Séminaire Lotharingien de Combinatoire **95B** (2026)
Article #107, 12 pp.

Proceedings of the 38th Conference on Formal Power Series and Algebraic Combinatorics (Seattle)

Introducing the Coxeter flag variety

Nantel Bergeron[*1], Lucas Gagnon[†2], Hunter Spink[‡3] Vasu Tewari[§4]

[1]*Department of Mathematics and Statistics, York University*
[2]*Department of Mathematics, University of Southern California*
[3]*Department of Mathematics, University of Toronto*
[4]*Department of Mathematical and Computational Sciences, University of Toronto Mississauga*

Abstract. Let W be a finite Weyl group and fix a Coxeter element $c \in W$. We construct a complex of toric varieties in the complete flag variety by setting all extremal generalized Plucker coordinates which are not c-noncrossing to zero. We describe the structure of this complex, including the corresponding moment polytopes for each c, which reflect the structure of c-clusters. We give a GKM description of the (T-equivariant) cohomology rings and show that its isomorphism type of this depends only on W.

Résumé. Soit W un groupe de Weyl fini et $c \in W$ un élément de Coxeter fixé. Nous construisons un complexe de variétés toriques en annulant toutes les coordonnées de Plücker généralisées extrémales qui ne sont pas c-non-croisées. Nous décrivons la structure de ce complexe, ainsi que les polyèdres du moment correspondants pour chaque c, lesquels reflètent la structure des c-clusters. Nous donnons ensuite une description GKM des anneaux de cohomologie (équivariante par rapport à T) et montrons que leur type d'isomorphisme dépend uniquement de W, et non de c.

Keywords: flag variety, noncrossing partition, Schubert, Richardson, Plucker, equivariant cohomology

1 Introduction

Let G be a reductive complex algebraic group of finite rank n and fix a Borel subgroup $B \subseteq G$ and a maximal torus $T \subseteq B$. The *generalized flag variety* is the coset space G/B. The starting point for any question about the geometry of G/B is the *Weyl group* $W = N_G(T)/T$. The Bruhat decomposition

$$G/B = \bigcup_{w \in W} \mathring{X}^w \qquad \text{with} \qquad \mathring{X}^w = BwB/B$$

[*]bergeron@yorku.ca
[†]lgagnon@usc.edu
[‡]hunter.spink@utoronto.ca
[§]vasu.tewari@utoronto.ca

gives rise to *Schubert varieties* $X^w = \overline{BwB}$ and *Richardson varieties* $X^v_u = X^v \cap w_\circ X^{w_\circ u}$, where $w_\circ \in W$ the longest element. Describing the interactions of these varieties using the combinatorics of W is a major focus of modern algebraic combinatorics, including in Schubert calculus, Kazhdan–Lusztig theory, and beyond.

A *(standard) Coxeter element* $c \in W$ is any product of all simple reflections in W. *Coxeter–Catalan combinatorics* is the study of a number of equinumerous families naturally generated by the choice of a pair (W, c), including:

(A) The lattice of *c-noncrossing partitions* $\mathrm{NC}(W, c) \subseteq W$ comprises all prefixes $\tau_1 \cdots \tau_i$ of minimal length reflection factorizations $c = \tau_1 \cdots \tau_n$ [4].

(B) The *(combinatorial) c-clusters* Cl_c are subsets of the almost positive roots that index the (algebraic) clusters in a cluster algebra [17].

(C) The *c-Cambrian congruence* $\equiv_c$ is a lattice quotient of the right weak order $\leqslant_R$ [16].

This abstract is an invitation to study a space $\mathrm{CFl}_c \subseteq G/B$ that enriches the Coxeter–Catalan combinatorics of (W, c) in much the same way that G/B enriches the general combinatorics of W. Complete results and proofs can be found in our paper [3].

We call CFl_c the *c-Coxeter flag variety*.[1] Of the possible definitions of CFl_c—we have one for each of (A)-(C) above—this abstract will use the *generalized Plücker embedding*

$$\begin{array}{rccc} \mathrm{Pl} : G/B & \hookrightarrow & \mathbb{P}(V^\lambda) \\ gB & \mapsto & [gv_\lambda], \end{array} \tag{1.1}$$

where $\lambda \in \hom_{\text{alg. grp.}}(T, \mathbb{C}^\times)$ is fixed to be regular and dominant, V^λ is the simple G-module with highest weight λ, and $v_\lambda \in V^\lambda$ is a highest weight vector. Each $w \in W$ indexes a one-dimensional *extremal weight space*

$$V^\lambda_{w\lambda} = \{v \in V^\lambda \mid hv = (\lambda)(wh) \text{ for all } h \in T\}.$$

The *extremal Plücker coordinates* Pl_w are the projective coordinates on G/B associated with each $V^\lambda_{w\lambda}$. For any $\mathcal{A} \subseteq W$, we can define a *$\mathcal{A}$-Plücker vanishing variety*

$$\mathrm{PV}_\mathcal{A} = \bigcap_{w \in \mathcal{A}} \{gB \in G/B \mid \mathrm{Pl}_w(gB) = 0\} \subseteq G/B.$$

Understanding arbitrary Plücker vanishing subvarieties is intrinsically impossible because of the key role they play in Mnëv's universality theorem [14].

Definition 1.1. The *c-Coxeter flag variety* is $\mathrm{CFl}_c \coloneqq \mathrm{PV}_{\mathrm{NC}(W,c)} \subseteq G/B$.

[1] We use the term "flag variety" to emphasize the remarkable combiantorial parallels between CFl_c and G/B. The inclusion $\mathrm{CFl}_c \subseteq G/B$ is also commensurate with the fact that the Coxeter–Catalan combinatorics of W are a proper subset of arbitrary Weyl group combinatorics.

