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Slit-slide-sew bijections for constellations and quasiconstellations

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Abstract. We extend so-called slit-slide-sew bijections to constellations and quasiconstellations. We present an involution on the set of hypermaps given with an orientation, one distinguished corner, and one distinguished edge leading away from the corner while oriented in the given orientation. This involution reverts the orientation, exchanges the distinguished corner with the distinguished edge in some sense, slightly modifying the degrees of the incident faces in passing, while keeping all the other faces intact.

The involution specializes into a bijection interpreting combinatorial identities and allows to recover the counting formula for constellations or quasiconstellations with a given face degree distribution.

Keywords: bijection, plane map, hypermap, constellation, map enumeration.

1 Introduction

In the present work, we pursue the investigation of so-called *slit-slide-sew* bijections, introduced in [1] on forests and plane quadrangulations, and then further developed in [2, 3] on plane bipartite and quasibipartite maps. Here, we focus on a generalization of the latter, called *constellations* and *quasiconstellations*.

Hypermaps. Recall that a *plane map* is an embedding of a finite connected graph (possibly with multiple edges and loops) into the sphere, considered up to orientation-preserving homeomorphisms. Now fix an integer $p \geq 2$. A (plane) *p-hypermap* is a plane map whose faces are shaded either *dark* or *light* in such a way that

- adjacent faces do not have the same shade (one is dark, the other light);
- each dark face has degree p.

These actually generalize maps, which correspond to 2-hypermaps. In the terminology of hypermaps, light faces generalize faces and might be called *hyperfaces*, whereas dark faces generalize edges and are called *hyperedges*. We do not use this terminology here.

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A (plane) *p-constellation* is a p-hypermap such that the degrees of its light faces are all multiples of p. In a *p-hypermap*, a light face whose degree is not a multiple of p will be called a *flawed face*. A p-constellation is thus a p-hypermap without flawed faces. A *quasi-p-constellation* is a p-hypermap with exactly two flawed faces. Note that, in a p-hypermap, the sum of the degrees of the light faces is necessarily a multiple of p, since it is equal to the sum of the degrees of the dark faces, which are all p. As a result, a p-hypermap cannot have a single flawed face and, in a quasi-p-constellation, the two flawed faces have, modulo p, degrees $+k$ and $-k$ for some $0 < k < p$.

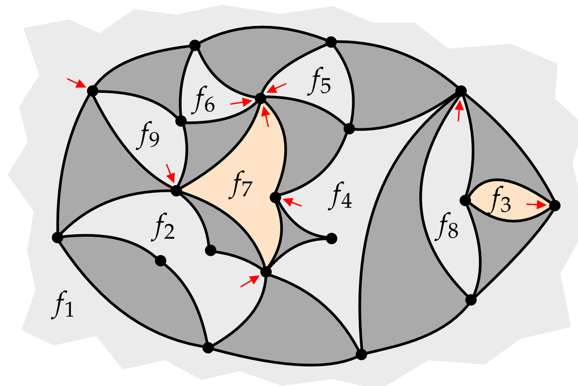


Figure 1: A quasi-3-constellation of type $(9, 6, 2, 6, 3, 3, 4, 3, 3)$. The two flawed faces, highlighted in orange, are f_3 and f_7. Every light face has a marked corner, always represented by a red arrowhead.

Enumeration. For an r-tuple $\boldsymbol{a} = (a_1, \ldots, a_r)$ of positive integers, let us denote by $C(\boldsymbol{a})$ the number of p-hypermaps with exactly r light faces, numbered $f_1, \ldots, f_r$ and of respective degrees $a_1, \ldots, a_r$, each bearing a marked corner[1]. The r-tuple $\boldsymbol{a}$ will be called the *type* of such p-hypermaps. See Figure 1. By elementary considerations and Euler's characteristic formula, the integers

$$E(\boldsymbol{a}) := \sum_{i=1}^{r} a_i\,, \qquad D(\boldsymbol{a}) := \frac{E(\boldsymbol{a})}{p}\,, \qquad \text{and} \qquad V(\boldsymbol{a}) := E(\boldsymbol{a}) - D(\boldsymbol{a}) - r + 2$$

are respectively the numbers of edges, dark faces, and vertices of p-hypermaps of type $\boldsymbol{a}$. Generalizing Tutte's so-called *formula of slicings* [6], it has been computed that, when at most two a_i's are not in $p\mathbb{N}$, that is, for p-constellations [4] or quasi-p-constellations [5], it holds that

$$C(\boldsymbol{a}) = c_{\boldsymbol{a}}\,\frac{\big(E(\boldsymbol{a}) - D(\boldsymbol{a}) - 1\big)!}{V(\boldsymbol{a})\,!}\prod_{i=1}^{r}\alpha(a_i), \quad \text{where } \alpha(x) := \frac{x!}{\lfloor x/p \rfloor!\,\big(x - \lfloor x/p \rfloor - 1\big)!}$$
$$\text{and } c_{\boldsymbol{a}} = \begin{cases} 1 & \text{if } p \text{ divides every } a_i \\ p-1 & \text{otherwise} \end{cases}. \tag{1.1}$$

