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Vanishing of Schubert coefficients in probabilistic polynomial time

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Abstract. The *Schubert vanishing problem* asks whether Schubert structure constants are zero. We give a complete solution of the problem from an algorithmic point of view, by showing that Schubert vanishing can be decided in probabilistic polynomial time.

Keywords: Schubert calculus, Schubert structure constants, Purbhoo's criterion, Schur P-functions, polynomial identity testing, randomized algorithm

1 Introduction

1.1 Vanishing of Schubert coefficients

Determining Schubert structure constants (*Schubert coefficients*) is one of the oldest and most celebrated problems in enumerative geometry, going back to Schubert's own work, see [35]. Motivated in part by *Hilbert's 15th Problem* aiming to make Schubert's work rigorous (see [14]), the area of Schubert calculus has exploded and developed rich connections with representation theory and algebraic combinatorics (see e.g. [1, 3]).

In this paper we study the *Schubert vanishing problem* which asks whether Schubert coefficients are zero. This problem has remained a major challenge for decades and remained unresolved despite significant study (see §1.2). We resolve it algorithmically, by showing that deciding Schubert vanishing can be done in probabilistic polynomial time. This is an ultimate result of a series of our previous papers [25, 26, 29].

We start with a general setup, see e.g. [4, 1] for the background. Let G be a connected and simply connected semisimple complex Lie group. Take $\mathsf{B} \subset \mathsf{G}$ and $\mathsf{B}_- \subset \mathsf{G}$ to be the Borel subgroup and opposite Borel subgroup, respectively. The *torus subgroup* is defined as $\mathsf{T} = \mathsf{B} \cap \mathsf{B}_-$. The *Weyl group* is defined as the normalizer $\mathcal{W} \cong N_{\mathsf{G}}(\mathsf{T})/\mathsf{T}$.

The *generalized flag variety* is defined as G/B. Recall that G/B has finitely many orbits under the left action of B_-. These are called *Schubert cells* and denoted Ω_w. Schubert cells are indexed by the Weyl group elements $w \in \mathcal{W}$.

The *Schubert varieties* X_w are the Zariski closures of Schubert cells Ω_w. The *Schubert classes* $\{\sigma_w\}_{w\in\mathcal{W}}$ are the Poincaré duals of Schubert varieties. These form a $\mathbb{Z}$–linear basis

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of the cohomology ring $H^*(\mathsf{G}/\mathsf{B})$. The *Schubert coefficients* $c_{u,v}^w$ are defined as structure constants:

$$\sigma_u \smile \sigma_v \;=\; \sum_{w \in \mathcal{W}} c_{u,v}^w \, \sigma_w \,. \tag{1.1}$$

Thus $c_{u,v}^w = [\sigma_{\mathrm{id}}]\, \sigma_u \smile \sigma_v \smile \sigma_{w_\circ w}$, where $w_\circ$ is the *long word* in $\mathcal{W}$. These are a special case of

$$c(u_1, u_2, \ldots, u_k) \;:=\; [\sigma_{\mathrm{id}}]\, \sigma_{u_1} \smile \sigma_{u_2} \smile \cdots \smile \sigma_{u_k}\,, \tag{1.2}$$

where $k \geq 3$. In particular, we have $c_{u,v}^w = c(u, v, w_\circ w)$. Additionally, these coefficients $c_{u,v}^w$ are the structure coefficients of *Schubert polynomials*, introduced by Lascoux–Schützenberger [21] for the $\mathsf{G} = \mathsf{GL}_n$ setting and generalized by Billey–Haiman [4] for classical groups G.

