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Ian G. Macdonald: Works of Art

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Abstract

Ian Macdonald's works changed our perspective on so many parts of algebraic combinatorics and formal power series. This talk will display some selected works of the art of Ian Macdonald, representative of different periods of his œuvre, and analyze how they resonate, both for the past development of our subject and for its future.

Contents

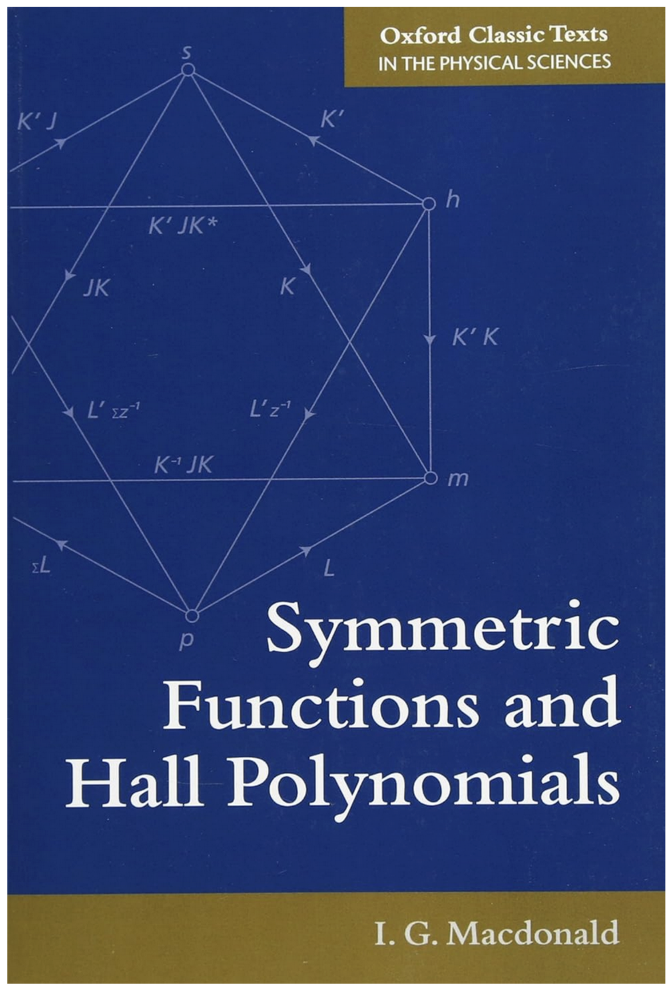
Acknowledgments. First and foremost my thanks go to Ian Macdonald for his teaching, companionship and for giving me bunches of handwritten notes and copies of his books over the years. I am so grateful that circumstances were such that I was able to convey these thanks directly to him in person in June 2023. I thank David Lumsden, Ziheng Zhou, Alex Shields and Dhruv Gupta for energy and insight as we worked through Ian Macdonald's unpublished manuscript on the n-line. I thank Chris Macdonald for reaching out to provide scans and files of the n-line manuscript. I am very grateful to Laura Colmenarejo, Persi Diaconis and Ole Warnaar for helpful suggestions and revisions on this tribute article.

AMS Subject Classifications: Primary 01A70; Secondary 05E05.

1 Preamble

This paper was prepared for the occasion of a lecture in tribute to Ian G. Macdonald, delivered at FPSAC 2024 in Bochum, Germany on 22 July 2024. I want to express thanks to the Executive Committee of FPSAC, the Organizing Committee of FPSAC 2024, and to the whole of our FPSAC 2024 community for making this lecture a possibility and for considering me for its delivery. Macdonald is my hero, and to be asked to play such a role in his legacy touches me deeply.

For this paper I have chosen a few selected topics from Macdonald's immense contributions to highlight (Macdonald polynomials, classification of affine root systems, cohomology and proof of the Weil conjectures for symmetric products of a curve, and the Clifford chain). I hope that you, as reader, will have your own favorites from Macdonald's œuvre and that your choices are equally stimulating, but different from mine. The final sections of this article highlight some of Macdonald's service contributions: as an influencer, as a translator, and as an expositor.



Ian G. Macdonald

The Symmetric Functions Bible

The image of Ian G. Macdonald is from https://sites.google.com/view/garsiafest/mementos

2 Tableaux and Macdonald polynomials

One of our favorite formulas is the formula for the Schur polynomial as a sum over semistandard Young tableaux (SSYTs),

$$s_\lambda = \sum_{T\in B(\lambda)} x^T, \qquad \text{where} \qquad \begin{matrix} B(\lambda) = \{\text{SSYTs of shape } \lambda\} \\ \text{and} \\ x^T = x_1^{(\#1\text{s in } T)} \cdots x_n^{(\#n\text{s in } T)}. \end{matrix}$$

It is most amazing that if $\delta = (n-1,\dots,2,1,0)$ then

$$s_\lambda = \frac{a_{\lambda+\delta}}{a_\delta}, \qquad \text{where} \qquad a_\mu = \sum_{w\in S_n} (-1)^{\ell(w)} w x^\mu$$

with $x^\mu = x_1^{\mu_1}\cdots x_n^{\mu_n}$ if $\mu = (\mu_1,\dots,\mu_n)$. This second formula for the Schur polynomial is the "Weyl character formula", which (in this type A case) was one of the first definitions of the Schur function (Jacobi 1841, according to Macdonald).

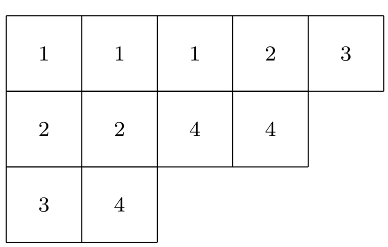
Macdonald pointed out something spectacular. The first formula for the Schur polynomial is the special case $q = t$ of the formula

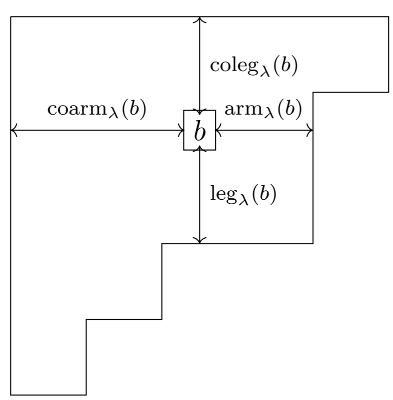
$$P_\lambda(q,t) = \sum_{T\in B(\lambda)} x^T \psi_T(q,t), \qquad \text{where } \psi_T(q,t) \text{ is given by (2.2) below,}$$

and the second formula for the Schur polynomial is a special case of

$$P_\lambda(q,qt) = \frac{A_{\lambda+\delta}(q,t)}{A_\delta(q,t)}, \qquad \text{where} \quad A_\mu(q,t) = \sum_{w\in S_n} (-t^{-\frac{1}{2}})^{\ell(w)} T_w E_\mu(q,t)$$

with T_w and $E_\mu(q,t)$ as defined (2.3) and (2.4) below. Maybe we think Schur polynomials are cool, but the Macdonald polynomials $P_\lambda(q,t)$ are two parameters cooler.

$T =$ 



$$P_\lambda(q,t) = \sum_{T\in B(\lambda)} x^T \psi_T(q,t).$$

2.1 (q,t)-hooks and the bosonic Macdonald polynomials $P_\lambda(q,t)$

Let $\lambda \in \mathbb{Z}^n_{\geq 0}$ with $\lambda_1 \geq \cdots \geq \lambda_n$ so that λ is a partition of length at most n.

A *box in* λ is a pair $b = (r,c)$ with $r \in \{1,\ldots,n\}$ and $c \in \{1,\ldots,\lambda_r\}$.

Identify λ with its set of boxes so that

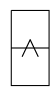
$$\lambda = \{(r,c) \mid (r,c) \text{ is a box in } \lambda\}.$$

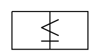
For a box $b = (r,c)$ in λ define

$$\begin{aligned} \mathrm{arm}_\lambda(b) &= \mathrm{arm}_\lambda(r,c) = \{(r,c') \in \lambda \mid c' > c\} \qquad \text{and} \\ \mathrm{leg}_\lambda(b) &= \mathrm{leg}_\lambda(r,c) = \{(r',c) \in \lambda \mid r' > r\}. \end{aligned}$$

A *SSYT (semistandard Young tableau) of shape* λ *filled from* $\{1,\ldots,n\}$ is a function

$$T\colon \lambda \to \{1,\ldots,n\} \qquad \text{such that}$$

(a) If $(r,c),(r+1,c) \in \lambda$ then $T(r,c) < T(r+1,c)$, 

(b) If $(r,c),(r,c+1) \in \lambda$ then $T(r,c) \leq T(r,c+1)$. 