We show that each c-Coxeter flag variety is an equidimensional complex of toric Richardson varieties indexed by c-Cambrian classes and, with affine pacing indexed by c-clusters. Moreover, the associated moment polytopes form a contractible polytopal complex of (W, c)-*polypositroids* in the sense of Lam–Postnikov [13].

In the special case of $G = \mathrm{GL}_{n+1}$, $W = S_{n+1}$, and $c = (n+1\ \cdots\ 2\,1)$, this variety was studied in [1, 2] as the *quasisymmetric flag variety* because of the following result.

Theorem 1.2 ([2, Theorem A])**.** If $G = \mathrm{GL}_{n+1}$, $W = S_{n+1}$, and $c = (n+1\ \cdots\ 2\,1)$, then

$$H^\bullet(\mathrm{CFl}_c) \cong \mathbb{Z}[x_1, \ldots, x_{n+1}]/\langle f(x_1, \ldots, x_{n+1}) - f(0, \ldots, 0) \mid f \in \mathrm{QSym}_{n+1}\rangle$$

where QSym_{n+1} is the quasisymmetric polynomial ring of Gessel and Stanley.

In this case we constructed a near-complete geometric and combinatorial analogue of Schubert calculus, replacing symmetric polynomials with quasisymmetric polynomials and Schubert polynomials with the *(equivariant) forest polynomials* of [1, 15].

The reader should consider how the results of [1, 2] might generalize to arbitrary Coxeter flag varieties, including analogues of Theorem 1.2 (see Section 7). However, even if one keeps $G = \mathrm{GL}_{n+1}$, varying "c" produces new and surprising structures which are not yet understood; see Appendix A for a concrete example.

2 Preliminaries

We review some standard preliminary material on Weyl groups and algebraic groups, directing the reader to [6, 20] for details. Let $\mathrm{Char}(T)$ be the group of algebraic characters of T. The Weyl group W acts on $\mathbb{E} = \mathbb{R} \otimes_{\mathbb{Z}} \mathrm{Char}(T)$ by reflections. The adjoint action of T decomposes the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ into weight spaces

$$\mathfrak{g} = \mathfrak{g}^T \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha,$$

and the set of nonzero weights is a root system Φ for W. The choice of B determines a set of simple roots Δ and positive roots Φ^+ in Φ.

For $\alpha \in \Phi^+$, let $s_\alpha \in W$ be the corresponding reflection. Let $\mathsf{S} = \{s_\alpha \mid \alpha \in \Delta\}$, the set of simple reflections, and $\mathsf{T} = \{s_\alpha \mid \alpha \in \Phi^+\}$, the set of all reflections. For $\tau \in \mathsf{T}$, let $r(\tau)$ be the associated root. For $w \in W$, a *reduced word* $\boldsymbol{w} = (s_1^{\boldsymbol{w}}, \ldots, s_\ell^{\boldsymbol{w}})$ is any minimal-length factorization

$$w = s_1^{\boldsymbol{w}} \cdots s_\ell^{\boldsymbol{w}} \text{ with } s_1^{\boldsymbol{w}}, \ldots, s_\ell^{\boldsymbol{w}} \in \mathsf{S}.$$

The *length* of w is $\ell(w)$, the number of simple reflections in any such factorization. The *inversions* of w are elements of $\mathrm{Inv}(w) = \{\tau \in \mathsf{T} \mid \ell(\tau w) < \ell(w)\}$. Let $\leqslant_B$ and $\leqslant_R$ be the strong and (right) weak Bruhat orders on W. Multiplication by the longest element $w_\circ \in W$ gives an antiautomorphism of these orders.

For $w \in W$ and any subset $C \subseteq r(\mathrm{Inv}(w))$ that is closed under addition in Φ, let

$$U_C = \prod_{\alpha \in C} \exp(\mathfrak{g}_\alpha) \qquad \text{where} \qquad \exp(X) = 1 + X + \frac{1}{2}X^2 + \cdots \tag{2.1}$$

and the product is taken in any fixed order. This gives a (non-canonical) T-equivariant isomorphism of varieties from U_C to $\bigoplus_{\alpha \in C} \mathfrak{g}_\alpha$. We also have a (canonical) isomorphism

$$U_{r(\mathrm{Inv}(w))} \cong \mathring{X}^w \qquad \text{given by} \qquad u \mapsto uwB,$$

so in all we have a T-equivariant coordinatization of each open Schubert cell $\mathring{X}^w$.

A *minimal reflection factorization* for $w \in W$ is any minimal factorization

$$w = \tau_{i_1} \cdots \tau_{i_k} \text{ with } \tau_{i_1}, \ldots, \tau_{i_k} \in \mathsf{T}.$$

The *absolute length* of w is $\ell_{\mathsf{T}}(w) = k$. The *absolute order* on W is the prefix order with respect to minimal reflection factorizations. For a Coxeter element $c \in W$, the *c-noncrossing partitions* are

$$\mathrm{NC}(W, c) = \{u \in W \mid u \leqslant_{\mathsf{T}} c\},$$

and the *positive noncrossing partitions* are

$$\mathrm{NC}(W, c)^+ = \{u \in \mathrm{NC}(W, c) \mid c \leqslant_B u\}.$$

The size of these sets are the *W-Catalan numbers* and the *positive W-Catalan numers*:

$$|\mathrm{NC}(W, c)| = \mathrm{Cat}_W \qquad \text{and} \qquad |\mathrm{NC}(W, c)^+| = \mathrm{Cat}_W^+ .$$

For $\leqslant_B$, $\leqslant_R$, or $\leqslant_{\mathsf{T}}$, we use a dot to indicate a covering relation, as in $\lessdot_B$, $\lessdot_R$, or $\lessdot_{\mathsf{T}}$. Every such relation corresponds to an edge in the Cayley graph $\mathrm{Cayley}(W)$ with respect to right multiplication by the generating set T. For an interval I in the Bruhat or absolute order, we write $\mathrm{Hasse}_{\mathsf{T}}(I)$ and $\mathrm{Hasse}_B(I)$ for the edge-labelled subgraph of $\mathrm{Cayley}(W)$ with vertex set I and edge set given by the covering relations of $\leqslant_{\mathsf{T}}$ or $\leqslant_B$, respectively.