By *Kleiman transversality* [13], the coefficients $c(u_1, \ldots, u_k)$ count the number of points in the intersection of generically translated Schubert varieties:

$$c(u_1, \ldots, u_k) \;=\; \#\big\{\, X_{u_1}\big(F_\bullet^{(1)}\big) \cap \cdots \cap X_{u_k}\big(F_\bullet^{(k)}\big) \,\big\}, \tag{1.3}$$

where $F_\bullet^{(i)}$ are generic flags. In particular, we have $c(u_1, \ldots, u_k) \in \mathbb{N}$. The *Schubert vanishing problem* is the decision problem

$$\textsc{SchubertVanishing} \;:=\; \big\{ c(u_1, \ldots, u_k) \stackrel{?}{=} 0 \big\},$$

where $u_1, \ldots, u_k \in \mathcal{W}$. We consider the problem only for classical types $Y \in \{A, B, C, D\}$, and use notation $\textsc{SchubertVanishing}(Y)$ to denote the Schubert vanishing problem in type Y.[1] These correspond to groups $\mathsf{G} \in \{\mathsf{SL}_n(\mathbb{C}), \mathsf{SO}_{2n+1}(\mathbb{C}), \mathsf{Sp}_{2n}(\mathbb{C}), \mathsf{SO}_{2n}(\mathbb{C})\}$, respectively. We will use $Y_n \in \{A_n, B_n, C_n, D_n\}$, where $n = n(\mathsf{G})$ is the *rank* of Lie group G. Recall that $c(u_1, \ldots, u_k) = 0$ for $k > \ell(w_\circ)$, assuming each $u_i \neq \mathrm{id}$. Thus we consider only the case when $k \leq \ell(w_\circ)$, where ℓ denotes the *length function* in $\mathcal{W}$.

Theorem 1.1 (Main theorem**).** *For $Y \in \{A, B, C, D\}$, the problem $\textsc{SchubertVanishing}(Y)$ can be decided in probabilistic polynomials time. More precisely, for all $k \geq 3$ and $\varepsilon > 0$, there is a probabilistic algorithm which inputs elements $u_1, \ldots, u_k \in Y_n$ and after $O\big(kn^{8.75} \log \frac{1}{\varepsilon}\big)$ arithmetic operations outputs either :*

- *$c(u_1, \ldots, u_k) > 0$, which holds with probability $P = 1$, or*
- *$c(u_1, \ldots, u_k) = 0$, which holds with probability $P > 1 - \varepsilon$.*

The proof is based on Purbhoo's criterion for Schubert vanishing [32, Cor. 2.6]. We show that the criterion is equivalent to the degeneracy of certain determinant with polynomial entries. It can then be tested in polynomial time whether this determinant is identically zero by a random substitution of variables. This explains the one-sided error in the algorithm, since finding a nonzero evaluation of the determinant *guarantees* positivity of the Schubert coefficient. This is an extended abstract of [28].

[1] For non-classical types E_6, E_7, E_8, F_4 and G_2, there is only a finite number of Schubert coefficients, so the problem is uninteresting from the computational complexity point of view.

1.2 Geometric background and motivation

The literature on Schubert calculus is much too large to be reviewed here. We refer to [1, 8, 16, 22] for geometric and combinatorial introductions, and to [17] for an overview.

Recall the *Grassmannian structure coefficients* in $H^*(\mathsf{G}/\mathsf{P})$ are special cases of the general structure constants in $H^*(\mathsf{G}/\mathsf{B})$. Thus by specializing our algorithm to the appropriate Grassmannian elements, we obtain a probabilistic poly-time algorithm to decide the vanishing of Grassmannian structure constants. We use notation $c^{\nu}_{\lambda,\mu}(Y)$ to denote these constants in type Y for $k = 3$.

In type A, we have $\mathsf{G}/\mathsf{P} \simeq \mathsf{Gr}_{k,n}$ is the ordinary Grassmannian, the space of k-dimensional planes in $\mathbb{C}^n$. Here, the Schubert structure constants $c^{\nu}_{\lambda\mu} = c^{\nu}_{\lambda\mu}(A)$ are the *Littlewood–Richardson* (LR) *coefficients*, which are extremely well studied in the literature, see e.g. [8]. Famously, LR coefficients are the structure constants of Schur polynomials and have several combinatorial interpretations.