Let

$$B(\lambda) = \{\text{SSYTs of shape } \lambda \text{ filled from } \{1,\ldots,n\}\}.$$

Let $T \in B(\lambda)$ and let $b \in \lambda$. Let $T(b)$ denote the entry in box b of T. Let $i \in \{1,\ldots,n\}$ with $i > T(b)$. Define the i*-restricted arm length*, i*-restricted leg length*, and i*-restricted* (q,t)*-hook length* by

$$\begin{aligned} a(b,<i) &= \mathrm{Card}\{b' \in \mathrm{arm}_\lambda(b) \mid T(b') < i\}, \\ l(b,<i) &= \mathrm{Card}\{b' \in \mathrm{leg}_\lambda(b) \mid T(b') < i\}, \end{aligned} \qquad \text{and} \qquad h_T(b,<i) = \frac{1 - t\cdot q^{a(b,<i)}t^{l(b,<i)}}{1 - q\cdot q^{a(b,<i)}t^{l(b,<i)}}. \tag{2.1}$$

For a column strict tableau $T \in B(\lambda)$ define

$$\psi_T(q,t) = \prod_{b\in\lambda} \psi_T(b), \qquad \text{where} \qquad \psi_T(b) = \prod_{\substack{i>T(b),\, i\in T(\mathrm{arm}_\lambda(b)) \\ i\notin T(\mathrm{leg}\lambda(b))}} \frac{h_T(b,<i)}{h_T(b,<i+1)}. \tag{2.2}$$

The *bosonic Macdonald polynomial* is $P_\lambda(q,t) \in \mathbb{C}[x_1,\ldots,x_n]$ given by

$$P_\lambda(q,t) = \sum_{T\in B(\lambda)} x^T \psi_T(q,t), \qquad \text{where} \quad x^T = x_1^{(\#1\text{s in } T)} \cdots x_n^{(\#n\text{s in } T)}.$$

The *Schur polynomial* is

$$s_\lambda = P_\lambda(t,t) = P_\lambda(0,0) = \sum_{T\in B(\lambda)} x^T.$$

2.2 Electronic and fermionic Macdonald polynomials

For $i \in \{1, \dots, n-1\}$ and $f \in \mathbb{C}[x_1, \dots, x_n]$ define

$$(s_i f)(x_1, \dots, x_n) = f(x_1, \dots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \dots, x_n)$$

and

$$T_i f = -t^{-\frac{1}{2}} f + (1+s_i) \frac{t^{-\frac{1}{2}} - t^{\frac{1}{2}} x_i^{-1} x_{i+1}}{1 - x_i^{-1} x_{i+1}} f.$$

If $w \in S_n$ and $w = s_{i_1} \cdots s_{i_\ell}$ is a reduced word for w as a product of s_is then write

$$T_w = T_{i_1} \cdots T_{i_\ell} \qquad \text{and} \qquad w = s_{i_1} \cdots s_{i_\ell}. \tag{2.3}$$

For $i \in \{1 \dots, n-1\}$ let

$$\partial_i = (1+s_i) \frac{1}{x_i - x_{i+1}}.$$

The *electronic Macdonald polynomial* $E_\mu = E_\mu(q,t)$ is recursively determined by

(E0) $E_{(0,\dots,0)} = 1$,

(E1) $E_{(\mu_n+1, \mu_1, \dots, \mu_{n-1})} = q^{\mu_n} x_1 E_\mu(x_2, \dots, x_n, q^{-1} x_1)$,

(E2) If $(\mu_1, \dots, \mu_n) \in \mathbb{Z}^n_{\geq 0}$ and $\mu_i > \mu_{i+1}$ then

$$E_{s_i \mu} = \Big(\partial_i x_i - t x_i \partial_i + \frac{(1-t) q^{\mu_i - \mu_{i+1}} t^{v_\mu(i) - v_\mu(i+1)}}{1 - q^{\mu_i - \mu_{i+1}} t^{v_\mu(i) - v_\mu(i+1)}} \Big) E_\mu, \tag{2.4}$$

where $v_\mu \in S_n$ is the minimal length permutation such that $v_\mu \mu$ is weakly increasing.

The monomial x^μ is $x^\mu = x_1^{\mu_1} \cdots x_n^{\mu_n}$. The world of Macdonald polynomials replaces the monomials x^μ with electronic Macdonald polynomials E_μ and replaces the action of permutations w by the operators T_w.

Let $\delta = (n-1, n-2, \dots, 2, 1, 0)$ and let $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$ with $\lambda_1 \geq \cdots \geq \lambda_n$. Let w_0 be the longest element of S_n so that $\ell(w_0) = \binom{n}{2}$. Then define

$$A_{\lambda+\delta}(q,t) = \sum_{w \in S_n} (-t^{-\frac{1}{2}})^{\ell(w) - \ell(w_0)} T_w E_{\lambda+\delta}(q,t) \qquad \text{and} \qquad a_{\lambda+\delta} = \sum_{w \in S_n} (-1)^{\ell(w) - \ell(w_0)} w x^{\lambda+\delta}.$$

The $A_{\lambda+\delta}(q,t)$ are the *fermionic Macdonald polynomials* (see [CR22, Intro] for motivation for the 'electronic', 'bosonic', 'fermionic' terminology which is in parallel analogy with the isomorphism between Heisenberg algebra representations on fermionic Fock space (an exterior algebra) and bosonic Fock space (a symmetric algebra) which appears, for example, in [Kac, § 14.10]).

2.3 The Weyl character formula

The "Weyl character formula" in the next theorem gives a formula for the bosonic Macdonald polynomial as a quotient of two fermionic Macdonald polynomials. When $q = t = 0$ this formula becomes the formula for the Schur function as a quotient of two determinants.

Theorem 2.1. *Let $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n_{\geq 0}$ with $\lambda_1 \geq \cdots \geq \lambda_n$.*

(a)

$$P_\lambda(q, qt) = \frac{A_{\lambda+\delta}(q,t)}{A_\delta(q,t)} \qquad \textit{and} \qquad s_\lambda = \frac{a_{\lambda+\delta}}{a_\delta}.$$

(b)

$$A_\delta(q,t) = \prod_{i<j} (x_i - t x_j) \qquad \textit{and} \qquad a_\delta = \prod_{i<j} (x_i - x_j).$$

3 Can you do type B?

Having worked something out for type A, a natural next problem for our community is to work it out for type B. Here Macdonald has something interesting to say.

Which type B?

Because, as Macdonald worked out in his 1972 paper on affine root systems,

there are 9 different type Bs.

A diagram showing these is given in Section 3.1.

But, there is something wonderful here. The type $(C^\vee, C)$ root system is one of the type Bs and

all other type Bs are obtained by specializations from type $(C^\vee, C)$.

This means that, if one wants to compute Macdonald polynomials for any one of the 9 different type Bs, then all one has to do, is compute the Macdonald polynomials for type $(C^\vee, C)$ and then specialize parameters as appropriate.

