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On an extension problem on the moment curve

Seunghun Lee*[1] and Eran Nevo†[2]

[1]*Department of Mathematics, Keimyung University, Daegu, South Korea.*
[2]*Mathematics Research Institute, Universidad de Valladolid, Spain and Einstein Institute of Mathematics, Hebrew University of Jerusalem, Israel, and CMSA, Harvard University, Cambridge, USA.*

Abstract. We show that for $2 \leq d \leq 4$, every finite geometric simplicial complex Δ in $\mathbb{R}^d$ with vertices on the moment curve can be extended to a triangulation T of the cyclic polytope C where Δ, T and C all have the same vertex set. Further, for $d \geq 5$ we construct complexes Δ for which no such triangulations T exist.

Keywords: moment curve, higher Stasheff-Tamari poset, triangulation, cyclic polytope, extension.

1 Introduction

We consider the following natural extension problem on the *moment curve*

$$\gamma_d = \{(t, t^2, \ldots, t^d) : \ t \in \mathbb{R}\}.$$

Question 1.1. *Does* every *geometric simplicial complex in* $\mathbb{R}^d$ *on a finite point set* $A \subseteq \gamma_d$ *extend to a triangulation of* $\mathrm{conv}(A)$ *without adding new vertices?*

Let us remark that this problem is in fact a combinatorial one, as intersections correspond to interlacing patterns (see Proposition 2.1) which are forbidden, and boundary faces are identified by the Gale evenness condition (e.g. [9]) and one checks that exactly these $(d-1)$-faces are contained in a unique d-simplex.

As we shall see, the answer to Question 1.1 depends on the dimension d. Considering a more general situation first, note that for $d = 2$ every simplicial complex on a finite point set $A \subseteq \mathbb{R}^2$ extends to a triangulation of $\mathrm{conv}(A)$, while for $d = 3$ there are finite point sets $A \subseteq \mathbb{R}^3$ in convex general position, and simplicial complexes on them, which do not extend to a triangulation of $\mathrm{conv}(A)$ without adding new vertices. As a simple

*seunghun.math@kmu.kr. Seunghun Lee was partially supported by the Institute for Basic Science (IBS-R029-C1) and the Department of Mathematical Sciences of Korea Advanced Institute of Science and Technology (BK21). This work was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MSIT) (RS-2026-25473635).

†nevo@math.huji.ac.il. Eran Nevo was partially supported by Israel Science Foundation grants ISF-2480/20 and ISF-687/24.

6-vertex example consider the Schönhardt's polyhedron [17], see [6, Lemma 3.6.2, Figure 3.43].

Perhaps surprisingly, we prove that the answer to Question 1.1 is Yes iff $d \leq 4$: For an affinely independent set $\sigma \subset \mathbb{R}^d$ of size at most $d+1$, we will frequently refer to the geometric simplex $\mathrm{conv}(\sigma)$ as σ when it is appropriate. We say that two simplices σ_1 and σ_2 in $\mathbb{R}^d$ *overlap* if the intersection of the simplices is not a face of each of them, namely

$$\mathrm{conv}(\sigma_1) \cap \mathrm{conv}(\sigma_2) \supsetneq \mathrm{conv}(\sigma_1 \cap \sigma_2).$$

In this definition, non-overlapping simplices may intersect, but only in a common face.

For convenience in the proof we use a different notation for dimension in our main result:

Theorem 1.2. *(i) For every $D \leq 4$ and every finite collection $\mathcal{F}$ of pairwise non-overlapping simplices in $\mathbb{R}^D$ on $A \subseteq \gamma_D$, $\mathcal{F}$ can be extended into a triangulation T of the cyclic polytope $\mathrm{conv}(A)$ such that A is exactly the vertex set of T.*

(ii) For every $D \geq 5$ and $n \geq D+3$, there exists a collection of pairwise non-overlapping D-simplices in $\mathbb{R}^D$ exactly on an n-vertex set $A \subseteq \gamma_D$ which cannot be extended into a triangulation of $\mathrm{conv}(A)$ without adding new vertices.

In order to prove Theorem 1.2(i) we will use the known facts that for $d = 2, 3$ the higher Stasheff-Tamari poset $\mathrm{HST}(n, d)$ is a lattice; see [10] and [8, Thm.3.6] respectively and Theorem 2.8. In order to prove Theorem 1.2(ii) we construct an example with $D = 5$ and $n = 8$, and then lift it to all $D \geq 5$ and $n \geq D+3$.