In a breakthrough, Knutson–Tao [18] established the *saturation conjecture* in type A:

$$c^{\lambda}_{\mu\nu}(A) > 0 \quad \Longleftrightarrow \quad c^{t\lambda}_{t\mu,t\nu}(A) > 0 \ \text{ for any } \ t \geq 1. \tag{1.4}$$

In [7, 23], the authors independently observed that the saturation property (1.4) implies that the vanishing of LR–coefficients can be solved by a linear program. This gives:

Theorem 1.2 ([18, 7, 23]). *The vanishing of LR-coefficients* $\{c^{\lambda}_{\mu\nu}(A) \stackrel{?}{=} 0\}$ *can be decided in deterministic polynomials time.*

This result is exceptional and despite several conjectural generalizations (see e.g. [12]), it does not seem to extend much beyond this setting.

The type B–D setting is also well studied. We refer to [4] for detailed overview of Schubert calculus in these types, to [33] for recursive Horn type formulas for the vanishing problem, and to [32] for detailed combinatorial vanishing investigation. See also [36] for discussion for non-maximal isotropic Grassmannians in types B–D.

In this case, we highlight those G/P that are maximal isotropic Grassmannians with respect to the appropriate skew–symmetric or symmetric bilinear form. In type C, we have $c^{\nu}_{\lambda,\mu}(C)$ as the structure constants of Q-Schur polynomials Q_λ [31]. Similarly, in types B/D, we have $c^{\nu}_{\lambda,\mu}(B)$ and $c^{\nu}_{\lambda,\mu}(D)$ as the structure constants of P-Schur polynomials P_λ, ibid. We refer to [11] for the extensive treatment of Q- and P-Schur polynomials in connection to projective representation theory of the symmetric group.

As noted in [34, Remark 7.7], the type B/C structure coefficients do not satisfy saturation.[2] Thus the arguments of [7, 23] may not be mirrored. In fact, our Theorem 1.1 gives the first poly-time algorithm for the vanishing of LR–coefficients in other types:

[2] We were unable to find in the literature a counterexample in type D.

Corollary 1.3. *For types $Y \in \{A, B, C, D\}$, the problem*

$$\text{LRVanishing}(Y) \ := \ \{c^{\lambda}_{\mu\nu}(Y) =^{?} 0\} \tag{1.5}$$

can be decided in probabilistic polynomials time when the input λ, μ, ν is in unary. More precisely, for Y_n, there is a probabilistic algorithm with one-sided error, and $O(n^{8.75})$ expected time.

1.3 Prior work

The literature on the vanishing of Schubert coefficients and its various extensions is too extensive to be fully reviewed. We summarize some relevant highlights below. Although many of these conditions hold for arbitrary types Y, we focus on the most well-studied setting: type A and $k = 3$.

(1) First, we overview sufficient conditions for the vanishing of coefficients $c^{w}_{u,v}$: [3]

- the *number of inversion condition* $\text{inv}(u) + \text{inv}(v) \neq \text{inv}(w)$ [21],
- the *number of descents condition* $\text{Des}(w) > \text{Des}(u) + \text{Des}(v)$ [21],
- *strong Bruhat order condition* $u \not\leqslant w$, see e.g. [39, §5.1],
- Knutson's *descent cycling condition* $\text{Des}(u) \cap \text{Des}(v) \cap \text{Des}(ww_{\circ}) \neq \varnothing$ [15],
- Billey–Vakil's *permutation array condition* [5, Thm 5.1],
- St. Dizier–Yong's condition on certain fillings of Rothe diagrams [39, Thm A], and
- Hardt–Wallach's condition on empty rows in Rothe diagrams [9, Cor. 5.12].