Each of the affine root systems of classical type is a subset of the $\mathbb{Z}$-vector space spanned by symbols $\varepsilon_1, \ldots, \varepsilon_n$ and $\frac{1}{2}\delta$,

$$V_\mathbb{Z} = \mathbb{Z}\text{-span}\{\varepsilon_1, \ldots, \varepsilon_n, \tfrac{1}{2}\delta\}.$$

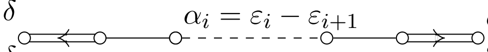
The affine Weyl group W is the group of $\mathbb{Z}$-linear transformations of $V_\mathbb{Z}$ generated by the transformations $s_0, s_1, \ldots, s_n$ given by: for $\lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n + \frac{k}{2}\delta$,

$$\begin{aligned}
s_0\lambda &= -\lambda_1\varepsilon_1 + \lambda_2\varepsilon_2 + \cdots + \lambda_n\varepsilon_n + \left(\tfrac{k}{2} + \lambda_1\right)\delta, \\
s_n\lambda &= \lambda_1\varepsilon_1 + \cdots + \lambda_{n-1}\varepsilon_{n-1} - \lambda_n\varepsilon_n + \tfrac{k}{2}\delta, \qquad \text{and} \\
s_i\lambda &= \lambda_1\varepsilon_1 + \cdots + \lambda_{i-1}\varepsilon_{i-1} + \lambda_{i+1}\varepsilon_i + \lambda_i\varepsilon_{i+1} + \lambda_{i+2}\varepsilon_{i+2} + \cdots + \lambda_n\varepsilon_n + \tfrac{k}{2}\delta,
\end{aligned}$$

for $i \in \{1, \ldots, n-1\}$. Each of the affine root systems of classical type is defined by which orbits of the affine Weyl group W that it contains. Let

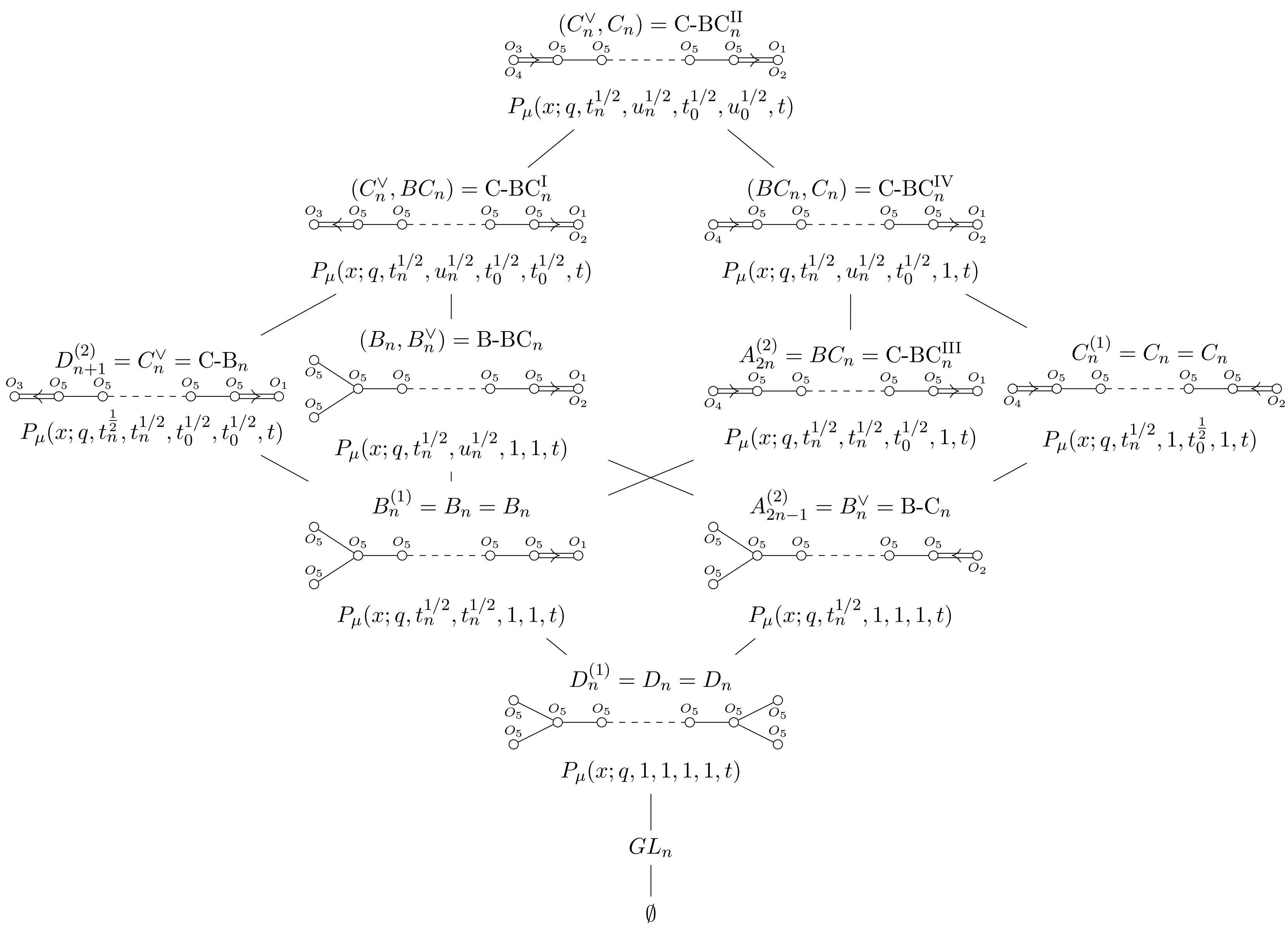
$$\begin{aligned}
O_1 &= W\cdot\alpha_n = W\cdot\varepsilon_n = \{\pm\varepsilon_i + r\delta \mid i \in \{1,\ldots,n\}, r\in\mathbb{Z}\}, \\
O_2 &= W\cdot 2\alpha_n = W\cdot 2\varepsilon_n = \{\pm 2\varepsilon_i + 2r\delta \mid i \in \{1,\ldots,n\}, r\in\mathbb{Z}\}, \\
O_3 &= W\cdot\alpha_0 = W\cdot(-\varepsilon_1 + \tfrac{1}{2}\delta) = \{\pm(\varepsilon_i + \tfrac{1}{2}(2r+1)\delta \mid i \in \{1,\ldots,n\}, r\in\mathbb{Z}\}, \\
O_4 &= W\cdot 2\alpha_0 = W\cdot(-2\varepsilon_1 + \delta) = \{\pm 2\varepsilon_i + (2r+1)\delta \mid i \in \{1,\ldots,n\}, r\in\mathbb{Z}\}, \\
O_5 &= W\cdot\alpha_1 = W\cdot(\varepsilon_1 - \varepsilon_2) = \left\{ \begin{array}{c} \pm(\varepsilon_i + \varepsilon_j) + r\delta \\ \pm(\varepsilon_i - \varepsilon_j) + r\delta \end{array} \;\middle|\; i,j \in \{1,\ldots,n\}, i<j, r\in\mathbb{Z} \right\},
\end{aligned}$$

where

$\alpha_0 = -\varepsilon_1 + \frac{1}{2}\delta$, $2\alpha_0 = -2\varepsilon_1 + \delta$  $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ … $\alpha_n = \varepsilon_n$, $2\alpha_n = 2\varepsilon_n$.

With these notations the irreducible affine root systems of classical type (and the appropriate specializations for obtaining the Macdonald polynomials of each type from the Macdonald polynomials of type $(C^\vee, C)$) are given by the following diagram. The middle notation for each root system is the notation in Macdonald [Mac03, § 1.3], the right notation is that of Bruhat and Tits [BT72] and the left notation is that of Kac [Kac, Ch. 6].

3.1 The poset of affine root systems of classical type



4 Circles and Lines

Though I don't travel often to England, whenever a trip did bring me to England I liked to try to stop in and visit Ian and his wife Greta if I could manage it. Greta passed away in 2019, and I saw Ian at his place two times after that. The last time was in June of 2023. When I first arrived, Ian emphatically told me he hadn't thought about mathematics in 15 years. He pointed to the Sudoku puzzles and the newspapers on his table as evidence. We chatted about mutual friends in mathematics and other memories.

