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Geometry of C-Matrices for Mutation-Infinite Quivers

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Abstract. The set of forks is a class of quivers introduced by M. Warkentin, where every connected mutation-infinite quiver is mutation equivalent to infinitely many forks. Let Q be a fork with n vertices, and $\boldsymbol{w}$ be a fork-preserving mutation sequence. We show that every c-vector of Q obtained from $\boldsymbol{w}$ is a solution to a quadratic equation of the form

$$\sum_{i=1}^{n} x_i^2 + \sum_{1 \leq i < j \leq n} \pm q_{ij} x_i x_j = 1,$$

where q_{ij} is the number of arrows between the vertices i and j in Q. From the proof of this result, when Q is a rank 3 mutation-cyclic quiver, every c-vector of Q is a solution to a quadratic equation of the same form.

Keywords: quivers, c-vectors, forks, quadratic equations

1 Introduction

The mutation of a quiver Q was discovered by S. Fomin and A. Zelevinsky in their seminal paper [12] where they introduced cluster algebras. It also appeared in the context of Seiberg duality [10]. The c-vectors (and C-matrices) of Q were defined through mutations in further developments of the theory of cluster algebras [13], and together with their companions, g-vectors (and G-matrices), played fundamental roles in the study of cluster algebras (for instance, see [7, 14, 19, 20, 22]). When Q is acyclic, positive c-vectors are actually real Schur roots, that is, the dimension vectors of indecomposable rigid modules over Q [5, 15, 25]. Moreover, they appear as the denominator vectors of non-initial cluster variables of the cluster algebra associated to Q [4].

Due to the multifaceted appearance of c-vectors in important constructions, there have been various results related to the description of c-vectors (or real Schur roots)

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of an acyclic quiver [1, 15, 16, 23, 24, 25]. In [18], K.-H. Lee and K. Lee conjectured a correspondence between real Schur roots of an acyclic quiver and non-self-crossing curves on a Riemann surface and proposed a new combinatorial/geometric description. The conjecture is now proven by A. Felikson and P. Tumarkin [9] for acyclic quivers with multiple edges between every pair of vertices. Recently, S. D. Nguyen [21] proved the conjecture for an arbitrary acyclic (valued) quiver.

For a given (not necessarily acyclic) quiver Q, the set of quivers that are mutation equivalent to Q is called the mutation equivalence class of Q and denoted by $\mathrm{Mut}(Q)$. The quiver Q is said to be *mutation-infinite* if $|\mathrm{Mut}(Q)|$ is not finite, and *mutation-finite* if $|\mathrm{Mut}(Q)| < \infty$. The mutation-finite quivers are completely classified, and relatively well studied. On the other hand, mutation-infinite quivers still await further investigations.

A reader-friendly version of our main theorem may be stated as follows.

Theorem 1.1. *Let n be any positive integer. Let P be a mutation-infinite connected quiver with n vertices. Then there exist an infinite number of pairs of a quiver $Q \in Mut(P)$ and $k \in \{1, ..., n\}$ such that every c-vector of Q obtained from any mutation sequence not starting with k is a solution to a quadratic equation of the form*

$$\sum_{i=1}^{n} x_i^2 + \sum_{1\le i<j\le n} \pm q_{ij} x_i x_j = 1, \tag{1.2}$$

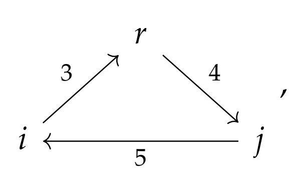
where q_{ij} is the number of arrows between the vertices i and j in Q. There does not seem to be a simple way of determining the exact signs of the $x_i x_j$ terms.

To state a more precise theorem, we need to recall the definition of forks. An *abundant quiver* is a quiver such that there are two or more arrows between every pair of vertices.

Definition 1.3. [26, Definition 2.1] A *fork* is an abundant quiver F, where F is not acyclic and where there exists a vertex r, called the point of return, such that

- For all $i \in F^-(r)$ and $j \in F^+(r)$ we have $f_{ji} > f_{ir}$ and $f_{ji} > f_{rj}$, where $F^-(r)$ is the set of vertices with arrows pointing towards r and $F^+(r)$ is the set of vertices with arrows coming from r.
- The full subquivers induced by $F^-(r)$ and $F^+(r)$ are acyclic.

An example of a fork is given by



where r is the point of return.

It is known that "most" quivers in $\mathrm{Mut}(Q)$ of any connected mutation-infinite quiver Q are forks, as Theorem 1.4 and Proposition 1.5 imply.

Theorem 1.4. [26, Theorem 3.2] *A connected quiver is mutation-infinite if and only if it is mutation-equivalent to a fork.*

Proposition 1.5. [26, Proposition 5.2] *Let G be the exchange graph of a connected mutation-infinite quiver. A simple random walk on G will almost surely leave the fork-less part and never come back.*

A *fork-preserving* mutation sequence is a reduced sequence of mutations that starts with a fork and does not mutate at its point of return. A more precise version of our main theorem is as follows.