Previous studies. The combinatorics of triangulations of the cyclic polytope $C(n, d)$ has been extensively studied. Edelman and Reiner [8] defined two partial orders on this set of triangulations; one is given by bistellar flips and the other by comparing the "heights of the liftings in $\mathbb{R}^{d+1}$" for those triangulations. These two orders define the so-called *(first and second) higher Stasheff-Tamari poset*. They showed that these two orders are equal for $d = 2, 3$. Since then there have been active research on this topic [16, 7, 18, 15, 19, 20]. Recently the two orders were proved to be the same for every dimension d, or equivalently, the intermediate space between any two sections can be triangulated in dimension $d+1$, by Williams [21]. Our result is a stronger version of this when $d = 2, 3$.

The combinatorial criterion for having overlapping simplices on the moment curve (see Proposition 2.1) is well-known and is one of the ways of defining the *alternating oriented matroids* [2, Section 9.4]. It is also closely related to recent works on geometric hypergraph colorings and their representations where a similar combinatorial property called the $(AB)^{l/2}$*-freeness* was introduced and has been studied [1, 11, 12, 5].

Outline. In Section 2 we review the higher Stasheff-Tamari poset $\mathrm{HST}(n, d)$, which is a partial order on the triangulations of the cyclic polytope $C(n, d)$, and analogously define a partial order on simplices in $\mathbb{R}^d$ by lifting to the moment curve one dimension

higher. In Section 3 we prove Theorem 1.2; to prove (i) in Subsection 3.1 we use the existence of certain triangulations in $\mathrm{HST}(n,2)$ and $\mathrm{HST}(n,3)$ – the proof sketch of the existence is shown right after that. Theorem 1.2(ii) is proved in Subsection 3.2 by means of constructions. In Section 4 we discuss complexity aspects of this work and end with related open problems.

2 Ordering simplices and triangulations by height

2.1 Ordering simplices by height

Since combinatorial properties of finite point sets $\gamma_d(t_1), \gamma_d(t_2), \ldots, \gamma_d(t_n)$ with $t_1 < t_2 < \cdots < t_n$ depend solely on the order, we will frequently abuse notation and refer to the point $\gamma_d(t_i)$ by its index i; so the whole point set is identified with $[n] := \{1, 2, \ldots, n\}$. *The* cyclic polytope is defined as $C(n,d) = \mathrm{conv}\{\gamma_d(i) : i \in [n]\}$ on the vertex set $[n]$. We similarly define $C(V,d)$ for a subset $V \subseteq [n]$.

For subsets σ and τ of $[n]$, we say that σ and τ are *k-interlacing* if there is a sequence $v_1 < v_2 < \cdots < v_k$ of elements of $\sigma \cup \tau$ which satisfies one of the following alternating conditions:

$$v_1 \in \sigma, v_2 \in \tau, v_3 \in \sigma \ldots \text{ (beginning with an element of } \sigma\text{), or} \qquad (\sigma)$$
$$v_1 \in \tau, v_2 \in \sigma, v_3 \in \tau \ldots \text{ (beginning with an element of } \tau\text{).} \qquad (\tau)$$

The following fact gives a combinatorial criterion for the overlapping condition on the moment curve and is well-known (for a proof, see e.g. [13, Lemma 2.5]).

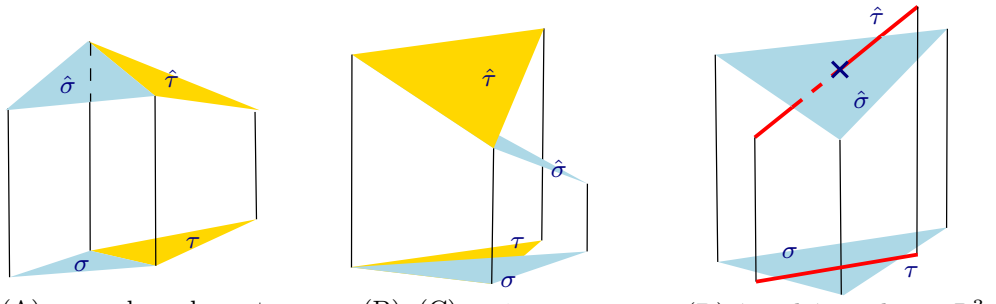
Proposition 2.1. *Simplices σ and τ on γ_d overlap in $\mathbb{R}^d$ if and only if they are $(d+2)$-interlacing.*