In addition to Grassmannian permutations, there are several other classes of permutations for which the Schubert coefficients have a proven combinatorial interpretation. In such cases, the combinatorial interpretation can provide sufficient conditions for non-vanishing. Notable examples of sufficient conditions include:

- Purbhoo's *root game condition* [32], and
- Knutson and Zinn-Justin's *tiling conditions* [19, 20].

(2) For purposes of computation, one may formulate algebraic systems for Schubert problems. Notably, in [5, Thm 5.4], Billey and Vakil give an algebraic system with exactly $c^{w}_{u,v}$ solutions for generic values of certain variables. Further, they describe the system of conditions for these choices being generic, assuming the set of solutions is 0-dimensional [5, Cor. 5.5]. In [10], Hein and Sottile introduced an algebraic system giving a practical algorithm for computing Schubert coefficients. This Hein–Sottile system has more variables compared to the Billey–Vakil system, and the Hein–Sottile equations have smaller (polynomial) size. Their system also assumes generic values of certain variables.

(3) Finally, we briefly summarize the previous understanding of the computational complexity of Schubert vanishing. From the literature, it follows that SchubertVanishing $\in$ PSPACE, although that bound was never explicitly written. In type A and $k = 3$, this was

[3] See §2.1 for notation and [1] for additional background.

observed by Morales as a consequence of the Postnikov–Stanley formula [30, Cor. 17.13], see [24, §10] for details. In combinatorial terminology, this formula shows $c^w_{u,v}$ has a *signed combinatorial interpretation* derived from the (usual) combinatorial interpretation for the Schubert–Kostka numbers in terms of pipe dreams [2].

For general $k \geq 4$, the result follows by taking convolutions. For other classical types, one can follow the approach above and combine the pipe dream construction in [38] with the effective Möbius inversion in [27, §2.2]. Another way to see that Schubert vanishing is in PSPACE is to use the recursive, but type-independent *Billey's formula* [6, Eq. (5.5)].

In recent work [29, Thm 1.1], we give a significant improvement:

$$(*) \qquad \textsc{SchubertVanishing}(Y) \in \mathsf{AM} \cap \mathsf{coAM} \quad \text{for all} \quad Y \in \{A, B, C, D\},$$

assuming the *Generalized Riemann Hypothesis*. The results in this abstract unconditionally strengthen the above result.

2 Preliminaries

2.1 Notation

We use $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $[n] = \{1, \ldots, n\}$. Unless stated otherwise, the underlying field is always $\mathbb{C}$. Let $e_1, \ldots, e_n$ denote the standard basis in $\mathbb{C}^n$. We use bold letters $\boldsymbol{x} = (x_1, x_2, \ldots)$ for collections of variables, and $\overline{x}$ for their evaluations. We write $f \equiv g$ for the equality of polynomials $f, g \in \mathbb{C}[\boldsymbol{x}]$.

For a Weyl group $\mathcal{W}$, we use $\ell(w)$ to denote the *length* of the element $w \in \mathcal{W}$ and $\mathrm{inv}(w)$ to denote its inversion set. For $\mathcal{W} = S_n$, let $\mathsf{Des}(w)$ denote the set of descents of w. The *long word* is an element $w_\circ \in \mathcal{W}$ of maximal length. We assume the reader is familiar with standard notation of barred and unbarred elements of the Weyl group $\mathcal{W} \simeq S_n \ltimes \mathbb{Z}_2^n$ in types B and C. We view this Weyl group as the group of *signed permutations* of $[n]$, see e.g. [1, §14.1.1] for further details.

Recall the following standard notation for almost simple algebraic groups. We have the *special linear group* $\mathsf{SL}_n(\mathbb{C})$, the *odd special orthogonal group* $\mathsf{SO}_{2n+1}(\mathbb{C})$, the *symplectic group* $\mathsf{Sp}_{2n}(\mathbb{C})$ and the *even special orthogonal group* $\mathsf{SO}_{2n}(\mathbb{C})$. These groups correspond to *root systems* A_n, B_n, C_n and D_n, and are called *groups of type* A, B, C and D, respectively.