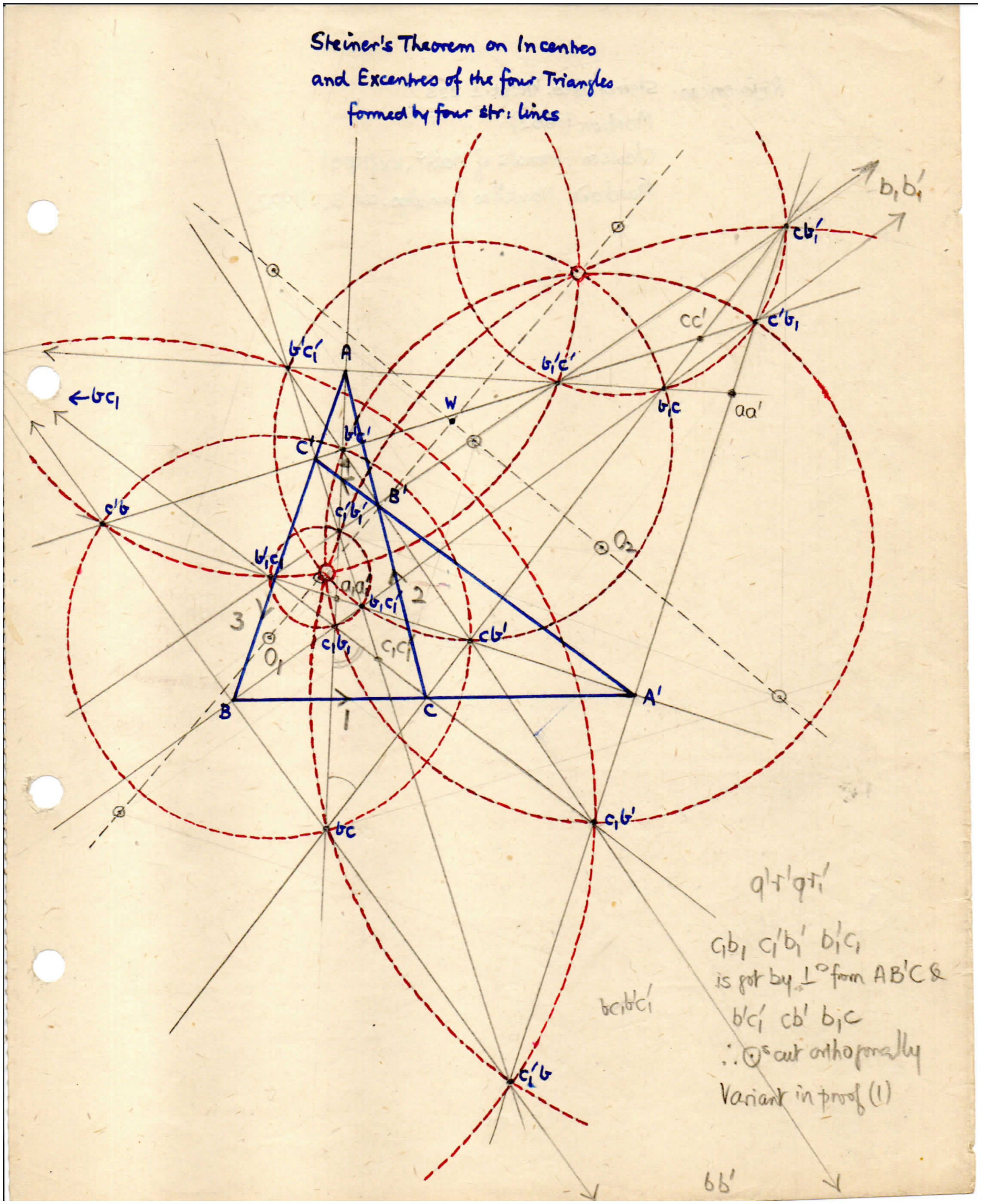
One evening during my visit, Ian and his sister and I went across the road for dinner – fish and chips and beer. During that dinner it came out in conversation with Ian's sister – Ian had indeed recently been fiddling with some mathematics, and Ian told me about the Clifford circle for the n-line. After dinner, when I was back at his place chatting; at some point, Ian lifted himself out of his chair, walked over to the other side of the room, picked up a manuscript, and gifted it to me. He explained that that was what he had been fiddling with and that it was a supplementary chapter to a book he had written just after high school on circles and triangles. It seems that the manuscript to the book was lost, but I was being given the supplementary chapter. I didn't quite know what to make of that, but I carefully packed it in my suitcase for my trip home.

After Ian passed away, his son kindly sent me scans of the original handwritten manuscript of the supplementary chapter and the tex source of the printed copy that Ian gave to me. Since the topic was lines and circles in the plane, I got a few undergraduates together to work through the manuscript.

The author of this manuscript was a talented math student right out of high school. He clearly had not read our key reference for symmetric functions – his notations for symmetric functions are certainly nonstandard for anyone that has read the Symmetric Functions Bible. This student shows a penchant for thorough work and thinking. For the first main theorem appearing in the manuscript he gives 6 or 7 different proofs, all from different points of view, before moving on to generalizations. This high school student is incredibly deft with classical and projective geometry and complex numbers (linear equations, determinants, lemniscates, cardiods, deltoids, Euler lines, coaxal systems, Newton identities for symmetric functions, etc.). Some of the induction proofs are a little bit clumsy – it seems that this student has not been formally taught 'proof by induction' like we might do in a first proof course for undergraduates. The command and thoroughness that this student exhibits extends to his referencing of the literature – in our modern times most of our community has no idea of the main players of classical intersection geometry any more. But this high school student was on top of this literature. If there were one piece of advice that I'd give to this student, it would be to read the books of Ian Macdonald and improve his writing style by emulating the master (admittedly, these books were not yet available).

After getting a feel for the contents of this high school student's manuscript I began to understand Macdonald's early trajectory in mathematics. He did Tripos at Cambridge and had some exposure to the professors there. Particularly from the vantage of Hodge, Pedoe and Todd, intersection theory and its connection to cohomology was "in the air" but not fully developed. Indeed, in his first published paper [Mac58], Macdonald thanks "Dr. J.A. Todd for his interest and helpful advice". By the time of his 1962 papers, Macdonald was clearly following the work of Grothendieck, and had understood that cohomology was an efficient way to compute intersections of the type that he had been computing in high school. In his paper [Mac62b] he already wields the tools of sheaves and cohomology like a master. It is truly amazing to see how this high school student's interest in intersections in classical geometry led him to the very forefront of the technology of cohomology and algebraic geometry that was being vigorously developed at the time.

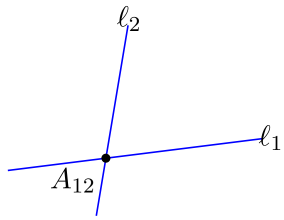
By 1962, without a Ph.D., Ian Macdonald was no longer a high school student, but had followed his nose to already become a mature mathematician of the highest caliber and a great expositor.



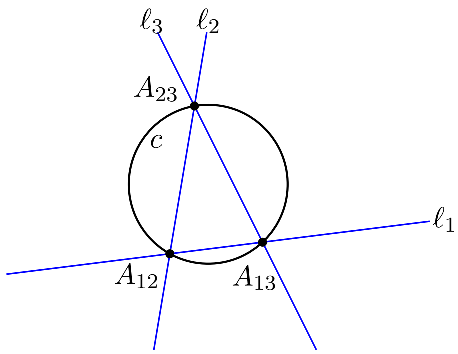
A diagram from Ian Macdonald's 1947 manuscript on the n-line

4.1 Clifford's n-line chain

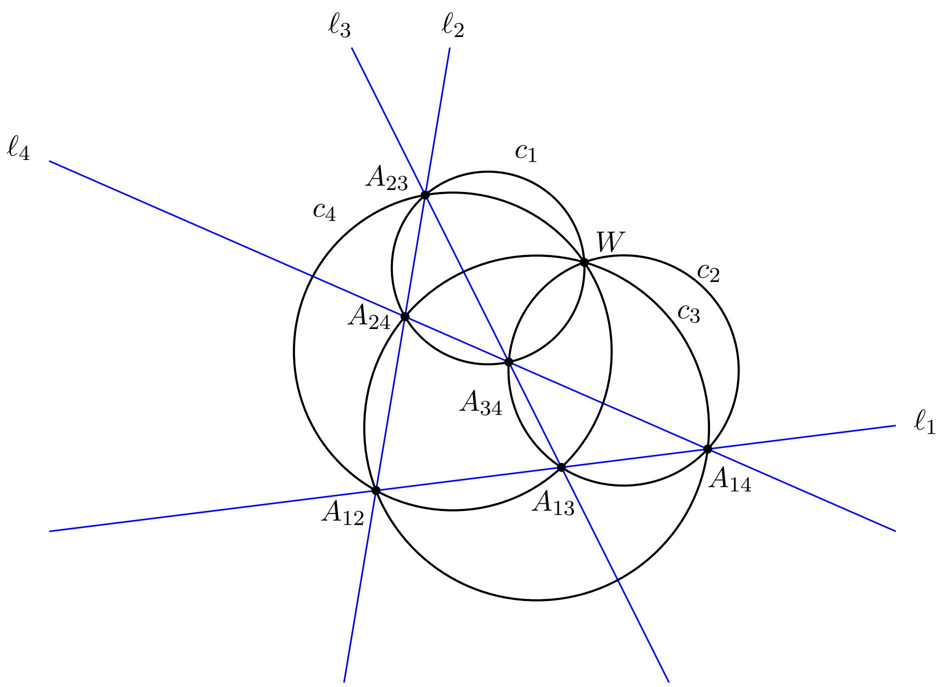
Two generic lines ℓ_1 and ℓ_2 intersect in a point A_{12}. The point A_{12} is the *Clifford point* of the 2-line.