A *Coxeter matroid* is a subset $\mathcal{M} \subset W$ such that $w\mathcal{M}$ has a unique Bruhat-maximum element for every $w \in W$. Recall that we have fixed a regular dominant weight $\lambda \in \mathbb{E}$. The Gelfand–Serganova theorem [7, Theorem 6.3.1] gives a bijection

$$\mathcal{M} \mapsto P_{\mathcal{M}} := \mathrm{conv}(\{w \cdot \lambda \mid w \in \mathcal{M}\}).$$

between Coxeter matroids and *Coxeter matroid polytopes*; we take the above map as our definition for Coxeter matroid polytopes.

For $Y \subseteq G/B$ a T-invariant subvariety, the *moment polytope* of Y is

$$\mu(Y) = \mathrm{conv}(\{w \cdot \lambda \mid w \in Y^T\}) \subseteq \mathbb{E}.$$

Proposition 2.1 ([9])**.** The moment polytope $\mu(Y)$ of an irreducible T-invariant variety $Y \subset G/B$ is a Coxeter matroid polytope.

The T-fixed points of G/B are wB/B for $w \in W$, so $\mu(G/B) = P_W$, the *W-permutahedron*. For a Richardson variety X_u^v, which has T-fixed point set $[u,v] = \{w \mid u \leqslant_B w \leqslant_B v\}$, the moment polytope is $\mu(X_u^v) = P_{[u,v]}$, the *twisted Bruhat interval polytope*.

Finally, say that a T-invariant subvariety $X \subset G/B$ is *rigid* if, for any T-invariant subvariety $Y \subseteq G/B$, $Y^T \subset X^T$ implies $Y \subset X$. The following is well-known to experts.

Theorem 2.2. If $\mathcal{A} \subseteq W$, then $\mathrm{PV}_{\mathcal{A}}$ is the union of all T-orbit closures $Y \subset G/B$ with $Y^T \subset \mathcal{A}$. As such it is the unique rigid variety with T-fixed point set $\mathcal{A}$.

3 Coxeter Richardson Varieties

A *c-Coxeter Richardson variety* is a Richardson variety of the form X_w^{wc}. These are n-dimensional toric varieties, which have been studied by a number of authors [8, 11, 12].

Fact 3.1 ([3, Theorem 7.1])**.** For $w \in W$, X_w^{wc} is nonempty if and only if $\ell(wc) = \ell(w) + \ell(c)$.

A *translated c-Coxeter Richardson variety* is the translation $w^{-1}X_w^{wc}$. In this section we investigate when two $w, w' \in W$ give the same translated variety; we find that this is determined by Reading's *c-Cambrian equivalence* [18].

Fix a reduced word $\boldsymbol{c}$ for c and let $\boldsymbol{c}^\infty$ denote the ∞-fold concatenation of $\boldsymbol{c}$. For $x \in W$, we let $\boldsymbol{x}_{c^\infty}$ denote the lexicographically first reduced word for x in $\boldsymbol{c}^\infty$. For any subword $\boldsymbol{x}$ of $\boldsymbol{c}^\infty$, we refer to the part of $\boldsymbol{x}$ in the ith copy of $\boldsymbol{c}$ in $\boldsymbol{c}^\infty$ as the *ith syllable of $\boldsymbol{x}$*.

Definition 3.2. A *c-sortable permutation* is a permutation x such that each syllable of $\boldsymbol{x}_{c^\infty}$ contains every letter in the following syllable. We denote the set of c-sortable permutations in W by $\mathrm{Sort}(W, c)$.

Example 3.3. Let $W = S_5$ and $c = s_2s_1s_3s_4 \in S_5$. Using one-line notation, the permutations $x = 35421$ and $y = 53421$ have c-sorting words

$$\boldsymbol{x} = s_2\ s_1\ s_3\ s_4 \mid s_2\ \cancel{s_1}\ s_3\ s_4 \mid s_2\ \cancel{s_1}\ \cancel{s_3}\ \cancel{s_4} \mid \cdots, \qquad \text{and}$$
$$\boldsymbol{y} = s_2\ s_1\ s_3\ s_4 \mid s_2\ \cancel{s_1}\ s_3\ s_4 \mid s_2\ s_1\ \cancel{s_3}\ \cancel{s_4} \mid \cdots.$$

Then x is c-sortable, but y is not, as $\{s_2, s_3, s_4\} \not\supseteq \{s_1, s_2\}$.

In [18, Theorem 1.1], Reading shows that for any $x \in W$ there is a unique $\leqslant_R$-maximum c-sortable element $\pi_\downarrow(x) \leqslant_R x$. The *$c$-Cambrian congruence* is the relation on W defined by $x \equiv_c y$ if $\pi_\downarrow(x) = \pi_\downarrow(y)$.

Example 3.4. Continuing Example 3.3 with $W = S_5$ and $c = s_2s_1s_3s_4 \in S_5$, $x = 35421$, and $y = 53421$, we have $x \equiv_c y$: x is covered by y in $\leqslant_R$, so it must be $\pi_\downarrow(y)$.

For $w \in W$, let $w_{op} := w^{-1}w_\circ$, and note that $X_w^{wc} \neq \varnothing$ if and only if $c \leqslant_R w_{op}$.

Theorem 3.5 ([3, Theorem 14.3 and Corollary 6.9])**.** If $u, v \in W$ have $c \leqslant_R u_{op}, v_{op}$, then the following are equivalent.