For $k \leq d+1$ and a simplex $\sigma = \{\gamma_d(t_1), \ldots, \gamma_d(t_k)\}$ on γ_d, we define the *height function* of σ, $h_\sigma : \mathrm{conv}(\sigma) \to \mathbb{R}$, by setting $h_\sigma(p)$, for each $p \in \mathrm{conv}(\sigma)$, to be the last coordinate of the point in $\mathrm{conv}(\hat{\sigma})$ corresponding to p via projection, where $\hat{\sigma} = \{\gamma_{d+1}(t_1), \ldots, \gamma_{d+1}(t_k)\}$. We call $\hat{\sigma}$ the *lifting of* σ. We define a relation $<_{d+1}$ on the set of simplices on γ_d. For two simplices $\sigma, \tau \subseteq \gamma_d$, we have $\sigma <_{d+1} \tau$ if and only if σ and τ overlap in $\mathbb{R}^d$ and $h_\sigma \leq h_\tau$ in the common domain.

Proposition 2.2. *For simplices σ and τ on γ_d, $\sigma <_{d+1} \tau$ if and only if σ and τ are $(d+2)$-interlacing by*

- *satisfying (σ) but not (τ) when d is even, and*
- *satisfying (τ) but not (σ) when d is odd.*

From Propositions 2.1 and 2.2, we have the following corollary, see Figure 1.



(A) σ and τ do not overlap in $\mathbb{R}^2$.

(B), (C) $\sigma <_3 \tau$

(D) $\hat{\sigma}$ and $\hat{\tau}$ overlap in $\mathbb{R}^3$.

Figure 1: Illustration of the mutually exclusive cases (A)-(D).

Corollary 2.3. *Given a pair of simplices σ and τ on γ_d, one of the following four mutually exclusive cases occurs:*

(A) σ and τ do not overlap in $\mathbb{R}^d$, or equivalently, σ and τ are not $(d+2)$-interlacing.

(B) $\sigma <_{d+1} \tau$, or equivalently, σ and τ are $(d+2)$-interlacing by satisfying the combinatorial condition of Proposition 2.2.

(C) $\sigma >_{d+1} \tau$, or equivalently, σ and τ are $(d+2)$-interlacing by satisfying the combinatorial condition of Proposition 2.2 with the roles of σ and τ interchanged.

(D) the liftings $\hat{\sigma}$ and $\hat{\tau}$ overlap in $\mathbb{R}^{d+1}$, or equivalently, σ and τ are $(d+3)$-interlacing.

For simplices $\sigma, \tau \subseteq \gamma_d$, we say $\sigma \preceq_{d+1} \tau$ if

- $\sigma = \tau$, or
- there is a positive integer k and a sequence of simplices $\sigma = \sigma_0 <_{d+1} \sigma_1 <_{d+1} \cdots <_{d+1} \sigma_k = \tau$ on γ_d.

Lemma 2.4. *The relation $\preceq_{d+1}$ is a partial order on the simplices on γ_d.*

Lemma 2.4 is used in the proof of Theorem 1.2(i).

Note that we never have $\tau <_{d+1} \sigma$ when $\sigma \preceq_{d+1} \tau$. It is easy to find examples of non-overlapping (in $\mathbb{R}^d$) $\sigma, \tau \subseteq \gamma_d$ with $\sigma \preceq_{d+1} \tau$. It is also not very difficult to find $\sigma, \tau \subseteq \gamma_d$ with $\sigma \preceq_{d+1} \tau$ and that the liftings $\hat{\sigma}$ and $\hat{\tau}$ overlap in $\mathbb{R}^{d+1}$; these suggest that there is subtle difference between the "above-below" relation $<_{d+1}$ and the order $\preceq_{d+1}$:

Example 2.5. Consider the following simplices on γ_2: $\sigma = \{1,4,6\}$, $e = \{2,7\}$ and $\tau = \{3,5,8\}$. Then $\sigma <_3 e <_3 \tau$ (so in particular $\sigma \preceq_3 \tau$) but σ and τ are 6-interlacing.

2.2 Height of a triangulation and submersion sets

Let $S(n,d)$ be the set of all triangulations on vertex set $[n]$ of the cyclic polytope $C(n,d)$.