Each pair of lines in a generic 3-line $\{\ell_1, \ell_2, \ell_3\}$ intersect in a point, and these three points determine a circle c. The circle c is the *Clifford circle of the* 3-*line.*



Each triple of lines in a generic 4-line $\{\ell_1, \ell_2, \ell_3, , \ell_4\}$ determines a Clifford circle, giving the circles c_1, c_2, c_3, c_4. These four circles intersect in a point W. The point W is the *Clifford point of the* 4-*line.*

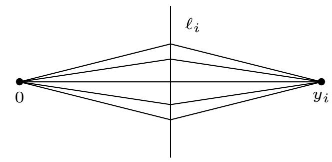


Each 4-tuple of lines in a generic 5-line $\{\ell_1, \ell_2, \ell_3, , \ell_4, \ell_5\}$ determines a Clifford point, giving the points p_1, p_2, p_3, p_4, p_5. These five points lie on a circle C. The circle C is the *Clifford circle of the* 5-*line.*
... and so on ...

4.2 Ian Macdonald's general formulation

Let $y_1, \ldots, y_n \in \mathbb{C}^\times$. For $i \in \{1, \ldots, n\}$ let ℓ_i be the line consisting of the points in $\mathbb{C}$ that are equidistant from 0 and y_i. The set of n lines is *the n-line* $\mathcal{L} = \{\ell_1, \ldots, \ell_n\}$, where

$$\ell_i = \{z \in \mathbb{C} \mid \bar{z} = t_i(z - y_i)\}, \qquad \text{where} \quad t_i = \frac{-\overline{y_i}}{y_i}.$$



For $k \in \{0, 1, \ldots, n-1\}$, define

$$c_k(\mathcal{L}) = \frac{y_1 t_1^{n-1-k}}{g_1(\mathcal{L})} + \frac{y_2 t_2^{n-1-k}}{g_2(\mathcal{L})} + \cdots + \frac{y_n t_n^{n-1-k}}{g_n(\mathcal{L})},$$

where

$$g_j(\mathcal{L}) = (t_j - t_1)(t_j - t_2)\cdots(t_j - t_{j-1})(t_j - t_{j+1})(t_j - t_{j+2})\cdots(t_j - t_n),$$

for $j \in \{1, \ldots, n\}$.

Theorem 4.1 (Clifford's chain)**.** *Let $\mathcal{L} = \{\ell_1, \ldots, \ell_n\}$ be an n-line (satisfying an appropriate genericity condition).*

Case n even: Each $(n-1)$-subset of the n-line determines a Clifford circle, and these n Clifford circles intersect in a unique point $p(\mathcal{L})$. Let $k \in \mathbb{Z}_{>0}$ such that $n = 2k$ and let $a_1, \ldots, a_{k-1} \in \mathbb{C}$ be given by

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{k-1} \end{pmatrix} = \begin{pmatrix} c_2(\mathcal{L}) & \cdots & c_k(\mathcal{L}) \\ \vdots & & \vdots \\ c_k(\mathcal{L}) & \cdots & c_{2k-2}(\mathcal{L}) \end{pmatrix}^{-1} \begin{pmatrix} -c_1(\mathcal{L}) \\ -c_2(\mathcal{L}) \\ \vdots \\ -c_{k-1}(\mathcal{L}) \end{pmatrix}.$$

Then

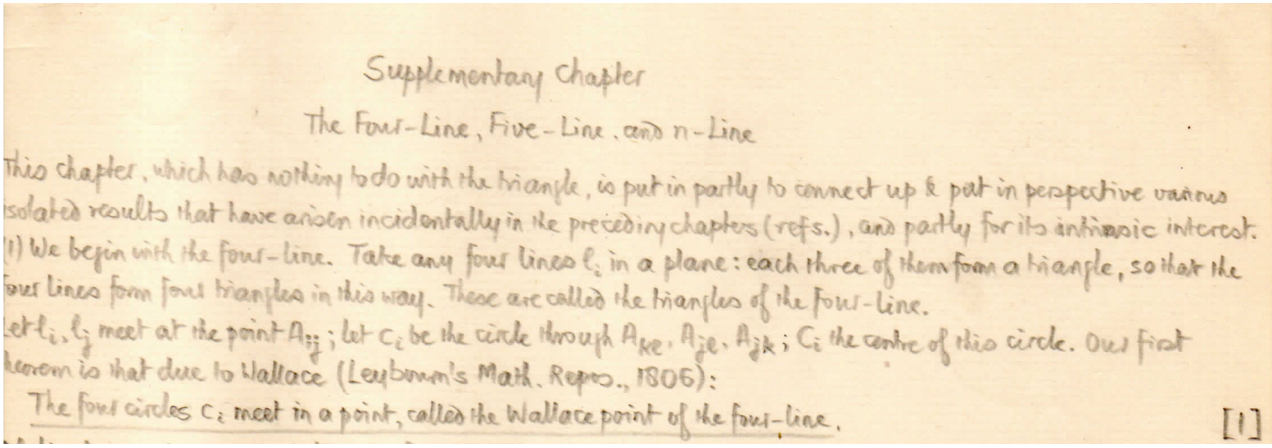
$$p(\mathcal{L}) = c_0(\mathcal{L}) + a_1 c_1(\mathcal{L}) + a_2 c_2(\mathcal{L}) + \cdots + a_{k-1} c_{k-1}(\mathcal{L})$$

is the Clifford point of the n-line $\mathcal{L} = \{\ell_1, \ldots, \ell_{2k}\}$.

Case n odd: Each $(n-1)$-subset of the n-line determines a Clifford point, and these n Clifford points lie on a unique circle $C(\mathcal{L})$. Let $k \in \mathbb{Z}_{>0}$ such that $n = 2k+1$. The Clifford circle $C(\mathcal{L})$ is given by

$$C(\mathcal{L}) = \{A(\mathcal{L}) - \theta B(\mathcal{L}) \mid \theta \in U_1(\mathbb{C})\}, \qquad \textit{where} \qquad U_1(\mathbb{C}) = \{\theta \in \mathbb{C} \mid \theta\bar{\theta} = 1\},$$

$$A(\mathcal{L}) = \frac{\det \begin{pmatrix} c_0(\mathcal{L}) & \cdots & c_{k-1}(\mathcal{L}) \\ c_1(\mathcal{L}) & \cdots & c_k(\mathcal{L}) \\ \vdots & & \vdots \\ c_{k-1}(\mathcal{L}) & \cdots & c_{2k-2}(\mathcal{L}) \end{pmatrix}}{\det \begin{pmatrix} c_2(\mathcal{L}) & \cdots & c_k(\mathcal{L}) \\ \vdots & & \vdots \\ c_k(\mathcal{L}) & \cdots & c_{2k-2}(\mathcal{L}) \end{pmatrix}} \qquad \textit{and} \qquad B(\mathcal{L}) = \frac{\det \begin{pmatrix} c_1(\mathcal{L}) & \cdots & c_k(\mathcal{L}) \\ \vdots & & \vdots \\ c_k(\mathcal{L}) & \cdots & c_{2k-1}(\mathcal{L}) \end{pmatrix}}{\det \begin{pmatrix} c_2(\mathcal{L}) & \cdots & c_k(\mathcal{L}) \\ \vdots & & \vdots \\ c_k(\mathcal{L}) & \cdots & c_{2k-2}(\mathcal{L}) \end{pmatrix}}.$$