To distinguish the types, we use parentheses or subscripts in LR and Schubert coefficients, e.g. $c^\lambda_{\mu\nu}(A)$ and $c_{\langle A \rangle}(u, v, w)$. We omit the dependence on the type when it is clear from the context.

2.2 Polynomial identity testing

In this paper we use the following textbook result:

Lemma 2.1 (Schwarz–Zippel Lemma)**.** *Let $Q \in \mathbb{C}[x_1, x_2, \ldots, x_n]$ be a non-zero polynomial with degree $d \geq 0$ over $\mathbb{C}$. Take $S \subset \mathbb{C}$ be a finite set. Then:*

$$\mathbf{P}\big[Q(c_1, c_2, \ldots, c_n) = 0\big] \ \leq \ \frac{d}{|S|},$$

where the probability is over random, independent and uniform choices of $c_1, c_2, \ldots, c_n \in S$.

This lemma is frequently used to test whether a polynomial given by an arithmetic circuit is identically zero. We refer to [37] for an extensive overview of complexity applications and many references.

2.3 Purbhoo's criterion

Take type $Y \in \{A, B, C, D\}$ and let $\mathsf{G} = \mathsf{G}_Y$ be the simply connected semisimple complex Lie group of type Y. Here G is a matrix group inside an ambient vector space V. Let N be the subgroup of unipotent matrices, giving

$$\mathsf{N} \subset \mathsf{B} \subset \mathsf{G} \subset V.$$

Let $\mathfrak{n}$ denote the Lie algebra of N. Again, we view $\mathfrak{n}$ as a subspace of V. Finally, for a Weyl group element $w \in \mathcal{W}$, let $R_w := \mathfrak{n} \cap (w\mathsf{B}_- w^{-1})$.

Lemma 2.2 (Purbhoo's criterion [32, Cor. 2.6])**.** *For generic $\rho_1, \ldots, \rho_k \in \mathsf{N} \subset \mathsf{G}$ and $u_1, \ldots, u_k \in \mathcal{W}$, we have:*

$$c(u_1, \ldots, u_k) \ > \ 0 \quad \Longleftrightarrow \quad \rho_1 R_{u_1} \rho_1^{-1} + \ldots + \rho_k R_{u_k} \rho_k^{-1} = \rho_1 R_{u_1} \rho_1^{-1} \oplus \ldots \oplus \rho_k R_{u_k} \rho_k^{-1}.$$

Generalizing the number of inversions condition, the *dimension condition* says that

$$c(u_1, \ldots, u_k) \ = \ 0 \quad \text{if} \quad \operatorname{inv}(u_1) + \cdots + \operatorname{inv}(u_k) \ \neq \ \dim(\mathfrak{n}). \tag{2.1}$$

Thus it suffices to restrict to the case $\operatorname{inv}(u_1) + \ldots + \operatorname{inv}(u_k) = \dim(\mathfrak{n})$. Then we consider the following specialization of Lemma 2.2:

Corollary 2.3. *For generic $\rho_1, \ldots, \rho_k \in \mathsf{N} \subset \mathsf{G}$ and $u_1, \ldots, u_k \in \mathcal{W}$ such that $\operatorname{inv}(u_1) + \ldots + \operatorname{inv}(u_k) = \dim(\mathfrak{n})$, we have:*

$$c(u_1, \ldots, u_k) \ > \ 0 \quad \Longleftrightarrow \quad \rho_1 R_{u_1} \rho_1^{-1} + \ldots + \rho_k R_{u_k} \rho_k^{-1} = \mathfrak{n}.$$

Using Corollary 2.3, it suffices to determine the dimension of the vector space

$$H \ := \ \rho_1 R_{u_1} \rho_1^{-1} + \ldots + \rho_k R_{u_k} \rho_k^{-1} \quad \text{for generic} \quad \rho_i.$$

3 General setup

In the setting of Purbhoo's criterion, we describe how to construct bases for R_{u_i} in §3.1. Then, in §3.2, we describe how to construct matrices ρ_i. In §3.3, we combine these constructions to obtain bases for each summand $\rho_i R_{u_i} \rho_i^{-1}$. From these, we obtain vectors π_j which generate H.