1. $u_{op} \equiv_c v_{op}$
2. $u^{-1}X_u^{uc} = v^{-1}X_v^{vc}$

Theorem 3.5 is proved independently as [8, Theorem 1.9]; in [3] we extend it to describe equivalence among $u^{-1}X_u^{uc'}$ for any $c' \leqslant_B c$. We sketch a few details from the proof that will be useful in later sections. The essential definition is that of a *translated Bruhat interval*: for $u, v \in W$, let

$$\overleftarrow{[u, v]} := u^{-1}[u, v] \qquad \text{where} \qquad [u, v] = \{w \mid u \leqslant_B w \leqslant_B v\}. \tag{3.1}$$

Proposition 3.6. For $u, v, w \in W$, the Plücker vanishing variety associated to $u[w, v]$ is uX_w^v, and therefore uX_w^v is rigid. In particular, this holds for $u = w^{-1}$ and $v = wc$.

Proof sketch. Translating by u in G/B also translates each T-fixed point and extremal Plücker coordinate by u. Moreover, if $Y \subseteq G/B$ is a T-invariant variety, then $Y^T = \{z \in W \mid Y \cap BzB \neq \varnothing\}$, so assuming $Y^T \subseteq [w, v]$ implies that $Y \subseteq X^v$ and $Y \subseteq w_\circ X^{w_\circ w}$. □

Say that two Bruhat intervals $[u, v]$ and $[u', v']$ are *shape equivalent* if multiplication by $u'u^{-1}$ gives an order homomorphism from $[u, v]$ to $[u', v']$. A consequence of the rigidity in Proposition 3.6 is the following geometric implication of shape equivalence.

Corollary 3.7. For $u, v, u', v' \in W$, the following are equivalent.

1. $[u, v]$ and $[u', v']$ are shape equivalent.
2. $\overleftarrow{[u, v]} = \overleftarrow{[u', v']}$
3. $u^{-1}X_u^v = (u')^{-1}X_{u'}^{v'}$

Proof sketch for Theorem 3.5. It is sufficient to show that $\overleftarrow{[u, uc]} = \overleftarrow{[v, vc]}$ if and only if $u_{op} \equiv_c v_{op}$. We show that $\overleftarrow{[u, vc]}$ is determined by a distinguished chain in $[u, uc]$, and that this chain is uniquely determined by the c-Cambrian equivalence class of u_{op}. □

4 Translated Richardsons and noncrossing partitions

In this section we explore how translated c-Coxeter Richardson varieties interact with noncrossing partitions.

Theorem 4.1 ([3, Theorem 7.1])**.** For any $w \in W$ we have an injective homomorphism

$$\begin{array}{rcl} \mathrm{Sh}_{w^{-1}} : \mathrm{Hasse}_B([w, wc]) & \to & \mathrm{Hasse}_{\mathsf{T}}(\mathrm{NC}(W, c)) \\ u & \mapsto & w^{-1}u \end{array}$$

Moreover $\mathrm{Hasse}_{\mathsf{T}}(\mathrm{NC}(W, c))$ is the union of the images of these maps.

Sketch of proof. Every maximal Bruhat chain $w \lessdot_B w\tau_1 \lessdot_B \cdots \lessdot_B wc$ corresponds to a maximal absolute chain $\mathrm{id} \lessdot_{\mathsf{T}} \tau_1 \lessdot_{\mathsf{T}} \cdots \lessdot_{\mathsf{T}} c$, proving the homomorphism. To see it is surjective, we do the reverse, finding intervals $[u, uc]$ that contain Bruhat chians mapping to every absolute chain, which is nontrivial. □

We now describe Reading's map from Cambrian classes to noncrossing partitions [17, Theorem 6.1] . The *cover reflections* for $x \in \mathrm{Sort}(W, c)$ are $\mathrm{Covers}(x) = \{xsx^{-1} \mid xs \lessdot_R x\} \subset \mathrm{Inv}(x)$. Then there is a unique order (up to trivial commutation) such that

$$\mathrm{nc}_c(x) := \prod_{\tau \in \mathrm{Covers}\, x} \tau \in \mathrm{NC}(W, c),$$

Moreover the map $\mathrm{nc}_c : \mathrm{Sort}(W, c) \to \mathrm{NC}(W, c)$ is a bijection, and restricts to a bijection

$$\{x \in \mathrm{Sort}(W, c) \mid x \geqslant_R c\} \to \mathrm{NC}(W, c)^+.$$

Example 4.2. Continuing Example 3.3 with $W = S_5$ and $c = s_2s_1s_3s_4 \in S_5$, and $x = 35421 \in \mathrm{Sort}(S_5, c)$, the cover reflections of x are $(4\,5)$, $(2\,4)$, and $(1\,2)$. Moreover,

$$\mathrm{nc}_c(x) = (2\,4)(1\,2)(4\,5) = (2\,4)(4\,5)(1\,2) = 41352.$$

We find a novel characterization of Reading's map using c-Coxeter Richardson varieties. Given $w \in \mathrm{NC}(W, c)$, define the *noncrossing inversion set* to be

$$\mathrm{Inv}_{\mathrm{NC}}(w) := \{\tau \in \mathrm{Inv}(w) \mid \tau w \in \mathrm{NC}(W, c)\}.$$

Theorem 4.3 ([3, Theorem 15.1])**.** For $w \in W$ with $\ell(wc) = \ell(w) + \ell(c)$:

1. the Bruhat maximum element of $\overleftarrow{[w, wc]}$ is $u = \mathrm{nc}_c(\pi_\downarrow(w_{op}))$, and
2. the neighbors of u in $\mathrm{Sh}_{w^{-1}}\big(\mathrm{Hasse}_B([w, wc])\big)$ are the n elements of $\{\tau u \mid \tau \in \mathrm{Inv}_{\mathrm{NC}}(u)\}$.