Supplementary Chapter

The Four-Line, Five-Line, and n-Line

This chapter, which has nothing to do with the triangle, is put in partly to connect up & put in perspective various solated results that have arisen incidentally in the preceding chapters (refs.), and partly for its intrinsic interest.
(1) We begin with the four-line. Take any four lines l_i in a plane: each three of them form a triangle, so that the our lines form four triangles in this way. These are called the triangles of the four-line.
et l_i, l_j meet at the point A_{ij}; let C_i be the circle through A_{kl}, A_{jl}, A_{jk}; c_i the centre of this circle. Our first heorem is that due to Wallace (Leybourn's Math. Repos., 1806):
The four circles C_i meet in a point, called the Wallace point of the four-line. [1]

The first page of the high school student's manuscript

5 The symmetric product of a curve Σ

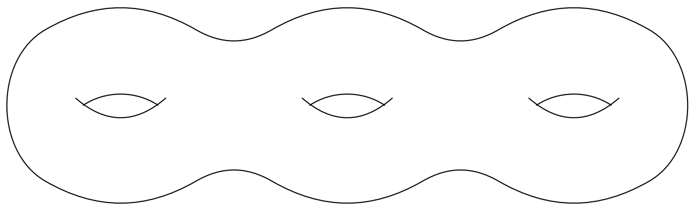
After high school Macdonald served in the military and then did the Mathematical Tripos at Trinity. After finishing at Cambridge in 1952, at the insistence of his father, Macdonald took competitive exams for a civil service job (i.e., a government job). He "stuck it out for five years" in a civil service job before leaving his good secure job for a temporary (1957–1960) position at Manchester and then another temporary position (1960–1963) at Exeter University. Then he became "Fellow and Tutor in Mathematics" at Magdalen College at Oxford until 1972.

In 1958, Macdonald's first paper appeared in Proceedings of the Cambridge Philosophical Society. Very likely this study arose as a continuation of his study of intersections of lines and circles. The paper is entitled *"Some enumerative formulae for algebraic curves"*. In Part I, Macdonald gives a generalization of de Jonquières formula and Part II makes contact with Schur functions and Schubert conditions in intersection theory. It shows a mastery of the methods of the classical Italian algebraic geometry school. This paper is a significant development of his high school knowledge of intersection theory. Even so, it hardly gives any hint of the amazing achievement that was to come next.

By 1962, Macdonald had understood that intersection numbers of families of curves could be computed by using cohomology as a tool. In his paper on the cohomology of symmetric products of an algebraic curve [Mac62b] he states "in particular we obtain natural proofs of the results of an earlier paper [Mac58] which were there obtained laboriously by classical methods."

5.1 The cohomology of a symmetric product of a curve

Let Σ be a curve.

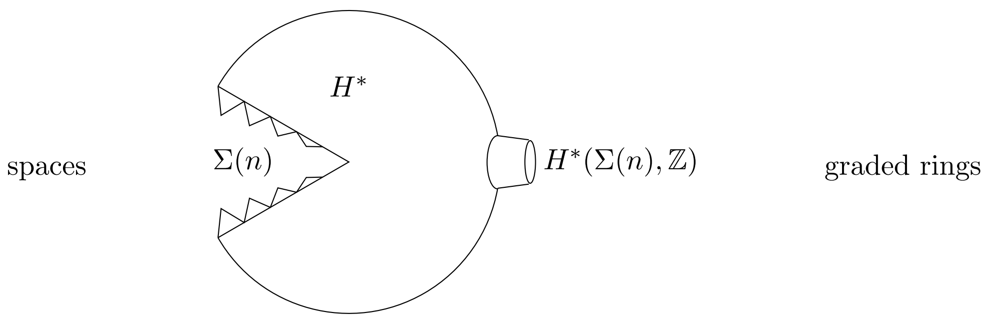


The nth symmetric power of Σ is

$$\Sigma(n) = \Sigma^n/S_n, \qquad \text{where} \quad w \cdot (p_1, \ldots, p_n) = (p_{w^{-1}(1)}, \ldots, p_{w^{-1}(n)}),$$

for $w \in S_n$ and $(p_1, \ldots, p_n) \in \Sigma^n$.

Cohomology is a creature (more precisely, a functor) that eats spaces and outputs graded rings.



In spite of the frightening teeth, cohomologies are really quite friendly (it is the spaces that are dangerous). How does one write down the cohomology $H^*(X;\mathbb{Z})$ of a space X? Well, $H^*(X,\mathbb{Z})$ is a graded ring and a graded ring is written down in a presentation by generators and relations. Macdonald's 1962 paper [Mac62b] gives an elegant presentation of the graded ring $H^*(\Sigma(n),\mathbb{Z})$, the cohomology of the nth symmetric product of a curve Σ.

Theorem 5.1. *Let Σ be a curve of genus g. The cohomology ring $H^*(\Sigma(n),\mathbb{Z})$ is the $\mathbb{Z}$-algebra presented by generators*

$$\xi_1, \ldots, \xi_g, \quad \xi'_1, \ldots, \xi'_g, \quad \eta$$

and relations

(a) If $i, j \in \{1, \ldots, g\}$ then

$$\xi_i\xi_j = -\xi_j\xi_i, \qquad \xi'_i\xi'_j = -\xi'_j\xi'_i, \qquad \xi_i\xi'_j = -\xi'_j\xi_i,$$

$$\xi_i\eta = \eta\xi_i, \qquad \xi'_i\eta = \eta\xi'_i,$$

(b) If $a, b, c, q \in \mathbb{Z}_{\geq 0}$ and $a + b + 2c + q = n + 1$ and $i_1, \ldots, i_a, j_1, \ldots, j_b, k_1, \ldots, k_c$ are distinct elements of $\{1, \ldots, g\}$ then

$$\xi_{i_1} \cdots \xi_{i_a} \xi'_{j_1} \cdots \xi'_{j_b} (\xi_{k_1}\xi'_{k_1} - \eta) \cdots (\xi_{k_c}\xi'_{k_c} - \eta)\eta^q = 0.$$

5.2 The Weil conjectures for the symmetric product $\Sigma(n)$ of a curve

Weil's famous conjectures about zeta functions of algebraic varieties are from his paper of 1949 [We49]. These conjectures stimulated a huge effort which included the development of étale cohomology and ℓ-adic cohomology. The Weil conjectures were proved in the 1960s and 70s: the proof of the rationality conjecture came in 1960 (Dwork), the proof of the functional equation and Betti numbers connection in 1965 (Grothendieck school) and the analogue of the Riemann hypothesis in 1974 (Deligne). In 1962 [Mac62b], as an application of his description of the cohomology of $\Sigma(n)$, Macdonald proved Weil's conjectures in an important special case: "... we calculate the zeta function of $\Sigma(n)$ and verify Weil's conjectures in this case."

The zeta function $Z(t)$ of an algebraic variety X is an exponential generating function for the number of points of X over the finite fields $\mathbb{F}_{q^n}$,

$$\frac{d}{dt}\log Z(t) = \sum_{n\in\mathbb{Z}_{>0}} \mathrm{Card}(X(\mathbb{F}_{q^n}))t^{n-1}.$$

Let Σ be a curve of genus g and assume that $\rho_1,\ldots,\rho_{2g}\in\mathbb{C}$ are such that

$$Z_1(t) = \frac{(1-\rho_1 t)\cdots(1-\rho_{2g}t)}{(1-t)(1-qt)} \qquad \text{is the zeta function of } \Sigma.$$

Let $\phi_0(t) = 1-t$ and, for $k\in\{1,\ldots,2g\}$, let

$$\phi_k(t) = \prod_{1\le i_1<\cdots<i_k\le 2g} (1-\rho_{i_1}\cdots\rho_{i_k}t).$$

Then define

$$F_k(t) = \begin{cases} \phi_k(t)\phi_{k-2}(t)\phi_{k-4}(t)\cdots, & \text{if } k\in\{0,1,\ldots,n\},\\ F_{2n-k}(q^{k-n}t), & \text{if } k\in\{n+1,\ldots,2n\}.\end{cases}$$

Corollary 5.2. *The Weil conjectures hold for $\Sigma(n)$. More specifically,*

(a) The zeta function of $\Sigma(n)$ is

$$Z_n(t) = \frac{F_1(t)F_3(t)\cdots F_{2n-1}(t)}{F_0(t)F_2(t)\cdots F_{2n}(t)}.$$

(b) The Riemann hypothesis for $\Sigma(n)$ holds:

$$\textit{All roots of } Z_n(t) \textit{ have absolute value in} \quad \{q^{-\frac{1}{2}\cdot 0}, q^{-\frac{1}{2}\cdot 1}, q^{-\frac{1}{2}\cdot 2},\ldots,q^{-\frac{1}{2}\cdot 2n}\}.$$

(c) The functional equation for $\Sigma(n)$ is

$$Z_n\big(\tfrac{1}{q^n t}\big) = (-q^{-\frac{1}{2}n}t)^{(-1)^n\binom{2g-2}{n}} Z_n(t).$$

6 I.G. Macdonald as influencer

6.1 Deligne–Lusztig 1976

In Lecture Notes in Math. **131**, T. Springer precisely states conjectures of Macdonald about complex representations of finite groups of Lie type. Looking back at these references, one gathers that the notes of Macdonald on Hall polynomials that were circulating in the late 1960's eventuallly became Chapter IV of his book on Symmetric functions and Hall polynomials. T. Springer's expositions appearing in [Spr70] make it clear that, by 1968, Ian Macdonald had understood how the type GL_n story from J.A. Green's 1955 paper could be reshaped for a statement for general Lie types. Macdonald's conjectures were proved by Deligne and Lusztig in 1976.