3.1 Root systems

The Weyl group $\mathcal{W}$ is generated by reflections r_γ, indexed by roots γ in a *root system* Φ. This root system Φ may be partitioned in terms of its positive and negative roots: $\Phi = \Phi_+ \sqcup \Phi_-$. In Table 1, we describe the positive roots Φ_+ in term of vectors e_i. Here e_i denotes the i-th elementary basis vector in the appropriate $\mathbb{C}^m$.

G	Φ_+	$U(\mathsf{G})$
SL_n	$\{e_i - e_j \,:\, 1 \le i < j \le n\}$	$\{(i,j) \,:\, 1 \le i < j \le n\}$
SO_{2n+1}	$\{e_i \pm e_j \,:\, 1 \le i < j \le n\} \cup \{e_i \,:\, i \in [n]\}$	$\{(i,j) \,:\, 1 \le i < j \le 2n+1-i\}$
SO_{2n}	$\{e_i \pm e_j \,:\, 1 \le i < j \le n\}$	$\{(i,j) \,:\, 1 \le i < j \le 2n-i\}$

Table 1: Positive roots and corresponding matrix entries.

Define the integer $N(\mathsf{G})$, where

$$N(\mathsf{G}) \;=\; \begin{cases} n & \text{if } \mathsf{G} = \mathsf{SL}_n(\mathbb{C}), \\ 2n+1 & \text{if } \mathsf{G} = \mathsf{SO}_{2n+1}(\mathbb{C}), \\ 2n & \text{if } \mathsf{G} = \mathsf{SO}_{2n}(\mathbb{C}). \end{cases}$$

To each $\gamma \in \Phi_+$, we may associate an $m \times m$ matrix, where $m = N(\mathsf{G})$. We define a distinguished subset $U(\mathsf{G}) \subset [m] \times [m]$ as outlined in Table 1. We then construct a bijection $\phi : U(\mathsf{G}) \to \Phi_+$, detailed below.

(A) For SL_n take $\phi(i,j) := e_i - e_j$.

(B) For SO_{2n+1} take

$$\phi(i,j) \;:=\; \begin{cases} e_i + e_j & \text{if } j \le n, \\ e_i - e_{2n+2-j} & \text{if } n+1 < j, \\ e_i & \text{if } j = n+1. \end{cases}$$

(D) For SO_{2n} take

$$\phi(i,j) := \begin{cases} e_i + e_j & \text{if } j \le n, \\ e_i - e_{2n+1-j} & \text{if } n < j. \end{cases}$$

For $\gamma \in \Phi_+$, set E'_γ to be the $m \times m$ matrix with a 1 in position $\phi^{-1}(\gamma)$ and 0 elsewhere. For SL_n take $\mathrm{E}_\gamma := \mathrm{E}'_\gamma$. For SO_{2n+1} and SO_{2n}, define $\mathrm{E}_\gamma := \mathrm{E}'_\gamma - \mathrm{D}_m(\mathrm{E}'_\gamma)^T\mathrm{D}_m$, where D_m is the antidiagonal matrix.

3.2 Generic unipotent subgroup elements

Let $m := N(\mathsf{G})$. We now describe how to construct an upper unitriangular $m \times m$ matrix K which lies in $\mathsf{N} \subset \mathsf{B} \subset \mathsf{G}$. Define:

$$\kappa_{ij} = \begin{cases} \alpha_{ij} & \text{if } i < j \text{ and } (i,j) \in U(\mathsf{G}), \\ z_{ij} & \text{if } i < j\,,\ (i,j) \notin U(\mathsf{G}), \text{ and } i + j \neq m+1, \\ 0 & \text{otherwise.} \end{cases}$$

Here we treat α_{ij} as parameters set $z_{ij} = -\alpha_{(m+1-j)(m+1-i)}$. Let $\kappa = (\kappa_{ij})$.