For a triangulation $T \in S(n,d)$, we can similarly define the height function $h_T : C(n,d) \to \mathbb{R}$ as the union of $h_\sigma : \mathrm{conv}(\sigma) \to \mathbb{R}$ for every $\sigma \in T$, that is, if a point $p \in \mathrm{conv}(\sigma)$, we let $h_T(p) = h_\sigma(p)$. For a triangulation $T \in S(n,d)$ and a simplex $\sigma \subseteq \gamma_d$, we say $T \leq_{d+1} \sigma$ ($\sigma \leq_{d+1} T$ resp.) if $h_T(p) \leq h_\sigma(p)$ ($h_\sigma(p) \leq h_T(p)$ resp.) for every point $p \in \mathrm{conv}(\sigma)$. By definition, for a triangulation $T \in S(n,d)$ and a simplex $\sigma \subseteq \gamma_d$, $\sigma \leq_{d+1} T$ if and only if for every $\tau \in T$ either σ and τ do not overlap in $\mathbb{R}^d$ or $\sigma <_{d+1} \tau$. A similar statment holds for $\sigma \geq_{d+1} T$.

The following generalizes [8, Propositions 3.2 and 4.1].

Lemma 2.6. *For every triangulation $T \in S(n,d)$ and every simplex $\sigma \subseteq \gamma_d$ there holds: $\sigma \leq_{d+1} T$ if and only if there are no $\sigma' \subseteq \sigma$ and $\tau \in T$ satisfying $\tau <_{d+1} \sigma'$ with $|\sigma'| = \lceil d/2 \rceil + 1$ and $|\tau| = \lfloor d/2 \rfloor + 1$.*

For two triangulations $T, T' \in S(n,d)$, say $T \leq_{d+1} T'$ if $h_T(p) \leq h_{T'}(p)$ for every point $p \in C(n,d)$. This relation $\leq_{d+1}$ is a partial order on $S(n,d)$, and was introduced by Edelman and Reiner [8] as the *second higher Stasheff-Tamari order*. This order (which coincides with their *first* higher Stasheff-Tamari order by [21]) defines the *higher Stasheff-Tamari poset* $\mathrm{HST}(n,d) = (S(n,d), \leq_{d+1})$. We similarly define $\mathrm{HST}(V,d)$ for a subset $V \subseteq [n]$.

The following result by Rambau [16, Theorem 1.1] will also be useful.

Theorem 2.7. *There is a surjective map Ψ from the set of maximal chains of $\mathrm{HST}(n,d)$ to $S(n,d+1)$ such that for a maximal chain $\mathcal{C} : T_1 \leq_{d+1} T_2 \leq_{d+1} \cdots \leq_{d+1} T_k$ in $\mathrm{HST}(n,d)$, the d-skeleton of $\Psi(\mathcal{C})$ is exactly the union of the liftings of all T_i (which in turn determines $\Psi(\mathcal{C})$).*

For a triangulation $T \in S(n,d)$, the *submersion set* $\mathrm{sub}(T)$ is the set

$$\mathrm{sub}(T) = \{\sigma \subseteq \gamma_d : \sigma \leq_{d+1} T, \dim(\sigma) = \lceil d/2 \rceil\}.$$

Submersion sets were used to provide the following encoding result which in particular implies that $\mathrm{HST}(n,d)$ is a lattice for $d = 2, 3$, see [8, Theorems 3.6, 4.9] and [10].

Theorem 2.8. *For $d = 2$ or 3, the poset map Φ from $\mathrm{HST}(n,d)$ to the poset $(\mathcal{P} = 2^{\binom{[n]}{\lceil d/2 \rceil + 1}}, \subseteq)$ given by $\Phi : T \mapsto \mathrm{sub}(T)$ is an injection, and the subposet of $\mathcal{P}$ induced by the image of Φ is a lattice where the meet $\mathrm{sub}(T_1) \wedge \mathrm{sub}(T_2)$ is given by the intersection $\mathrm{sub}(T_1) \cap \mathrm{sub}(T_2)$. In particular, $\mathrm{HST}(n,d)$ is a lattice.*

In order to prove that Φ is injective, one of the key ideas in [8] is to extract "maximal" elements from *diagonal-closed* or *triangle-closed* families, and (re)construct the triangulation from them. We reformulate this idea for our purpose.

Lemma 2.9. *For $T \in S(n,d)$ and a simplex $\tau \subseteq \gamma_d$ with $\tau \leq_{d+1} T$, τ is a face of T if and only if there are no $\sigma \in \mathrm{sub}(T)$ such that $\tau <_{d+1} \sigma$.*

3 Proof of Theorem 1.2

3.1 Proof of Theorem 1.2(i)

The following theorems are useful in the proof of Theorem 1.2(i); their proof are sketched later, after they are used.

