front matter+38. DOI.

[5] R. Britto, F. Cachazo, B. Feng, and E. Witten. "Direct proof of the tree-level scattering amplitude recursion relation in Yang-Mills theory". *Phys. Rev. Lett.* **94**.18 (2005), pp. 181602, 4. DOI.

[6] C. Even-Zohar, T. Lakrec, M. Parisi, R. Tessler, M. Sherman-Bennett, and L. Williams. "Cluster algebras and tilings for the m=4 amplituhedron". 2024. arXiv:2310.17727. Link.

[7] C. Even-Zohar, M. Parisi, M. Sherman-Bennett, R. Tessler, and L. Williams. "Plabic tangles and cluster promotion maps". *J. Algebra* **695** (2026), pp. 14–102. DOI.

[8] S. Fomin, L. Williams, and A. Zelevinsky. "Introduction to cluster algebras: Chapters 1-3". arXiv:1608.05735 [math.CO]. arXiv:1608.05735.

[9] S. Fomin and A. Zelevinsky. "Cluster algebras. I. Foundations". *J. Amer. Math. Soc.* **15**.2 (2002), 497–529 (electronic). DOI.

[10] C. Fraser. "Quasi-homomorphisms of cluster algebras". *Adv. in Appl. Math.* **81** (2016), pp. 40–77. DOI.

[11] V. F. R. Jones. "Planar algebras, I". *New Zealand J. Math.* **52** (2021 [2021–2022]), pp. 1–107. DOI.

[12] A. Knutson, T. Lam, and D. E. Speyer. "Positroid varieties: juggling and geometry". *Compos. Math.* **149**.10 (2013), pp. 1710–1752. Link.

[13] T. Lam. "Amplituhedron cells and Stanley symmetric functions". *Comm. Math. Phys.* **343**.3 (2016), pp. 1025–1037. DOI.

[14] T. Lam. "Totally nonnegative Grassmannian and Grassmann polytopes". *Current developments in mathematics 2014*. Int. Press, Somerville, MA, 2016, pp. 51–152.

[15] G. Lusztig. "Total positivity in reductive groups". *Lie theory and geometry*. Vol. 123. Progr. Math. Birkhäuser Boston, Boston, MA, 1994, pp. 531–568.

[16] A. Postnikov. "Total positivity, Grassmannians, and networks". 2006. arXiv:math/0609764. Link.

[17] A. Postnikov. "Positive Grassmannian and polyhedral subdivisions". *Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. IV. Invited lectures*. World Sci. Publ., Hackensack, NJ, 2018, pp. 3181–3211.

[18] J. Scott. "Grassmannians and cluster algebras". *Proc. London Math. Soc. (3)* **92**.2 (2006), pp. 345–380. DOI.

[19] N. J. Sloane et al. "The Online Encyclopedia of Integer Sequences". Link.