[1] Recall that a *corner* is an angular sector delimited by two consecutive edges around a vertex.

Combinatorial identities. In the present work, we give a bijective interpretation for the following combinatorial identity, which transfers one degree from one face to another.

Proposition 1 (Transferring one degree from f_1 to f_2)**.** *Let $\boldsymbol{a} = (a_1, \ldots, a_r)$ be an r-tuple of positive integers such that $a_1 \geq 2$, and with coordinates equal modulo p to*

(i) either $(k, -k, 0, \ldots, 0)$ for some $k \in \{0, \ldots, p-1\}$,

(ii) or $(1, 0, \ldots, 0, -1, 0, \ldots, 0)$, with the -1 in any position from 3 to r.

Let also $\tilde{\boldsymbol{a}} = (\tilde{a}_1, \ldots, \tilde{a}_r) := (a_1 - 1, a_2 + 1, a_3, \ldots, a_r)$. Then the following identity holds:

$$\big(a_1 - \lceil a_1/p \rceil\big)\,\big(a_2 + 1\big)\,\mathrm{C}(\boldsymbol{a}) = \big(\tilde{a}_1 + 1\big)\,\big(\tilde{a}_2 - \lceil \tilde{a}_2/p \rceil\big)\,\mathrm{C}(\tilde{\boldsymbol{a}})\,. \tag{1.2}$$

To obtain (1.2) from (1.1), one might first observe that, for any $x \in \mathbb{N}$,

$$\frac{\alpha(x)}{\alpha(x-1)} = d_x\,\frac{x}{x - \lceil x/p \rceil} \qquad \text{where} \qquad d_x = \begin{cases} p-1 & \text{if } p \mid x \\ 1 & \text{if } p \nmid x \end{cases},$$

and then that, in both cases (*i*) and (*ii*), $c_{\boldsymbol{a}} d_{a_1} = c_{\tilde{\boldsymbol{a}}} d_{\tilde{a}_2} = p - 1$.

We furthermore treat the case of a degree 1-face, which may easily be obtained as above.

Proposition 2 (Transferring the degree of a degree 1-face f_1 to f_2)**.** *Let $\boldsymbol{a} = (1, a_2, \ldots, a_r)$ and $\tilde{\boldsymbol{a}} = (\tilde{a}_2, \ldots, \tilde{a}_r) := (a_2 + 1, a_3, \ldots, a_r)$ be respectively an r-tuple and an $r-1$-tuple of positive integers, both having at most two coordinates not lying in $p\mathbb{N}$. Then the following identity holds:*

$$\big(a_2 + 1\big)\,\mathrm{C}(\boldsymbol{a}) = V(\tilde{\boldsymbol{a}})\,\big(\tilde{a}_2 - \lceil \tilde{a}_2/p \rceil\big)\,\mathrm{C}(\tilde{\boldsymbol{a}})\,. \tag{1.3}$$

It is easy to see that the number of p-constellations with exactly one light face of degree pn is equal to the known number of p-ary trees with n nodes. Using this as initial condition, Propositions 1 and 2 provide yet another proof of (1.1).

Methodology. In order to bijectively interpret (1.2) and (1.3), the idea is to distinguish elements, such as edges, vertices, faces, corners, etc., in such a way that each side of an equation of interest counts maps given with such distinguished elements. Remark that we will always use the word "distinguished" to designate these extra elements, keeping the word "marked" only for the marked corners, which we see as inherent to the hypermaps into consideration.

Once both sides of the considered equation are properly interpreted as cardinalities of sets of maps with distinguished elements, we bijectively go from one set to the other as follows. Using the distinguished elements, we construct a directed path in the map, called *sliding path*. We then slit the map along this sliding path and sew back together the sides of the slit after sliding by one unit, in the sense that the left side of the i-th edge is sewn back on the right side of the $i \pm 1$-th edge (the ± 1 being the same for all edges and determined by some rule). This mildly modifies the map along the path but does not affect its faces, except the two that are around the extremities of the sliding path. In the process, new distinguished elements naturally appear in the resulting map; these allow us to recover the sliding path in order to slide back.