For $\mathsf{G} = \mathsf{SL}_n(\mathbb{C})$, set $\mathsf{K} := \mathrm{I}_m + \kappa$. For $\mathsf{G} = \mathsf{SO}_{2n+1}(\mathbb{C})$ and $\mathsf{G} = \mathsf{SO}_{2n}(\mathbb{C})$, we use the Cayley transform to construct $\mathsf{K} \in \mathsf{G}$ from κ. In particular, set $\mathsf{K} = (\mathrm{I}_m + \kappa)^{-1}(\mathrm{I}_m - \kappa)$. By construction, K is upper unitriangular. It is straightforward to confirm $\mathsf{K} \in \mathsf{G}$ and that such elements are dense in N. To check $\mathsf{K} \in \mathsf{G}$, one need only confirm $\mathsf{K}^T \cdot \mathrm{D}_m \cdot \mathsf{K} = \mathrm{D}_m$, and note that $\det(\mathsf{K}) = 1$.

3.3 Main construction

Let $m = N(\mathsf{G})$. With Corollary 2.3 in mind, consider the vector space

$$H \,=\, \rho_1 R_{u_1} \rho_1^{-1} \,+\, \ldots \,+\, \rho_k R_{u_k} \rho_k^{-1}.$$

Let $d := \dim \mathsf{G}/\mathsf{B}$ and note $\dim \mathfrak{n} = d$. Using the dimension condition, we assume

$$\mathrm{inv}(u_1) \,+\, \ldots \,+\, \mathrm{inv}(u_k) \,=\, d. \tag{3.1}$$

Additionally, we assume $\mathrm{inv}(u_i) \ge 1$ for each $i \in [k]$. Combining these assumptions gives $k \le d$. We also assume $k \ge 3$, as taking $k \le 2$ is trivial.

First recall the definition, $R_w = \mathfrak{n} \cap (w\mathsf{B}_- w^{-1})$ for $w \in \mathcal{W}$. Alternatively, R_w is the subspace of $\mathfrak{n}$ generated by basis elements E_γ (see §3.1) for $\gamma \in \Phi_+(w)$, where

$$\Phi_+(w) \,:=\, \big\{\beta \in \Phi_+ \,:\, w^{-1}\beta \notin \Phi_+\big\}.$$

Now we construct bases for the spaces R_{u_i} for $i \in [k]$:

$$S_{u_i} := \{x_{\gamma,i} \mathrm{E}_\gamma \,:\, \gamma \in \Phi_+(u_i)\}.$$

Note the number of positive roots $|\Phi_+| = d = O(m^2)$. Let $\boldsymbol{x} := \{x_{\gamma,i}\}$ be the set of those variables appearing in the above collection. Since (2.1) holds, we have the following:

$$\sum_{i=1}^{k} \mathrm{inv}(u_i) = \sum_{i=1}^{k} \dim(R_{u_i}) = \sum_{i=1}^{k} |S_{u_i}| = d. \tag{3.2}$$

Then construct generic matrices $\rho_1, \ldots, \rho_k \in \mathsf{N}$ as outlined in §3.2, using the formal parameters $\alpha_{j\ell}^{(i)}$. Define $\overline{\alpha} := \{\alpha_{j\ell}^{(i)}\}$ to be the set of parameters, respectively, appearing in some ρ_i, where $i \in [k]$. Then $|\overline{\alpha}| \leq k \cdot m^2 = O(m^4)$.