Annals of Mathematics, **103** (1976), 103–161

Representations of reductive groups over finite fields

By P. Deligne and G. Lusztig

Introduction

Let us consider a connected, reductive algebraic group G, defined over a finite field $\mathbf{F}_q$, with Frobenius map F. We shall be concerned with the representation theory of the finite group G^F, over fields of characteristic 0.

In 1968, Macdonald conjectured, on the basis of the character tables known at the time (GL_n, Sp_4), that there should be a well defined correspondence which, to any F-stable maximal torus T of G and a character θ of T^F in general position, associates an irreducible representation of G^F; moreover, if T modulo the centre of G is anisotropic over $\mathbf{F}_q$, the corresponding representation of G^F should be cuspidal (see Seminar on algebraic groups and related finite groups, by A. Borel et al., Lecture Notes in Mathematics, 131, pp. 117 and 101). In this paper we prove Macdonald's conjecture. More precisely, for T as above and θ an arbitrary character of T^F we construct virtual representations R_T^θ which have all the required properties.

6.2 Maulik–Yun 2013

It is still the case that most topologists and geometers view Macdonald's computation of the cohomology of the symmetric product of a curve [Mac62b] as his most well known achievement. In recent years the study of moduli spaces of curves and related cohomological Hall algebras has become an important part of geometry and mathematical physics, and Macdonald's study of symmetric products of curves continues to be an important stimulus for research in this direction today.

J. reine angew. Math., Ahead of Print
DOI 10.1515/crelle-2012-0093

Journal für die reine und angewandte Mathematik

Macdonald formula for curves with planar singularities

By *Davesh Maulik* at Columbia and *Zhiwei Yun* at Stanford

Abstract. We generalize Macdonald's formula for the cohomology of Hilbert schemes of points on a curve from smooth curves to curves with planar singularities: we relate the cohomology of the Hilbert schemes to the cohomology of the compactified Jacobian of the curve. The new formula is a consequence of a stronger identity between certain perverse sheaves defined by a family of curves satisfying mild conditions. The proof makes essential use of Ngô's support theorem for compactified Jacobians and generalizes this theorem to the relative Hilbert scheme of such families. As a consequence, we give a cohomological interpretation of the numerator of the Hilbert-zeta function of curves with planar singularities.

1. Introduction

Let C be a smooth projective connected curve over an algebraically closed field k. Let $\mathrm{Sym}^n(C)$ be the n-th symmetric product of C. Macdonald's formula [21] says there is a canonical isomorphism between graded vector spaces,

$$\mathrm{H}^*(\mathrm{Sym}^n(C)) \cong \mathrm{Sym}^n(\mathrm{H}^*(C)) = \bigoplus_{i+j\leq n, i,j\geq 0} \bigwedge^i (\mathrm{H}^1(C))[-i-2j](-j). \tag{1.1}$$

Here [?] denotes the cohomological shift and (?) denotes the Tate twist. This formula respects

6.3 Casselman 2012

An announcement of Macdonald's computation of the spherical function for p-adic groups appeared in 1968, and the full details appeared in his book published by the University of Madras in 1971. From the point of view of symmetric function theory, Macdonald proved that the favorite formula [Mac, Ch. III, (2.1)] for the Hall-Littlewood polynomial

$$P_\lambda(x;t) = \frac{1}{v_\lambda(t)} \sum_{w\in S_n} w\Big(x_1^{\lambda_1}\cdots x_n^{\lambda_n} \prod_{i<j} \frac{x_i - tx_j}{x_i - x_j}\Big)$$

generalizes to all Lie types and is a formula for the spherical function for G/K where G is the corresponding p-adic group $G = G(\mathbb{Q}_p)$ and $K = G(\mathbb{Z}_p)$ is a maximal compact subgroup of G.

1:17 p.m. August 25, 2012 [Macdonald]

Remarks on Macdonald's book on p-adic spherical functions

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When Ian Macdonald's book **Spherical functions on a group of *p*-adic type** first appeared, it was one of a very small number of publications concerned with representations of $\mathfrak{p}$-adic groups. At just about that time, however, the subject began to be widely recognized as indispensable in understanding automorphic forms, and the literature on the subject started to grow rapidly. Since it has by now grown so huge, in discussing here the subsequent history of some of Macdonald's themes I shall necessarily restrict myself only to things closely related to them. This will be no serious restriction since some of the most interesting problems in all of representation theory—among others, those connected with Langlands' 'fundamental lemma'—are concerned with $\mathfrak{p}$-adic spherical functions. Along the way I'll reformulate from a few different perspectives what his book contains. I'll begin, in the next section, with a brief sketch of the main points, postponing most technical details until later.

Throughout, suppose k to be what I call a $\mathfrak{p}$-adic field, which is to say that it is either a finite extension of some $\mathbb{Q}_p$ or the field of Laurent polynomials in a single variable with coefficients in a finite field. Further let

- $\mathfrak{o}$ = the ring of integers of k;
- $\mathfrak{p}$ = the maximal ideal of $\mathfrak{o}$;
- ϖ = a generator of $\mathfrak{p}$;
- $q = |\mathfrak{o}/\mathfrak{p}|$, so that $\mathfrak{o}/\mathfrak{p} \cong \mathbb{F}_q$.

Let $\mathbb{D}$ be a field of characteristic 0, which will play the role of coefficient field in representations. The minimal requirement on $\mathbb{D}$ is that it contain $\sqrt{q}$, but it will in the long run be convenient to assume that it is algebraically closed. It may usually be taken to be $\mathbb{C}$, but I want to emphasize that special properties of $\mathbb{C}$ are rarely required.

In writing this note I had one major decision to make about what class of groups I would work with. What made it difficult was that there were conflicting goals to take into account. On the one hand, I wanted to be able to explain a few basic ideas without technical complications. For this reason, I did not want to deal with arbitrary reductive groups, because even to state results precisely in this case would have required much distracting effort—effort, moreover, that would have just duplicated things explained very well in Macdonald's book. On the other, I wanted to illustrate some of the complexities that Macdonald's book confronts. In the end, I chose to restrict myself to **unramified** groups. *I will suppose throughout this account that G is a reductive group defined over k arising by base extension from a smooth reductive scheme over $\mathfrak{o}$.* I hope that the arguments I present here are clear enough that generalization to arbitrary reductive groups will be straightforward once one understands their fine structure. I also hope that the way things go with this relatively simple class of groups will motivate the geometric treatment in Macdonald's book, which although extremely elegant is somewhat terse and short of examples. I'll say something later on in the section on root data about their structure.

Upon learning that I was going to be writing this essay, Ian Macdonald asked me to mention that Axiom V in Chapter 2 of his book is somewhat stronger than the corresponding axiom of Bruhat-Tits, and not valid for the type C–B_2 in their classification. Deligne pointed this out to him, and made the correction:

> Axiom V. *The commutator group $[U_\alpha, U_\beta]$ for $\alpha, \beta > 0$ is contained in the group generated by the U_γ with $\gamma > 0$ and not parallel to α or β.*

6.4 V. Kac, *Infinite dimensional Lie algebras*, Cambridge University Press, 1982

Macdonald's work on p-adic groups drew him into the combinatorics of affine root systems and he made a thorough classification and study of affine root systems and affine Weyl groups, resulting in his 1972 paper entitled *"Affine root systems and Dedekind's η-function"*. This study brought him into contact with affine Kac–Moody Lie algebras and formulas for characters of their representations.