Organization of the paper. The remainder of the document is structured in the following manner. We start by giving in Section 2 the definitions and conventions we use, as well as a combinatorial interpretation of the prefactor $\big(a - \lceil a/p \rceil\big)$ appearing in the identities (1.2) and (1.3). We then present in Section 3 our bijective interpretation of these identities through a more general involution on the set of maps given with an orientation, a distinguished corner, and a distinguished edge satisfying an extra constraint.

2 Preliminaries

Distinguishing a corner. Following previous works on slit-slide-sew bijections, we use the convention, depicted in Figure 2, that the marked corner of a face creates two possible corners to distinguish. One might think of the marked corner as a dangling half-edge, with one corner on each side. As a result, a face of degree a bearing its marked corner has $a+1$ possible corners to distinguish.

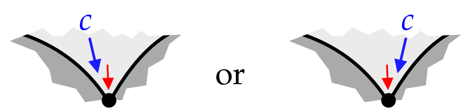


Figure 2: Distinguishing a corner around the marked corner.

Edge orientation. As is customary, we will orient the edges of the hypermaps we consider, in such a way that light faces always lie to the same side of the oriented edges (and thus dark faces always lie to the same other side). These orientations will be called the *light-left orientation* when the light faces[2] all lie to the left, and the *light-right orientation* when the light faces all lie to the right. In other words, in the light-right orientation, the edges are oriented clockwise around light faces and counterclockwise around dark faces. See Figure 3. We will need to use both orientations in the present paper. We will always clearly mention which orientation we use whenever it matters. Without specific mention, both orientations can be used. **Once one of the two possible orientations is fixed**, we will use the following conventions.

[2]Recall that the light faces are the main objects of focus.

Given an edge e, we will respectively denote by e^- and e^+ the origin and end of the edge e, oriented as convened. The corner *preceding* e is defined as the corner c_e delimited by e and the edge that precedes e in the contour of the incident light face, in the convened orientation. Similarly, we denote by c^+ the vertex incident to a corner c.

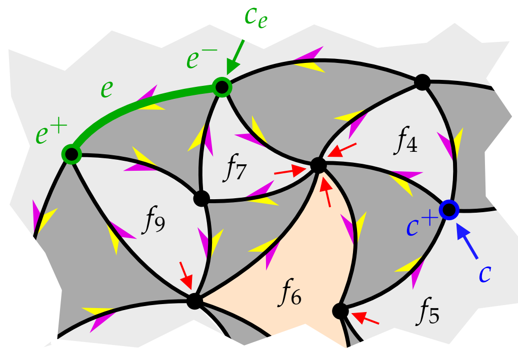


Figure 3: Edge orientation and definitions. Here, the light-right orientation is depicted.

Paths. A *path* from a vertex v to a vertex v' is a finite sequence $\wp = (e_1, e_2, \ldots, e_k)$ of edges such that $e_1^- = v$, for $1 \le i \le k-1$, $e_i^+ = e_{i+1}^-$, and $e_k^+ = v'$. Its *length* is the integer k, which we denote by $[\wp] := k$. A path is called *simple* if the vertices it visits are all distinct.

Beware that a path is only made of edges oriented in the convened orientation. In other words, edges cannot be used "backward." In particular, this means that all the faces lying to the left of a path are of the same shade (either all light or all dark), whereas all the faces lying to its right are of the other shade. The side of the path where the faces are all light will be called its *light side*, whereas the other side will be called its *dark side*.

Directed metric and geodesics. We will use the directed metric associated with the convened orientation: given two vertices v, v' in a p-hypermap, we denote by $\vec{\mathrm{d}}(v, v')$ the smallest k for which there exists a path from v to v' of length k. (We put an arrow on top in the notation to keep in mind that this is only a directed metric.) A *geodesic* from v to v' is such a path.