Theorem 3.1. *Let $\sigma \subseteq \gamma_2$ be an edge or a triangle whose vertices are contained in $[n]$. Then there is $T(\sigma) \in S(n,2)$ such that $\sigma \in T(\sigma)$ and for every triangle or edge τ different from σ which does not overlap σ in $\mathbb{R}^2$ or does satisfy $\tau <_3 \sigma$, we have $\tau \leq_3 T(\sigma)$.*

Theorem 3.2. *Let $\sigma, \tau_1, \tau_2, \dots, \tau_m \subseteq \gamma_3$ be distinct triangles whose liftings are pairwise non-overlapping in $\mathbb{R}^4$ and whose vertices are contained in $[n]$. We further assume that $\min(\sigma) \leq \min(\tau_i)$ for every $i \in [m]$, and $\min(\sigma) = \min(\tau_i)$ implies $\max(\sigma) \geq \max(\tau_i)$ for every $i \in [m]$. Then there is $T \in S(n,3)$ such that $\sigma \in T$ and $\tau_i \leq_4 T$ for every $i \in [m]$.*

Proof sketch of Theorem 1.2(i). We may assume $\mathcal{F}$ consists only of triangles and edges when $D = 3$, and may assume $\mathcal{F}$ consists only of triangles when $D = 4$; we omit the easy proof of this reduction. Let $d = D - 1$ and $m = |\mathcal{F}|$. We project the simplices in $\mathcal{F}$ into $\mathbb{R}^d$ by ignoring the last coordinate. Abusing notation, we also identify such projections with subsets of $[n]$.

We order these projections as $\sigma_1, \sigma_2, \dots, \sigma_m$ to satisfy the following conditions: When $D = 3$, we require that for every $1 \leq i < j \leq m$ either $\sigma_i <_D \sigma_j$ or σ_i and σ_j do not overlap in $\mathbb{R}^d$; such an ordering can be given as a linear extension of the partial order $\preceq_D$ by Lemma 2.4. When $D = 4$, we require that whenever $i < j$ we have $\min(\sigma_j) \leq \min(\sigma_i)$, and if $\min(\sigma_i) = \min(\sigma_j)$ then $\max(\sigma_i) \leq \max(\sigma_j)$. By Theorems 3.1 and 3.2, we obtain $T_i \in S(n,d)$ for every $i \in [m]$ such that $\sigma_i \in T_i$ and $\sigma_j \leq_D T_i$ for $j < i$.

Now we use the lattice property of $\mathrm{HST}(n,d)$ in Theorem 2.8. For every $k \in [m]$, let $S_k = T_k \wedge T_{k+1} \wedge \cdots \wedge T_m$ where $\wedge$ denotes the meet operation. Clearly these S_k form a chain in $\mathrm{HST}(n,d)$; considering a maximal chain in $\mathrm{HST}(n,d)$ containing all S_k gives a triangulation $\widetilde{T} \in S(n,D)$ which contains the liftings of all S_k by Theorem 2.7.

It remains to show that S_k contains σ_k for every $k \in [m]$. It is enough to show that for every $\lceil d/2 \rceil$-dimensional face τ of σ_k, we have $\tau \in \mathrm{sub}(S_k)$ and τ satisfies the right-hand side condition of Lemma 2.9 with T replaced by S_k; once it is shown, then by Lemma 2.9 every $\lceil d/2 \rceil$-dimensional face of σ_k belongs to S_k, which implies $\sigma_k \in S_k$.

By Theorem 2.8, we have

$$\mathrm{sub}(S_k) = \mathrm{sub}(\bigwedge_{i=k}^{m} T_i) = \bigcap_{i=k}^{m} \mathrm{sub}(T_i). \tag{3.1}$$

Since $\sigma_k \in T_k$ and $\sigma_k \leq_D T_i$ for every $i \in [k,m]$, we have $\tau \in \mathrm{sub}(T_i)$ for every $i \in [k,m]$, which together with (3.1) implies that $\tau \in \mathrm{sub}(S_k)$. Now assume for a contradiction (by

Lemma 2.9), that there is another $\sigma' \in \mathrm{sub}(S_k)$ such that $\tau <_{d+1} \sigma'$. In particular, σ' and τ overlap in $\mathbb{R}^d$. Now by (3.1), we have $\sigma' \in \mathrm{sub}(T_k)$. By definition of $\mathrm{sub}(T_k)$ we have $\sigma' <_D \tau$, so we have a contradiction since $\sigma' <_D \tau$ and $\tau <_D \sigma'$ are exclusive events by Corollary 2.3. □