Sketch of proof. By the Coxeter matroid property, it is sufficient to prove 2. We do so by explicitly constructing intervals $[v, vc]$ which are shape equivalent to $[w, wc]$ and contain, in a clear way, the covering relation $v\tau u \lessdot_B vu$. Key to our proof are results of Biane–Josuat-Vergès [5] and Reading–Speyer [19]. □

5 Cluster charts for CFl_c

We now characterize CFl_c using the combinatorics of clusters. As defined in [17, Section 7], the *c-cluster fan* is a complete simplicial fan in $\mathbb{R}\Phi$ whose ray generators comprise the *almost positive roots* $\Phi_{\geqslant -1} = \Phi^+ \cup -\Delta$. The *positive cluster fan* is the set

$$Cl_c^+ = \{\mathcal{F} \cap \Phi^+ \mid \mathcal{F} \in Cl_c\},$$

where as is customary we identify each cone $\mathcal{F} \in Cl_c$ with its collection of ray generators.

Biane and Josuat-Vergès [5] construct a bijection between $\mathrm{NC}(W,c)$ and Cl_c^+ by taking each $u \in \mathrm{NC}(W,c)$ to the roots corresponding to its *right* inversions. We exploit the fact that $Cl_c^+ = Cl_{c^{-1}}^+$ to create a "left" version of this map:

$$\mathrm{Clust} : \mathrm{NC}(W,c) \to Cl_c^+, \qquad \text{with} \qquad \mathrm{Clust}(u) = \{r(\tau) \mid \tau \in \mathrm{Inv}_{\mathrm{NC}}(u)\}.$$

Restricting to $\mathrm{NC}(W,c)^+$ gives a bijection to the maximal cones in Cl_c^+, which subdivides the positive root cone $\mathbb{R}_{\geqslant 0}\Delta$ and contains every positive root as a ray generator.

Definition 5.1. For $u \in \mathrm{NC}(W,c)$ we define the *Coxeter Schubert cell* by

$$\mathring{X}^u_{\mathrm{NC}} := U_{\mathrm{Clust}(u)} B/B \subset BuB/B.$$

Explicit examples of $\mathring{X}^u_{\mathrm{NC}}$ can be found in Appendix A. If we coordinatize $\mathring{X}^u$ as the T-module $\bigoplus_{\alpha \in r(\mathrm{Inv}(w))} \mathfrak{g}_\alpha$ as in Equation (2.1), then $\mathring{X}^u_{\mathrm{NC}} \subset \mathring{X}^u$ is the coordinate subspace $\bigoplus_{\alpha \in \mathrm{Clust}(u)} \mathfrak{g}_\alpha$, which is a T-submodule with characters $\mathrm{Clust}(u)$.

Theorem 5.2 ([3, Theorem 9.3])**.** We have $\bigcup_{\substack{w \in W \\ w \leqslant_B wc}} w^{-1} X^{wc}_w = \mathrm{CFl}_c = \bigsqcup_{u \in \mathrm{NC}(W,c)} \mathring{X}^u_{\mathrm{NC}}$.

As we already know that $w^{-1}X^{wc}_w \subseteq \mathrm{CFl}_c$, we prove Theorem 5.2 by establishing the second equality, and that each $\mathring{X}^u_{\mathrm{NC}}$ belongs to some $w^{-1}X^{wc}_w$.

Proposition 5.3. If $u \in \mathrm{NC}(W,c)$, then $\mathring{X}^u_{\mathrm{NC}} = BuB \cap \mathrm{CFl}_c$. Otherwise, $BuB \cap \mathrm{CFl}_c = \varnothing$.

The proof of this fact uses a key property of the cluster fan. Say that a subset $C \subset \Phi^+_W$ of roots is *strongly closed* if $\mathbb{R}_{\geqslant 0} C \cap \Phi = C$. A direct calculation ([3, Proposition 9.8]) shows that for any $w \in W$ and any subset $C \subset \mathrm{Inv}(w)$ such that $r(C)$ is strongly closed,

$$U_{r(C)} wB/B = BwB \cap \bigcap_{\alpha \in \mathrm{Inv}(w) \setminus C} \{\mathrm{Pl}_{s_\alpha w} = 0\}.$$

Fact 5.4 (*Cluster cone property*)**.** For $u \in \mathrm{NC}(W,c)$ the subset $\mathrm{Clust}(u)$ is strongly closed. Therefore Proposition 5.3 is true.

For the next result, recall the contents of Theorem 4.3.

Proposition 5.5. Take $w \in W$ with $\ell(wc) = \ell(w) + \ell(c)$, and let $u \in \mathrm{NC}(W,c)^+$ be the Bruhat-maximum element of $\overleftarrow{[w, wc]}$. Then the $w^{-1}X^{wc}_w$ is the closure of $\mathring{X}^u_{\mathrm{NC}}$.

To prove Theorem 5.2, we show an analogue of Proposition 5.5 for the $\mathring{X}^u_{\mathrm{NC}}$ for $u \in \mathrm{NC}(W,c) \setminus \mathrm{NC}(W,c)^+$ by applying the proposition to each parabolic subgroup of W.