There are generally several geodesics from a given vertex v to a target vertex v'. Among all of these, one will be of particular interest in this work: the *lightest geodesic*, constructed as follows. It is only well defined from a starting edge or corner e_0 such that $e_0^+ = v$. (The starting element e_0 does not belong to the path.) Then, provided e_0, e_1, ..., e_j have already been constructed and the path is not complete (that is, $e_j^+ \neq v'$), we set the subsequent edge e_{j+1} as the one, among the edges e such that $e^- = e_j^+$ and $\vec{\mathrm{d}}(e^+, v') = \vec{\mathrm{d}}(e_j^+, v') - 1$, that comes first while turning around e_j^+ in the direction incoming edge, light face. See Figure 4. In other words, the lightest geodesic is the leftmost geodesic if the convened orientation is the light-left orientation and the rightmost geodesic if the convened orientation is the light-right orientation.

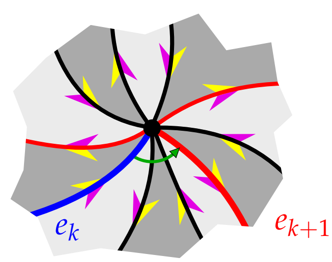


Figure 4: Definition of the lightest geodesic. The edges going closer to v' are in red.

Edge types. Given a fixed vertex v in a p-hypermap, we may differentiate three types of edges: an edge e is said to be

- *leaving* v if $\vec{\mathsf{d}}(v, e^+) = \vec{\mathsf{d}}(v, e^-) + 1$;
- *approaching* v if $\vec{\mathsf{d}}(v, e^+) = \vec{\mathsf{d}}(v, e^-) + 1 - p$;
- *irregular with respect to* v if $\vec{\mathsf{d}}(v, e^+) - \vec{\mathsf{d}}(v, e^-) \not\equiv 1 \bmod p$.

Observe that $1 - p \le \vec{\mathsf{d}}(v, e^+) - \vec{\mathsf{d}}(v, e^-) \le 1$ since there is always a path of length 1, namely the path consisting of the single edge e, as well as a path from e^+ to e^- of length $p - 1$, made of all the other edges incident to the dark face incident to e. As a result, if e is irregular with respect to v, then it holds that $\vec{\mathsf{d}}(v, e^+) - \vec{\mathsf{d}}(v, e^-) \in \{2 - p, 3 - p, \dots, 0\}$.

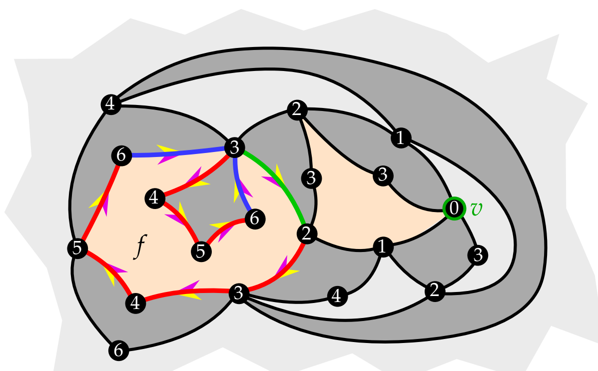


Figure 5: The different types of edges incident to a flawed face.The distances to v are written in the vertices. Around f, the $(10 - \lceil 10/4 \rceil) = 7$ red edges are leaving v; the $\lfloor 10/4 \rfloor = 2$ blue edges are approaching v; the green edge is irregular with respect to v.

The following proposition gives the number of each type among edges incident to a given face in a p-constellation or a given flawed face in a quasi-p-constellation; this provides an interpretation to the prefactor $(a - \lceil a/p \rceil)$ appearing in (1.2) and (1.3). We refer the reader to the extended version of this paper for a proof.

Proposition 3. *We consider a vertex v and a light face f of degree a in a p-hypermap.*

(1) If the p-hypermap is a p-constellation then, among the a edges incident to f, $a - a/p$ are leaving v; a/p are approaching v; none are irregular with respect to v.

(2) If the p-hypermap is a quasi-p-constellation and f a flawed face then, among the a edges incident to f, $(a - \lceil a/p \rceil)$ are leaving v; $\lfloor a/p \rfloor$ are approaching v; one is irregular with respect to v.

3 Bijective interpretation

3.1 Slit slide sew

Let us first describe the operation at the heart of our construction. See Figure 6. Assume that, on some p-hypermap $\mathbf{m}$, we have a simple path $\wp = (e_1, e_2, \ldots, e_k)$ linking some corner c in some light face f to a different corner c' in some light face f' (which may possibly be equal to f), that is, such that $e_1^- = c^+$ and $e_k^+ = c'^+$. We may then follow $\wp$, **entering from the corner** c and **exiting through the corner** c'. This creates a simple path on the sphere, starting inside the face f and finishing inside f'. We may slit the sphere along this path, thus doubling the sides of the path. In the hypermap $\mathbf{m}$, this doubles the path $\wp$, making up two copies, one incident to light faces and the slit, and one incident to dark faces and the slit. We denote by $\boldsymbol{\ell} = (\ell_1, \ldots, \ell_k)$ the former and by $\boldsymbol{d} = (d_1, \ldots, d_k)$ the latter.