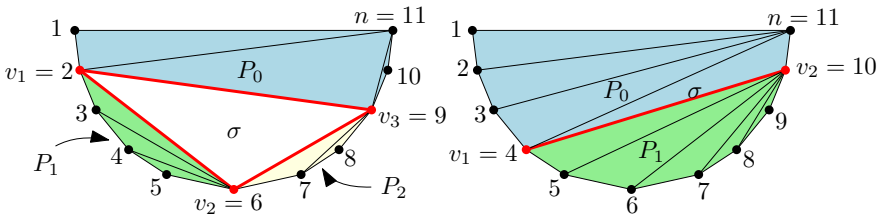


Figure 2: Extension of σ into a triangulation $T(\sigma) \in S(n, 2)$.

Proof sketch of Theorem 3.1. For a given edge or a triangle $\sigma \subseteq \gamma_2$ from Theorem 3.1, we add in each polygon divided by σ all the diagonals which are incident with the "maximum" vertex in that polygon, see Figure 2. □

Note that the join of all the triangulations in the lattice $\mathrm{HST}(n, 2)$ which contain σ again contains σ – indeed, this join equals $T(\sigma)$ in Theorem 3.1. However, this is not the case for $\mathrm{HST}(n, 3)$, which leads us to take a more dynamic approach for proving Theorem 3.2.

Example 3.3. In $\mathrm{HST}(7, 3)$, the following two triangulations T and T' are maximal among all triangulations in $S(7, 3)$ which contain the simplex $\sigma = \{2, 5, 6, 7\}$, see [8, Figure 4(b)]:

$$T = \{\{1,2,3,5\},\{1,2,5,6\},\{1,2,6,7\},\{1,3,4,5\},\{2,3,5,7\},\{2,5,6,7\},\{3,4,5,7\}\}$$
$$T' = \{\{1,2,3,4\},\{1,2,4,5\},\{1,2,5,6\},\{1,2,6,7\},\{2,3,4,7\},\{2,4,5,7\},\{2,5,6,7\}\}$$

We give a proof sketch of Theorem 3.2. Let $\sigma = \{v_1 < v_2 < v_3\}, \tau_1, \ldots, \tau_m \subseteq \gamma_3$ be the given triangles from Theorem 3.2. By Lemma 2.6, it is enough to find a triangulation T which contains σ and has no edges which are 5-interlacing with any τ_i.

For a simplicial polytope P and a point q outside P in $\mathbb{R}^d$ such that $\{q\} \cup V$ is in general position, we say that a face τ (as a vertex subset) of P *is visible from* q if $\mathrm{conv}(\tau \cup \{q\})$ does not intersect the interior of P. For a triangulation of T of P (without adding new vertices), the *cone of* T *over* q, denoted by $\mathrm{cone}(T, q)$, is the triangulation of $\mathrm{conv}(P \cup \{q\})$ given by the set of faces

$$T \cup \{\tau \cup \{q\} : \tau \text{ is a face of } P \text{ visible from } q\}.$$

We define four intervals as follows, see Figure 3.

$$J_L = \{i \in [n] : i < v_2\} \neq \emptyset, \qquad J_R = \{i \in [n] : i > v_3\},$$
$$I_M = \{i \in [n] : v_2 < i < v_3\}, \text{ and} \qquad J_M = I_M \cup \{v_2, v_3\} \neq \emptyset.$$

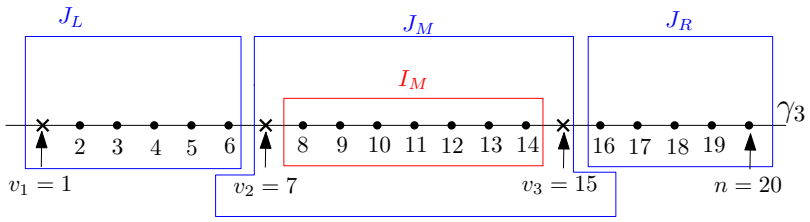


Figure 3: Illustrations of $\sigma = \{v_1, v_2, v_3\}$ and the intervals J_L, J_R and $I_M \subset J_M$.