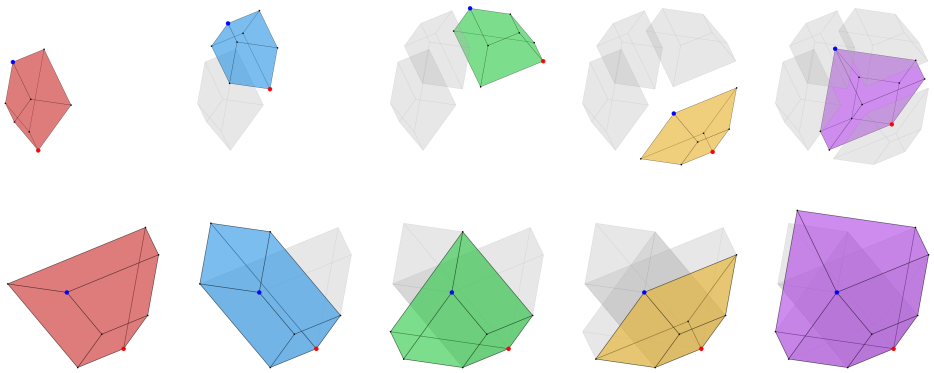


Figure 1: With $c = s_1 s_3 s_2 \in S_4$, the cumulative overlay of moment polytopes for untranslated (top) and translated (bottom) c-Coxeter Richardson varieties. Each translated polytope is distinct, as the c-Cambrian congruence is trivial on $\{w \mid w_{op} \leqslant_R c\}$.

6 The Coxeter Flag Variety as a toric complex

Using Theorem 4.1 and 5.2, we have an abstract polytopal complex whose 1-skeleton is $\mathrm{Hasse}_{\mathsf{T}}(\mathrm{NC}(W,c))$, namely

$$\mathrm{Complex}(\mathrm{CFl}_c) = \left(\bigsqcup_{w \leqslant_R wc} w^{-1} P_{[w,wc]} \right) \Big/ \left\{ \begin{array}{l} \text{identify any faces } F_1 \text{ and } F_2 \\ \text{having the same vertex sets.} \end{array} \right\}. \tag{6.1}$$

Corollary 6.1. CFl_c is a toric complex which is combinatorially equivalent under the moment map to $\mathrm{Complex}(\mathrm{CFl}_c)$.

By decomposing $\mathrm{Complex}(\mathrm{CFl}_c)$ according to our affine paving, one can show that the $\mathrm{Complex}(\mathrm{CFl}_c)$ is contractable. However, the type of $\mathrm{Complex}(\mathrm{CFl}_c)$ depends heavily on c; for instance if $G = \mathrm{GL}_3$ and $c = s_3 s_2 s_1$, every constituent will be a cube, but if $c = s_2 s_1 s_3$, one constituent will be a tetragonal trapezohedron; see Appendix A and Figure 1.

Recall that a Coxeter matroid polytope is a (W,c)-*polypositroid* if its defining inequalities have the form $\mathbf{x} \cdot \vec{n} \leqslant a$ with $(\mathrm{id} - c)\vec{n}$ parallel to a root [13].

Theorem 6.2 ([3, Theorem 8.5])**.** All faces in $\mathrm{Complex}(\mathrm{CFl}_c)$ are (W,c)-polypositroids.

7 The cohomology of CFl_c

We now describe the equivariant cohomology of CFl_c with respect to the *adjoint torus* $T_{ad} = T/Z(G)$; this choice is made to resolve divisibility issues that arise for general T. In this cohomology theory we have $H^\bullet_{T_{ad}}(pt) = \mathbb{Z}[t_\alpha \mid \alpha \in \Delta]$.

Our main tool is a framework known as GKM theory [10]; we now describe the relevant details. For $\mathcal{A} \subseteq W$, let the *GKM cohomology ring* be the $H^\bullet_{T_{ad}}(pt)$-algebra

$$H^\bullet_{T_{ad}}(\mathcal{A}) = \{(f_w)_{w\in\mathcal{A}} \in \left(H^\bullet_{T_{ad}}(pt)\right)^{\mathcal{A}} \mid t_\alpha \text{ divides } f_w - f_{s_\alpha w} \text{ whenever } w, s_\alpha w \in \mathcal{A}\}.$$

Under a certain condition, which we will refer to as having a *good affine paving* (see [2, §11.1] for applicable criteria), we have an isomorphism $H^\bullet_{T_{ad}}(\mathrm{PV}_\mathcal{A}) \cong H^\bullet_{T_{ad}}(\mathcal{A})$. This is a free $H^\bullet_{T_{ad}}(pt)$-module, and the ordinary (singular) cohomology ring can be obtained as

$$H^\bullet(\mathrm{PV}_\mathcal{A}) \cong H^\bullet_{T_{ad}}(\mathrm{PV}_\mathcal{A})/\mathfrak{m} \qquad \text{where} \qquad \mathfrak{m} = (t_\alpha \mid \alpha \in \Delta).$$

Theorem 7.1 ([3, Theorem 9.3])**.** The decomposition $\mathrm{CFl}_c = \bigsqcup_{u\in \mathrm{NC}(W,c)} \mathring{X}^u_{\mathrm{NC}}$ from Theorem 5.2 is a good affine paving. As a consequence,

$$H^\bullet_{T_{ad}}(\mathrm{CFl}_c) \cong H^\bullet_{T_{ad}}(\mathrm{NC}(W,c)) \qquad \text{and} \qquad H^\bullet(\mathrm{CFl}_c) \cong H^\bullet_{T_{ad}}(\mathrm{CFl}_c)/\mathfrak{m}.$$

Corollary 7.2. For Coxeter elements $c, c' \in W$, we have $H^\bullet(\mathrm{CFl}_c) \cong H^\bullet(\mathrm{CFl}_{c'})$

Proof. All Coxeter elements are conjugate so $c = uc'u^{-1}$ for some $u \in W$. The map

$$H^\bullet_{T_{ad}}(\mathrm{CFl}_c) \to H^\bullet_{T_{ad}}(\mathrm{CFl}_{c'}) \qquad \text{defined by} \qquad (f_w)_{w\in\mathrm{NC}(W,c)} \mapsto \left(f_{uwu^{-1}}(t^{u\cdot\alpha})\right)_{uwu^{-1}\in\mathrm{NC}(W,c')}$$

is a ring isomorphism that maps $\mathfrak{m}H^\bullet_{T_{ad}}(\mathrm{CFl}_c)$ to $\mathfrak{m}H^\bullet_{T_{ad}}(\mathrm{CFl}_{c'})$. □