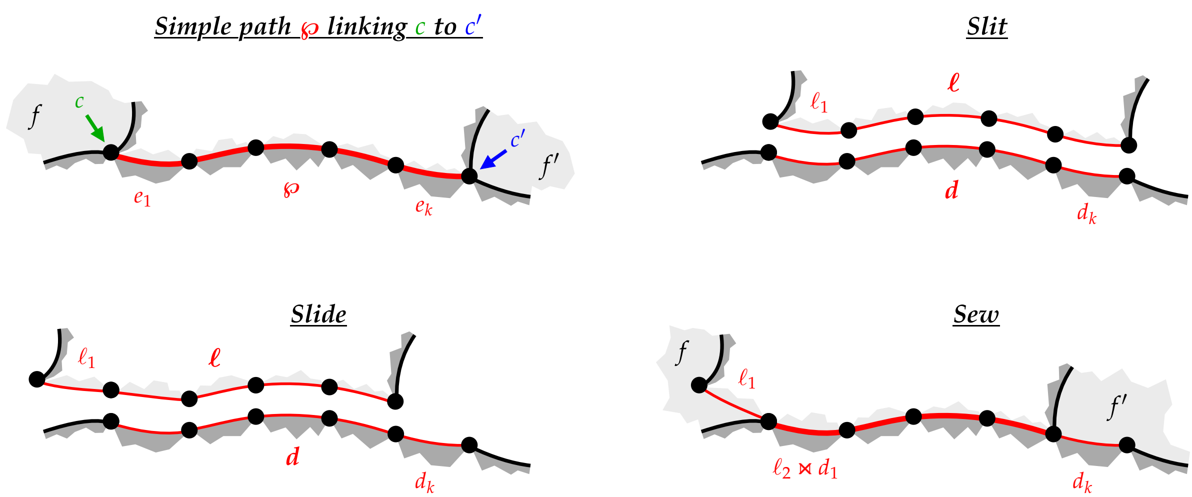


Figure 6: The slit-slide-sew operation on a p-hypermap.

Note that the data of $\wp$ is not sufficient to properly define this operation; one needs to know from which corner to enter $\wp$ in order to decide if an edge incident to e_1^- becomes incident whether to ℓ_1^- or to d_1^-. Similarly, one needs to know through which corner to exit $\wp$.

We then sew back $\boldsymbol{\ell}$ onto $\boldsymbol{d}$ but only after sliding by one unit, in the sense that we match ℓ_{i+1} with d_i, for every $1 \leq i \leq k-1$. For further reference, we denote by $\ell_{i+1} \bowtie d_i$ the resulting edge. Observe that, except from f and f', the faces are not altered by the process. Observe also that ℓ_1 and d_k are not matched with anything:

- d_k is still incident to the original dark face and is now also incident to f';
- ℓ_1 is still incident to the original light face and is now also incident to f.

Consequently, the result is no longer a p-hypermap since ℓ_1 is incident to light faces from both sides. However, in the case where ℓ_1 is actually a dangling edge (an edge with one extremity of degree 1), removing it provides a p-hypermap. This happens if and only if c is the corner preceding e_1 ; this will always be the case in the present work.

3.2 Face of degree two or more

We now present the bijective interpretation for the identity (1.2) of Proposition 1.

Involution. We define a mapping Φ on the set $\mathcal{H}$ of quadruples $(\mathcal{o}, \mathbf{m}, c, e)$, where

- $\mathcal{o}$ is an orientation (either light-left or light-right);
- $\mathbf{m}$ is a p-hypermap;
- c is a distinguished corner of some light face;
- e is a distinguished edge leaving c^+ in the orientation $\mathcal{o}$.

We break down the process into the following steps. See Figure 7.

1. Reorientation
From now on, we convene to use the reverse orientation, which we denote by $\tilde{\mathcal{o}}$.

2. Sliding path
We consider the corner c_e preceding e and the lightest geodesic γ from c_e to c^+.

3. Slitting, sliding, sewing
We slit, slide, sew along γ from c_e to c as described in the previous section: along γ, the light side of an edge is now matched with the dark side of the previous edge.