Proof sketch of Theorem 3.2. We may assume that $v_1 = 1$; this reduction is via the coning operation described above. Let $V_0 = [n] \setminus I_M$ (we may assume that $|V_0| \geq 4$). Let $T_0 = T_M^{V_0}$ be the maximum triangulation of $\mathrm{HST}(V_0, 3)$ on vertex set V_0. By Gale's evenness criterion, σ is contained in T_0. Further the set of edges of T_0 is given as

$$\{\{i, n\}, \{1, j\}, \{k, s(k)\} : i \in V_0 \setminus \{n\}, j \in V_0 \setminus \{1\}, k \in V_0 \setminus \{1, n\}\}. \tag{3.2}$$

None of the edges in (3.2) is 5-interlacing with any triangle τ_i when I_M is empty, so we can take $T = T_0$ in the assertion of the theorem in this case. Thus, we may assume that $l := |I_M| > 0$. Still the edges other than $\{v_2, v_3\} \subseteq J_M$ cannot 5-interlace any of the given triangles τ_i.

Now, we find a certain total order $q_1, q_2, \ldots, q_l$ on I_M and use it to find a desirable triangulation by inductive coning: For $i \in [l]$, let $V_i = V_{i-1} \cup \{q_i\}$ and $T_i = \mathrm{cone}(T_{i-1}, q_i) \in S(V_i, 3)$. The edges newly added at the ith step to form T_i are exactly $\{1, q_i\}$, $\{p(q_i), q_i\}$, $\{q_i, s(q_i)\}$ and $\{q_i, n\}$ where $p(q_i)$ and $s(q_i)$ are the immediate predecessor and successor of q_i in V_i, respectively. Note that these edges are exactly the edges of the cyclic polytope $P_i = \mathrm{conv}(V_i)$ which are not in T_{i-1}. We can find a suitable order on I_M so that these edges never 5-interlace any of the given triangles τ_i; the details of constructing such order on I_M are delicate and are omitted for space reasons. □

3.2 Proof of Theorem 1.2(ii)

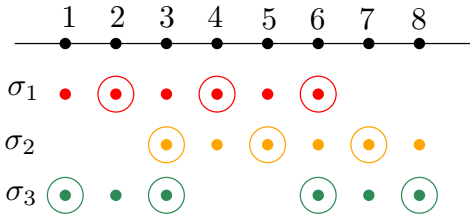


Figure 4: Illustration of $\mathcal{F}$. Circled elements in each σ_i describe the only possible choice of elements which can form a 7-interlacing pattern with other simplices.

Proof sketch of Theorem 1.2(ii). On the moment curve in $\mathbb{R}^5$, consider the family

$$\mathcal{F} = \{\sigma_1 = \{1,2,3,4,5,6\},\ \sigma_2 = \{3,4,5,6,7,8\},\ \sigma_3 = \{1,2,3,6,7,8\}\}$$

which is not extendable to a triangulation in $S(8,5)$, see Figure 4. Indeed, one cannot add a new 5-simplex in $[8]$ to $\mathcal{F}$ without creating a 7-interlacing with some element of $\mathcal{F}$. To increase the number of vertices n while maintaining dimension D, we add the convex hull of the new vertex $n+1$ and each facet of the cyclic polytope $C(n,D)$ which are visible from $n+1$. To increase both n and D, we take the join $\mathcal{F}' * \{\{n+1\}\}$ from a previous family on γ_D by adding a new vertex $n+1$. □

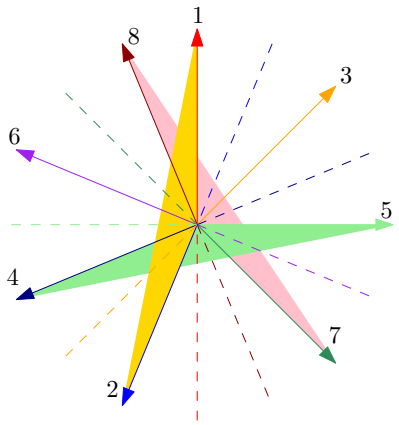


Figure 5: Description of the Gale transform of $\mathcal{F}$. The dual cones to σ_1, σ_2 and σ_3 are spanned by $\{7,8\}$, $\{1,2\}$ and $\{4,5\}$, respectively.

Remark 3.4 (Communicated with Francisco Santos)**.** An alternative way to see non-extendability when $n = D+3$ can be provided via Gale transform, see Figure 5. It is known that when $n = D+3$ every triangulation over n points in $\mathbb{R}^D$ is regular. A given list of D-simplices is extendable to a regular triangulation without adding new vertices if and only if the intersection of all dual cones (the dual cone of σ is the cone at the Gale dual over the complement $[n] \setminus \sigma$) has a full-dimensional intersection. Therefore, it is reduced to checking intersection of the dual cones in $\mathbb{R}^2$. For more backgrounds, consult [6].