Thus Theorem 1.2 also describes $H^\bullet(\mathrm{CFl}_c)$ for any $c \in S_{n+1}$. However, the isomorphism does not map classes $[w^{-1}X^{wc}_w]$ to classes $[v^{-1}X^{vc'}_v]$, so the forest polynomials in [1, 2] are no longer a Schubert-type basis. In [3, §12], we find a family of "flow-up bases" for each $H^\bullet(\mathrm{CFl}_c)$ and $H^\bullet_{T_{ad}}(\mathrm{CFl}_c)$ with a Schubert-like property, but a combinatorial formula for these elements is unknown outside the quasisymmetic case.

Question 7.3. What are the forest polynomials for each Coxeter element $c \in S_{n+1}$? Is there an analogue of Theorem 1.2 and related results for other reductive groups?

A Examples in type A

The article [3] gives examples from all classical groups; here we consider $G = \mathrm{GL}_{n+1}(\mathbb{C})$. In this case $W = S_{n+1}$, the symmetric group generated by $s_1, \ldots, s_n$ with $s_i = (i, i+1)$. Further $\Phi = \{e_i - e_j \mid i \neq j\} \subseteq \mathbb{E} = \mathbb{R}^{n+1}$ and the weight lattice is $\mathbb{Z}^{n+1}$. The Lie algebra is $\mathfrak{g} = \mathrm{Mat}_{n+1}(\mathbb{C})$, with weight space $\mathfrak{g}_{e_i-e_j}$ being the i,j coordinate subspace. Choosing B to be all upper triangular matrices in G gives $\Phi^+ = \{e_i - e_j \mid i < j\}$. We can choose regular dominant weight $\lambda = (n, n-1, \ldots, 0) \in \mathbb{E}$, which has extremal Plücker functions

$$\mathrm{Pl}_w(M) = \prod_{i=1}^{n} \det\left(M_{w(i),j}\right)_{1\leqslant i,j\leqslant k}.$$

The $U_{r(\mathrm{Inv}(w))}$ chart of a Schubert cell $\mathring{X}^w$ is obtained from the permutation matrix of w by replacing each 0 that is not right or below a 1 with an $*$ (free entry). For example,

$$\underbrace{\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}}_{123}, \quad \underbrace{\begin{bmatrix} * & 1 & \\ 1 & & \\ & & 1 \end{bmatrix}}_{213}, \quad \underbrace{\begin{bmatrix} 1 & & \\ & * & 1 \\ & 1 & \end{bmatrix}}_{132}, \quad \underbrace{\begin{bmatrix} * & * & 1 \\ 1 & & \\ & 1 & \end{bmatrix}}_{231}, \quad \underbrace{\begin{bmatrix} * & 1 & \\ * & & 1 \\ 1 & & \end{bmatrix}}_{312} \quad \text{and} \quad \underbrace{\begin{bmatrix} * & * & 1 \\ * & 1 & \\ 1 & & \end{bmatrix}}_{321}.$$

To illustrate the extremal Plücker functions, let a, b, and c to be the $(1,2)$, $(2,1)$, and $(1,1)$ entries of $\mathring{X}^{321}$. Then the extremal Plücker functions on this chart are

$$(\mathrm{Pl}_{123}, \mathrm{Pl}_{213}, \mathrm{Pl}_{132}, \mathrm{Pl}_{231}, \mathrm{Pl}_{312}, \mathrm{Pl}_{321}) = (c(c-ab), b(c-ab), -ac, -b, -a, -1).$$

We now give two example of Coxeter flag varieties. First, we describe the Coxeter flag variety for $c = s_1 s_2 = 231$, which is isomorphic to the "quasisymmetric" case, in which the closure of every $\mathring{X}^u_{\mathrm{NC}}$ chart has a cubical moment polytope:

$$\underbrace{\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}}_{123}, \quad \underbrace{\begin{bmatrix} * & 1 & \\ 1 & & \\ & & 1 \end{bmatrix}}_{213}, \quad \underbrace{\begin{bmatrix} 1 & & \\ & * & 1 \\ & 1 & \end{bmatrix}}_{132}, \quad \underbrace{\begin{bmatrix} * & * & 1 \\ 1 & & \\ & 1 & \end{bmatrix}}_{231}, \quad \text{and} \quad \underbrace{\begin{bmatrix} * & 0 & 1 \\ * & 1 & \\ 1 & & \end{bmatrix}}_{321}.$$

Taking $c = s_1 s_3 s_2 = 2413 \in S_4$ departs from the "quasisymmetric" case. The charts are

$$\underbrace{\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}}_{1234}, \quad \underbrace{\begin{bmatrix} * & 1 & & \\ 1 & & & \\ & & 1 & \\ & & & 1 \end{bmatrix}}_{2134}, \quad \underbrace{\begin{bmatrix} 1 & & & \\ & * & 1 & \\ & 1 & & \\ & & & 1 \end{bmatrix}}_{1324}, \quad \underbrace{\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & * & 1 \\ & & 1 & \end{bmatrix}}_{1243}, \quad \underbrace{\begin{bmatrix} 1 & & & \\ & * & 1 & \\ & * & & 1 \\ & 1 & & \end{bmatrix}}_{1423}, \quad \underbrace{\begin{bmatrix} * & 1 & & \\ 1 & & & \\ & & * & 1 \\ & & 1 & \end{bmatrix}}_{2143}, \quad \underbrace{\begin{bmatrix} * & * & 1 & \\ 1 & & & \\ & 1 & & \\ & & & 1 \end{bmatrix}}_{2314},$$