4. Output
The unmatched light side of the first edge of γ yields a dangling edge; we remove it and denote the resulting corner by $\tilde{c}$. We denote the edge corresponding to the unmatched dark side of the final edge of γ by $\tilde{e}$. We let $\tilde{\mathbf{m}}$ be the resulting map. Finally, the output of the construction is the quadruple $\Phi(\mathcal{o}, \mathbf{m}, c, e) := (\tilde{\mathcal{o}}, \tilde{\mathbf{m}}, \tilde{c}, \tilde{e})$.

Theorem 4. *The mapping $\Phi\colon \mathcal{H} \to \mathcal{H}$ is an involution.*

We refer the reader to the extended version of this work for the proof of Theorem 4.

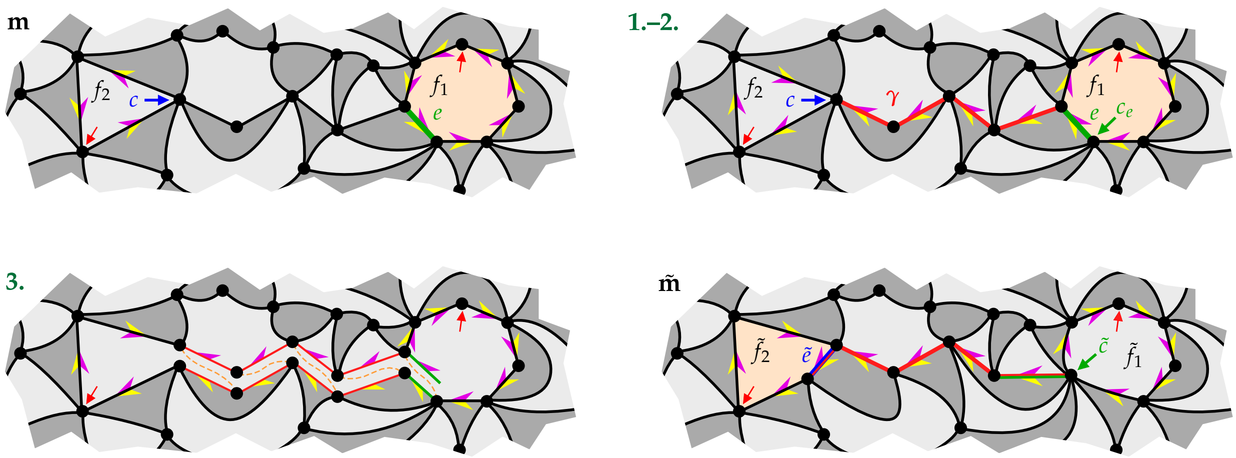


Figure 7: The involution $\Phi\colon \mathcal{H} \to \mathcal{H}$. Only the orientation around the faces of interest and along γ are depicted. **Top left.** The input. **Top right.** We changed the orientation and defined the sliding path γ. **Bottom left.** We slit along the path. The dashed lines indicate to sew back after sliding. **Bottom right.** The output.

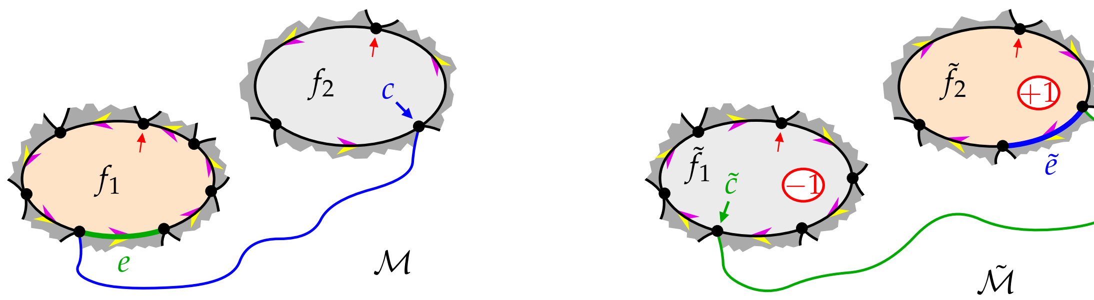
Specialization. We now see how Φ specializes into a bijection interpreting (1.2). We let

$$\boldsymbol{a} = (a_1, \ldots, a_r) \qquad \text{and} \qquad \tilde{\boldsymbol{a}} = (\tilde{a}_1, \ldots, \tilde{a}_r) := (a_1 - 1, a_2 + 1, a_3, \ldots, a_r)$$

be as in the statement of Proposition 1. Note that this means that p-hypermaps of type $\boldsymbol{a}$ are either p-constellations or quasi-p-constellations whose **first** face is flawed. Similarly, p-hypermaps of type $\tilde{\boldsymbol{a}}$ are either p-constellations or quasi-p-constellations whose **second** face is flawed.