4 Final remarks

Algorithmic aspects. Theorem 1.2(i) raises the question of how fast the extension can be computed:

Conditions	$n = d+2$	$n = d+3$	$n = d+4$	$n = d+5$
Condition (4.1) on HST(n,d)	YES	NO [8]	-	-
Extendability in $\mathbb{R}^D = \mathbb{R}^{d+1}$	YES	YES	NO	NO
HST(n,d) being a lattice	YES	YES [21]	?	NO [7, 19]

Table 1: Comparison between the lattice conditions and extendability. Here $D = d+1$.

Problem 4.1. *For $d = 3, 4$, given a geometric complex Δ on n-vertex set V on the moment curve γ_d, find the running time of computing a triangulation $T \in S(V,d)$ of the cyclic d-polytope* conv(V) *which is an extension of Δ, without adding new vertices; namely in the worst case, as n tends to infinity.*

For Problem 4.1, a greedy algorithm can be given which runs in $O(n^5)$ time.

Quantitative aspects. Theorem 1.2(ii) raises the question of how many new vertices suffice in order to extend:

Problem 4.2. *Let $d \geq 5$. Running over all geometric complexes Δ on an n-vertex set V on the moment curve γ_d, find the minimum number $m(d,n)$ of new vertices that suffices for extension of Δ to a triangulation of* conv(V)*, in the worst case.*

By Theorem 1.2(ii), for every $d \geq 5$ and $n \geq d+3$ we have $m(d,n) \geq 1$, but we were unable to find a better bound. There are studies on convex partitions of polyhedra [3, 4], but in this context one may subdivide simplices of Δ, which we forbid in our setting.

Connection to the lattice property. Recall that when $d = 2, 3$, Theorem 2.8 guarantees that HST(n,d) is a lattice where the meet is given by submersion sets to satisfy

$$\text{sub}(T_1 \wedge T_2) = \text{sub}(T_1) \cap \text{sub}(T_2) \text{ for every } T_1, T_2 \in S(n,d). \tag{4.1}$$

Condition (4.1) was important in our proof of Theorem 1.2(i).

However, imposing (4.1) for HST(n,d) is not necessary for extendability at one higher dimension $D = d+1$: While it is not difficult to see that given pairwise non-overlapping simplices on γ_D on $n \leq D+2 = d+3$ vertices are extendable to a triangulation in $S(n,D)$, it was shown that (4.1) is not satisfied when $d = D-1$ is 4 or 5 and $n = d+3 = D+2$, see [8, Section V, Remark 1]. Still, HST$(d+3,d)$ is a lattice (communicated from Nicholas J. Williams) and this can be seen, for instance, by the explicit description of this poset [21, Lemma 3.1]. However for $d = 4, 5$, HST$(d+5,d)$ is not a lattice [7, 19]. See Table 1 for a summary of known results.

Question 4.3. *Is* HST$(d+4,d)$ *a lattice for every $d \geq 4$?*

Tightness for embedded d-simplices on γ_d. Our construction for Theorem 1.2(ii) gives a non-extendable example with respect to d-simplices that has three d-simplices on γ_d (and on $d+3$ vertices), for every $d \geq 5$.

Problem 4.4. *For $d \geq 5$, is there a collection of two d-simplices on γ_d which cannot be extended to a triangulation of the convex hull of their union, without adding new vertices?*

Note that no such examples with one d-simplex exist, even if the vertex set includes extra vertices from the moment curve:

Lemma 4.5. *Let $d \geq 2$ and $\sigma \subseteq V \subseteq \gamma_d$ such that V is finite and $|\sigma| = d+1$. Then the simplex σ can be extended to a triangulation of $\operatorname{conv}(V)$ without adding vertices not in V.*

Proof. Order the vertices in $V \setminus \sigma$ in an arbitrary order $v_1, \ldots, v_l$. Let $T_0 = \{\sigma' : \sigma' \subseteq \sigma\}$, and $T_i = \operatorname{cone}(T_{i-1}, v_i)$ for $i \in [l]$. Then T_i is a triangulation of $\operatorname{conv}(\sigma \cup \{v_1, \ldots v_i\})$, hence T_l is a desired triangulation. □

Remark 4.6. Lemma 4.5 answers [14, Question 5.4(1)] in the affirmative.

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