$$\underbrace{\begin{bmatrix} 1 & & & \\ & * & * & 1 \\ & 0 & 1 & \\ & 1 & & \end{bmatrix}}_{1432}, \quad \underbrace{\begin{bmatrix} * & 0 & 1 & \\ * & 1 & & \\ 1 & & & 1 \end{bmatrix}}_{3214}, \quad \underbrace{\begin{bmatrix} * & * & 1 & \\ 1 & & & \\ & * & & 1 \\ & 1 & & \end{bmatrix}}_{2413}, \quad \underbrace{\begin{bmatrix} * & * & * & 1 \\ 1 & & & \\ & & 1 & \\ & 1 & & \end{bmatrix}}_{2431}, \quad \underbrace{\begin{bmatrix} * & 0 & 1 & \\ * & 1 & & \\ * & & & 1 \\ 1 & & & \end{bmatrix}}_{4213}, \quad \underbrace{\begin{bmatrix} * & 0 & 1 & \\ * & * & & 1 \\ 1 & & & \\ & 1 & & \end{bmatrix}}_{3412}, \quad \text{and} \quad \underbrace{\begin{bmatrix} * & 0 & * & 1 \\ * & 1 & & \\ 0 & & 1 & \\ 1 & & & \end{bmatrix}}_{4231}.$$

The moment polytopes for the top-dimensional chart closures are shown in Figure 1, and we observe that $\mathring{X}^{3412}_{\mathrm{NC}}$ corresponds to the (purple) tetragonal trapezohedron, with vertices corresponding to the permutations 1234, 1243, 1423, 1432, 2134, 2143, 2314, 2413, 3214, and 3412.

Acknowledgement We thank Allen Knutson and Arun Ram for helpful conversations and an anonymous referee for their helpful suggestions on this extended abstract.

References

[1] N. Bergeron, L. Gagnon, P. Nadeau, H. Spink, and V. Tewari. "Equivariant quasisymmetry and noncrossing partitions". 2025. arXiv:2504.15234. Link.

[2] N. Bergeron, L. Gagnon, P. Nadeau, H. Spink, and V. Tewari. "The quasisymmetric flag variety: a toric complex on noncrossing partitions". 2025. arXiv:2508.12171. Link.

[3] N. Bergeron, L. Gagnon, H. Spink, and V. Tewari. "The Coxeter Flag Variety". 2026. arXiv:2601.23111. Link.

[4] D. Bessis. "The dual braid monoid". *Ann. Sci. École Norm. Sup. (4)* **36**.5 (2003), pp. 647–683.

[5] P. Biane and M. Josuat-Vergès. "Noncrossing partitions, Bruhat order and the cluster complex". *Ann. Inst. Fourier (Grenoble)* **69**.5 (2019), pp. 2241–2289. DOI.

[6] A. Björner and F. Brenti. *Combinatorics of Coxeter groups*. Vol. 231. Graduate Texts in Mathematics. Springer, New York, 2005, pp. xiv+363.

[7] A. V. Borovik, I. M. Gelfand, and N. White. *Coxeter matroids*. Vol. 216. Progress in Mathematics. Birkhäuser, Boston, MA, 2003.

[8] C. Defant, M. Sherman-Bennett, and N Williams. "Permutahedron Triangulations via Total Linear Stability and the Dual Braid Group". 2025. arXiv:2509.11497. Link.

[9] I. M. Gelfand, R. M. Goresky, R. D. MacPherson, and V. V. Serganova. "Combinatorial geometries, convex polyhedra, and Schubert cells". *Adv. in Math.* **63**.3 (1987), pp. 301–316.

[10] M. Goresky, R. Kottwitz, and R. MacPherson. "Equivariant cohomology, Koszul duality, and the localization theorem". *Invent. Math.* **131**.1 (1998), pp. 25–83. DOI.

[11] M. Harada, T. Horiguchi, M. Masuda, and S. Park. "The volume polynomial of regular semisimple Hessenberg varieties and the Gelfand-Zetlin polytope". *Proc. Steklov Inst. Math.* **305** (2019), pp. 318–344.

[12] A. Knutson, M. Sanchez, and M. Sherman-Bennett. "Permutahedral subdivisions and class formulas from Coxeter elements". *Sém. Lothar. Combin.* **93B** (2025), Art. 8, 12.

[13] T. Lam and A. Postnikov. "Polypositroids". *Forum Math. Sigma* **12** (2024), Paper e42, 67 pp.

[14] N. E. Mnëv. "The universality theorems on the classification problem of configuration varieties and convex polytopes varieties". *Topology and geometry—Rohlin Seminar*. Vol. 1346. Lecture Notes in Math. Springer, Berlin, 1988, pp. 527–543.

[15] P. Nadeau and V. Tewari. "Forest polynomials and the class of the permutahedral variety". *Adv. Math.* **453** (2024), Paper No. 109834, 33 pp. DOI.

[16] N. Reading. "Cambrian lattices". *Adv. Math.* **205**.2 (2006), pp. 313–353. DOI.

[17] N. Reading. "Clusters, Coxeter-sortable elements and noncrossing partitions". *Trans. Amer. Math. Soc.* **359**.12 (2007), pp. 5931–5958. DOI.

[18] N. Reading. "Sortable elements and Cambrian lattices". *Algebra Universalis* **56**.3-4 (2007), pp. 411–437. DOI.

[19] N. Reading and D. E. Speyer. "Sortable elements in infinite Coxeter groups". *Trans. Amer. Math. Soc.* **363**.2 (2011), pp. 699–761. DOI.

[20] T. A. Springer. *Linear algebraic groups*. Second. Vol. 9. Progress in Mathematics. Birkhäuser, Boston, MA, 1998.