We fix an orientation $\mathcal{o}$ and define the following sets, whose cardinalities are respectively the left-hand side and the right-hand side of (1.2), by Proposition 3 (recall also the convention at the begining of Section 2 for distinguishing corners).

- We let $\mathcal{M}$ be the set of p-hypermaps of type $\boldsymbol{a}$ carrying
  - one distinguished corner c in the **second** face,
  - one distinguished edge e incident to the **first** face and leaving c^+, for the **orientation** $\mathcal{o}$.
- We let $\tilde{\mathcal{M}}$ be the set of p-hypermaps of type $\tilde{\boldsymbol{a}}$ carrying
  - one distinguished corner $\tilde{c}$ in the **first** face,
  - one distinguished edge $\tilde{e}$ incident to the **second** face and leaving $\tilde{c}^+$, for $\tilde{\mathcal{o}}$.



Here, $p = 3$, we are in the case (ii) of Proposition 1, and $\circlearrowleft =$ light-left.

The pictograph above summarizes the definitions of $\mathcal{M}$ and $\tilde{\mathcal{M}}$. The red ± 1 on the right shows the increase or decrease of the degree of the face in $\tilde{\mathcal{M}}$ in comparison with the one of the corresponding face in $\mathcal{M}$. In order to avoid confusion, we denote the first and second faces of maps in $\mathcal{M}$ by f_1 and f_2 as before, and use $\tilde{f}_1$ and $\tilde{f}_2$ instead, for maps in $\tilde{\mathcal{M}}$. The paths symbolize the fact that the edges are leaving the corners.

Remark 1. Note that the convention on the orientation of edges is not the same in the definitions of the sets $\mathcal{M}$ and $\tilde{\mathcal{M}}$. This clearly bears no effects from an enumeration point of view but is of crucial importance for our bijections.

Corollary 5. *The mapping Φ specializes into a bijection from $\{(\circlearrowleft, \mathbf{m}, c, e) : (\mathbf{m}, c, e) \in \mathcal{M}\}$ onto $\{(\tilde{\circlearrowleft}, \tilde{\mathbf{m}}, \tilde{c}, \tilde{e}) : (\tilde{\mathbf{m}}, \tilde{c}, \tilde{e}) \in \tilde{\mathcal{M}}\}$, thus providing a bijection between $\mathcal{M}$ and $\tilde{\mathcal{M}}$.*

3.3 Face of degree one

We proceed to the bijective interpretation for the identity (1.3) of Proposition 2, which works in a similar fashion as before.

Setting. Let $\boldsymbol{a} = (1, a_2, \dots, a_r)$ and $\tilde{\boldsymbol{a}} = (\tilde{a}_2, \dots, \tilde{a}_r) := (a_2 + 1, a_3, \dots, a_r)$ be tuples of positive integers, both with at most two coordinates not lying in $p\mathbb{N}$. In order not to be confused by the index shift in $\tilde{\boldsymbol{a}}_2$, we denote the faces of p-hypermaps of type $\tilde{\boldsymbol{a}}$ by $\tilde{f}_2, \dots, \tilde{f}_r$. In particular, p-hypermaps of type $\tilde{\boldsymbol{a}}$ are either p-constellations, or are quasi-p-constellations whose face $\tilde{f}_2$ (the one with degree $\tilde{a}_2$) is flawed. We fix an orientation $\circlearrowleft$ and define the following sets, whose cardinalities are the sides of (1.3), again by Proposition 3 for the right-hand side.

- We let $\mathcal{N}$ be the set of p-hypermaps of type $\boldsymbol{a}$ carrying
  - one distinguished corner c in the face f_2.
- We let $\tilde{\mathcal{N}}$ be the set of p-hypermaps of type $\tilde{\boldsymbol{a}}$ carrying
  - one distinguished vertex $\tilde{v}$,
  - one distinguished edge $\tilde{e}$ incident to $\tilde{f}_2$ and leaving $\tilde{v}$ for $\tilde{\circlearrowleft}$.