For each $i \in [k]$, construct bases for summands $\rho_i R_{u_i} {\rho_i}^{-1}$:

$$T_{u_i} := \rho_i S_{u_i} {\rho_i}^{-1} = \{\rho_i \cdot g \cdot {\rho_i}^{-1} \,:\, g \in S_{u_i}\}.$$

Using (3.2), we find $|T_{u_1}| + \ldots + |T_{u_k}| = d$.

Let τ be the map on $m \times m$ matrices defined by restricting to the matrix entries in positions $U(\mathsf{G})$. By their definition, matrices in $\mathfrak{n}$ are determined by their entries in positions $U(\mathsf{G})$. Thus, $\dim(T_{u_i}) = \dim\left(\tau(T_{u_i})\right)$ for each $i \in [k]$. Take

$$T := \bigcup_{i \in [k]} \tau(T_{u_i}),$$

and let $T = \{\pi_i \,:\, i \in [d]\}$. Using the fact that $|U(\mathsf{G})| = d$, we view each $\pi_i \in T$ as a d-vector. Finally, consider the $d \times d$ matrix M with column vectors π_j. Using Purbhoo's criterion, the Schubert vanishing problem reduces to determining if the matrix M is singular. Since M is a matrix with entries polynomials in $\mathbb{Z}[\overline{\alpha}, \boldsymbol{x}]$, we can apply the Schwarz–Zippel Lemma 2.1.

3.4 Algorithm outline for Theorem 1.1

We give a concise outline of the algorithm in all types A, B/D. Note that the type B and C vanishing problems are equivalent by [4, Thm 3].

Input: $u_1, \ldots, u_k \in \mathcal{W}$
Decide: $\left[c(u_1, \ldots, u_k) =^? 0\right]$

- Let

$$p := \begin{cases} \frac{3}{2}n(n^2-1)+1 & \text{if} \quad \mathsf{G} = \mathsf{SL}_n \\ 3\left\lfloor \frac{m}{2} \right\rfloor^2 (2m+1)+1 & \text{if} \quad \mathsf{G} = \mathsf{SO}_m \end{cases}$$

- For all $i \in [k]$:
  - Generate strictly upper triangular matrices κ_i with random entries $\alpha_{j\ell}^{(i)} \in [p]$, see §3.2.
  - Compute K_i using κ_i and set $\rho_{u_i} := \mathsf{K}_i$, see §3.2.
  - Compute inverse matrices $\rho_{u_i}^{-1}$.
- For all $i \in [k]$ and $\gamma \in \Phi_+(u_i)$:
  - Compute matrices $\overrightarrow{x_{\gamma,i}}\,\mathrm{E}_\gamma$ with random values $\overrightarrow{x_{\gamma,i}} \in [p]$, see §3.1.
  - Compute matrices $T_{\gamma,i} = \rho_{u_i}\big(\overrightarrow{x_{\gamma,i}}\,\mathrm{E}_\gamma\big)\rho_{u_i}^{-1}$.
  - Record the entries of $T_{\gamma,i}$ in positions $U(\mathsf{G})$ as a vector $v_{\gamma,i}$.[4]
- Let M be the matrix with column vectors $v_{\gamma,i}$, over all $\gamma \in \Phi_+(u_i)$ and $i \in [k]$.

Output:

$$\begin{cases} c(u_1,\ldots,u_k) \;=\; 0 & \text{if} \;\; \det(M) = 0, \\ c(u_1,\ldots,u_k) \;>\; 0 & \text{if} \;\; \det(M) \neq 0. \end{cases}$$

The algorithm above is always correct when it outputs $[c(u_1,\ldots,u_k) > 0]$ and has a probability of error $\leq \frac{1}{3}$ when it outputs $[c(u_1,\ldots,u_k) = 0]$. Repeating the algorithm s times reduces this probability to $\frac{1}{3^s} \leq \varepsilon$ for $s = \lceil \log_3 \frac{1}{\varepsilon} \rceil$.

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[4] Formally, we need to do this under a fixed order. Take, e.g., a lexicographic order on matrix positions.

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