At about the same time Macdonald [1972] obtained his remarkable identities. In this work he undertook to generalize the Weyl denominator

xiv *Introduction*

identity to the case of affine root systems. He remarked that a straightforward generalization is actually false. To salvage the situation he had to add some "mysterious" factors, which he was able to determine as a result of lengthy calculations. The simplest example of Macdonald's identities is the famous Jacobi triple product identity:

$$\prod_{n\geq 1}(1-u^nv^n)(1-u^{n-1}v^n)(1-u^nv^{n-1}) = \sum_{m\in\mathbb{Z}}(-1)^m u^{\frac{1}{2}m(m+1)}v^{\frac{1}{2}m(m-1)}.$$

The "mysterious" factors which do not correspond to affine roots are the factors $(1-u^nv^n)$.

After the appearance of the two works mentioned above very little remained to be done: one had to place them on the desk next to one another to understand that Macdonald's result is only the tip of the iceberg—the representation theory of Kac–Moody algebras. Namely, it turned out that a simplified version of Bernstein–Gelfand–Gelfand's proof may be applied to the proof of a formula generalizing Weyl's formula, for the formal character of the representation π_Λ of an arbitrary Kac–Moody algebra $\mathfrak{g}'(A)$ corresponding to a symmetrizable matrix A. In the case of the simplest 1-dimensional representation π_0, this formula becomes the generalization of Weyl's denominator identity. In the case of an affine Lie algebra, the generalized Weyl denominator identity turns out to be equivalent to the Macdonald identities. In the process, the "mysterious" factors receive a

7 I.G. Macdonald as translator

One of Ian Macdonald's great silent contributions to the mathematical community was his work as a translator.

7.1 I.G. Macdonald as translator: Bourbaki

I.G. Macdonald was the first translator of Bourbaki into English. It is not clear how much of Bourbaki Macdonald translated as the publisher did not list the translator in the published English versions. A best guess is that the volumes which appeared in English between 1966 and 1974 were translated by Macdonald. These volumes comprise more than 2500 pages.

Bourbaki, General Topology Parts I and II 1966, vii+437 pp. and iv+363 pp.

Bourbaki, Theory of Sets 1968, viii+414 pp.

Bourbaki, Commutative Algebra 1972, xxiv+625 pp.

Bourbaki, Algebra 1974, xxiii+709 pp.

7.2 I.G. Macdonald as translator: Dieudonné

I.G. Macdonald's work as a translator of Dieudonné's Treatise on Analysis is documented in [Mar] and he is explicitly listed as translator in the English version of Dieudonné's Panorama of Pure Mathematics, which appeared in 1982. Together, these volumes amount to more than 2300 pages.

Dieudonné, Foundations of Modern Analysis 1960 and 1969, xiv+361 pp.

Dieudonné, Treatise on Analysis Vol. II 1970 and 1976, xviii+387 pp.

Dieudonné, Treatise on Analysis Vol. III 1972, xvii+388 pp.

Dieudonné, Treatise on Analysis Vol. IV 1974, xiv+444 pp.

Dieudonné, Treatise on Analysis Vol. V 1977, xiv+243 pp.

Dieudonné, Treatise on Analysis Vol. VI 1978, xi+239 pp

Dieudonné, A panorama of pure mathematics 1982, x+289 pp.

8 I.G. Macdonald for my students

Every once in a while, not infrequently, a student comes by my office and says "I'd like to learn about **Lie groups**, do you have a reference that you can recommend?" I usually find myself saying, "How about the notes of Macdonald?"

> Algebraic structure of Lie groups, Cambridge University Press, 1980.
> https://doi.org/10.1017/CBO9780511662683.005

Every once in a while, not infrequently, a student comes by my office and says "I'd like to learn about **algebraic groups**, do you have a reference that you can recommend?" I usually find myself saying, "How about the notes of Macdonald?"

> Linear algebraic groups,
> in Lectures on Lie Groups and Lie Algebras, Cambridge University Press 1995
> https://doi.org/10.1017/CBO9781139172882

Every once in a while, not infrequently, a student comes by my office and says "I'd like to learn about **reflection groups**, do you have a reference that you can recommend?" I usually find myself saying, "How about the notes of Macdonald?"

> Reflection groups, unpublished notes 1991.
> Available at http://math.soimeme.org/~arunram/resources.html

Every once in a while, not infrequently, a student comes by my office and says "I'd like to learn about **algebraic geometry**, do you have a reference that you can recommend?" I usually find myself saying, "How about the book of Macdonald?"

> Algebraic Geometry - Introduction to schemes, published by W.A. Benjamin 1968.
> Available at http://math.soimeme.org/~arunram/resources.html

Every once in a while, not infrequently, a student comes by my office and says "I'd like to learn about **Haar measure, spherical functions and harmonic analysis**, do you have a reference that you can recommend?" I usually find myself saying, "How about the book of Macdonald?"

> Spherical functions on a group of p-adic type, University of Madras 1971.
> Available at http://math.soimeme.org/~arunram/resources.html

Every once in a while, not infrequently, a student comes by my office and says "I'd like to learn about **Kac–Moody Lie algebras**, do you have a reference that you can recommend?" I usually find myself saying, "How about the notes of Macdonald?"

> Kac–Moody Lie algebras, unpublished notes 1983.
> Available at http://math.soimeme.org/~arunram/resources.html

Every once in a while, not infrequently, a student comes by my office and says "I'd like to learn about **flag varieties and Schubert varieties**, do you have a reference that you can recommend?" I usually find myself saying, "How about the notes of Macdonald?"

> Notes on Schubert polynomials: Appendix: Schubert varieties.
> Published by LACIM 1991.
> Available at http://math.soimeme.org/~arunram/resources.html

9 I.G. Macdonald as an author of books

"If you see a gap in the literature, write a book to fill it." – I.G. Macdonald

Atiyah-Macdonald, Introduction to commutative algebra 1969

Spherical functions on a group of p-adic type 1971

Symmetric functions and Hall polynomials First Edition 1979

Kac–Moody Lie algebras: unpublished notes 1983

Hypergeometric functions: unpublished notes 1987

Reflection groups: unpublished notes 1991

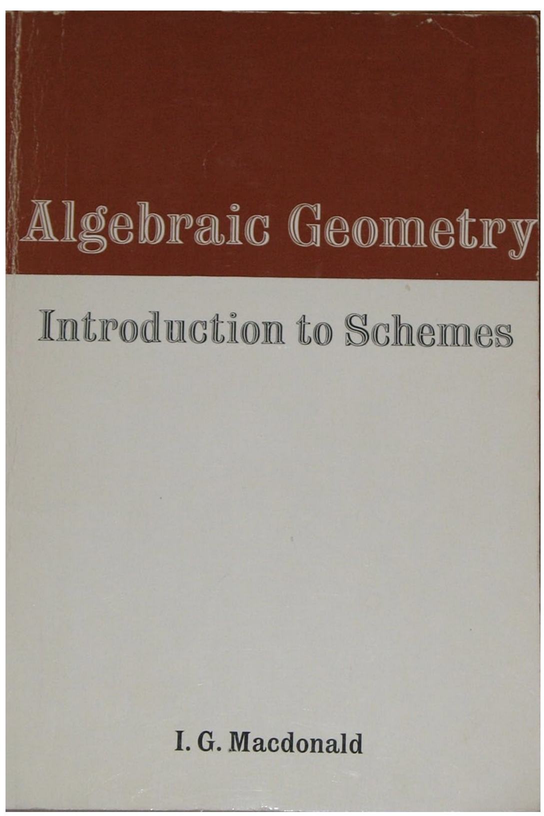
Schubert polynomials 1991

Symmetric functions and Hall polynomials Second Edition 1995

Linear algebraic groups: in Lectures on Lie groups and Lie algebras 1995

Affine Hecke algebras and orthogonal polynomials 2003

9.1 The first book: Algebraic Geometry - Introduction to Schemes 1968



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