We put f_1 on the pictograph since we think of it as the "missing" distinguished element for $\mathcal{N}$. Note that we do not need to specify an orientation for maps in $\mathcal{N}$; we will however use the orientation $\mathcal{o}$ for these maps in due time. The bijections between $\mathcal{N}$ and $\tilde{\mathcal{N}}$ can be thought of as degenerate versions of the one of the previous section. Here, we do not have an involution; we need to describe both mappings. We break them down into similar steps as above. See Figure 8.

Suppressing a face. We consider $(\mathbf{m}, c) \in \mathcal{N}$.

1. From this point on, we use the reverse orientation $\tilde{\mathcal{o}}$.
2. We consider the lightest geodesic γ from the unique corner of f_1 to c^+.
3. We denote by d_0 the unique edge incident to f_1. We slit, slide, sew along γ from the unique corner of f_1 to c as described in Section 3.1, while furthermore matching the unmatched light side of the first edge with d_0.
4. We set $\Psi_-(\mathbf{m}, c) := (\tilde{\mathbf{m}}, \tilde{v}, \tilde{e})$, where $\tilde{\mathbf{m}}$ is the resulting map, $\tilde{e}$ is the edge corresponding to the unmatched dark side of the final edge of γ, and $\tilde{v}$ is the origin of γ.

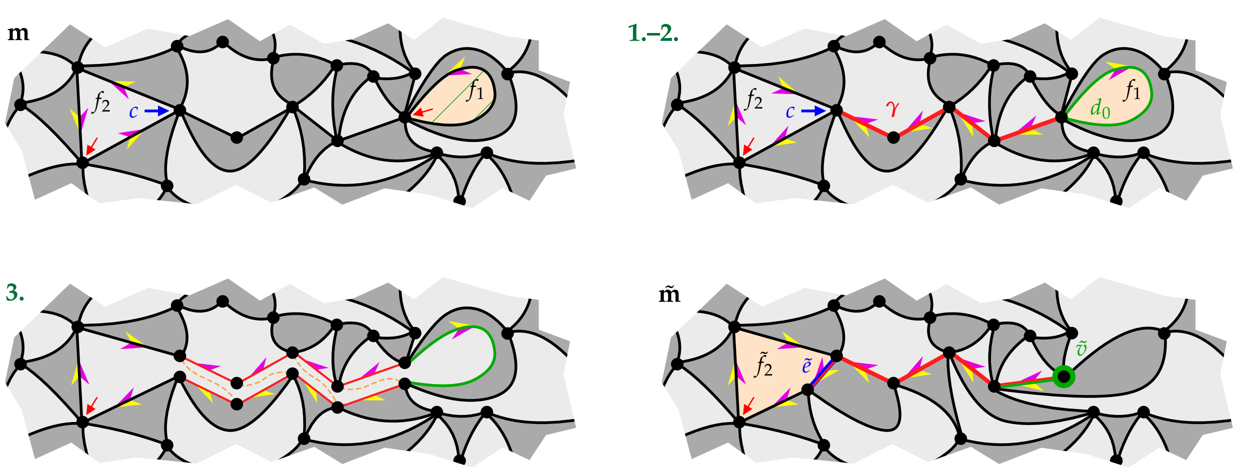


Figure 8: The bijection in the case of a degree 1-face, from $\mathcal{N}$ to $\tilde{\mathcal{N}}$.

Adding a face. We consider $(\tilde{\mathbf{m}}, \tilde{v}, \tilde{e}) \in \tilde{\mathcal{N}}$.

1. From this point on, we use the orientation $\mathcal{O}$.
2. We consider the lightest geodesic $\tilde{\gamma}$ from the corner $c_{\tilde{e}}$ preceding $\tilde{e}$ to $\tilde{v}$.
3. We slit $\tilde{\mathbf{m}}$ along $\tilde{\gamma}$, entering from $c_{\tilde{e}}$ and stopping at $\tilde{v}$, **without disconnecting the map** at $\tilde{v}$, slide by one unit, and sew back as before. Now the unmatched dark side of the final edge creates a loop enclosing an extra face, which we denote by f_1 and mark at its unique corner.
4. We replace $\tilde{\ell}_1$ with a corner c, let $\mathbf{m}$ be the resulting map, and set $\Psi_+(\tilde{\mathbf{m}}, \tilde{v}, \tilde{e}) := (\mathbf{m}, c)$.

Theorem 6. *The mappings $\Psi_- \colon \mathcal{N} \to \tilde{\mathcal{N}}$ and $\Psi_+ \colon \tilde{\mathcal{N}} \to \mathcal{N}$ are well defined and inverse bijections.*

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