Theorem 1.6. *Let Q be a fork, and let $\boldsymbol{w}$ be a fork-preserving mutation sequence. Every c-vector of Q obtained from $\boldsymbol{w}$ is a solution to a quadratic equation of the form* (1.2)*.*

A quiver Q is called *mutation-acyclic* if it is mutation-equivalent to an acyclic quiver, else it is called *mutation-cyclic*. Notably, we have discovered a counterexample to Theorem 1.6 for truly arbitrary mutation-sequences $\boldsymbol{w}$ in the case of quivers on four vertices (to appear in the full version of this abstract [8]), but the proof of the theorem provides a stronger corollary in the three vertex case. Ahmet Seven informed us that he had independently discovered this result.

Corollary 1.7. *Let Q be a mutation-cyclic quiver with 3 vertices. Then every c-vector of Q is a solution to a quadratic equation of the form* (1.2) *with $n = 3$.*

As a byproduct of our proof, we also obtain the following theorem, which is closely related to a result of Fomin and Neville [11, Lemma 6.14].

Theorem 1.8. *Let $\boldsymbol{w}$ be a fork-preserving mutation sequence. The sign-vector (see Definition 2.3) of $C^{\boldsymbol{w}}$ depends only on the signs of entries of initial exchange matrix B. In other words, the sign-vector is independent of the number of arrows between vertices of the initial quiver Q.*

Corollary 1.9. *Let n be any positive integer, and let Q be a fork with n vertices. For each fork-preserving mutation sequence $\boldsymbol{w}$ from Q, the corresponding n-tuple of reflections $(r_1^{\boldsymbol{w}}, r_2^{\boldsymbol{w}}, \dots, r_n^{\boldsymbol{w}})$ (see Definition 2.6) depends only on the signs of entries of the initial exchange matrix B.*

From this, we are able to prove that the product of reflections is equal to a Coxeter element. More precisely, we have the following.

Theorem 1.10. *Let n be any positive integer, and let Q be a fork with n vertices. For each fork-preserving mutation sequence $\boldsymbol{w}$ from Q, we have*

$$r_{\lambda(1)}^{\boldsymbol{w}} \dots r_{\lambda(n)}^{\boldsymbol{w}} = r_{\rho(1)} \dots r_{\rho(n)}$$

for some permutations $\lambda, \rho \in \mathfrak{S}_n$, where $\mathfrak{S}_n$ is the symmetric group on $\{1, \dots, n\}$ and $r_1, \dots, r_n$ are the initial reflections, where λ is determined by $\boldsymbol{w}$ and ρ is fixed by the first mutation of $\boldsymbol{w}$.

Corollary 1.11. *Let n be any positive integer, and let Q be a fork with n vertices. For each fork-preserving mutation sequence $\boldsymbol{w}$ from Q, there exist pairwise non-crossing and non-self-crossing admissible curves $\eta_i^{\boldsymbol{w}}$ (see Definition 2.10) such that $r_i^{\boldsymbol{w}} = \nu(\eta_i^{\boldsymbol{w}})$ for every $i \in \{1, ..., n\}$.*

The above results are explored more thoroughly in our forthcoming paper [8], and they all rely heavily on our use of l-vectors and generalized intersection matrices.

2 Preliminaries

2.1 C-matrices

Let n be a positive integer. If $B = [b_{ij}]$ is an $n \times n$ skew-symmetric matrix, then B is in correspondence with a quiver Q on n vertices: if $b_{ij} > 0$ and $i \neq j$, then Q has b_{ij} arrows from vertex i to vertex j. The statements of some theorems have been formulated in terms of Q; however, we prefer to work with B since the description of c-vectors is more clear in this setting. Also, for a nonzero vector $c = (c_1, \ldots, c_n) \in \mathbb{Z}^n$, we write $c > 0$ if all c_i are non-negative, and $c < 0$ if all c_i are non-positive.

Assume that $M = [m_{ij}]$ is an $n \times 2n$ matrix with integer entries. Let $\mathcal{I} := \{1, 2, \ldots, n\}$ be the set of indices. For $\boldsymbol{w} = [i_1, i_2, \ldots, i_\ell]$, $i_j \in \mathcal{I}$, we define the matrix $M^{\boldsymbol{w}} = [m_{ij}^{\boldsymbol{w}}]$ inductively: the initial matrix is M for $\boldsymbol{w} = [\,]$, and assuming we have $M^{\boldsymbol{w}}$, define the matrix $M^{\boldsymbol{w}[k]} = [m_{ij}^{\boldsymbol{w}[k]}]$ for $k \in \mathcal{I}$ with $\boldsymbol{w}[k] := [i_1, i_2, \ldots, i_\ell, k]$ by

$$m_{ij}^{\boldsymbol{w}[k]} = \begin{cases} -m_{ij}^{\boldsymbol{w}} & \text{if } i = k \text{ or } j = k, \\ m_{ij}^{\boldsymbol{w}} + \operatorname{sgn}(m_{ik}^{\boldsymbol{w}}) \max(m_{ik}^{\boldsymbol{w}} m_{kj}^{\boldsymbol{w}}, 0) & \text{otherwise,} \end{cases} \tag{2.1}$$

where $\operatorname{sgn}(a) \in \{1, 0, -1\}$ is the signature of a. The matrix $M^{\boldsymbol{w}[k]}$ is called the *mutation of $M^{\boldsymbol{w}}$* at index (or label) k, $\boldsymbol{w}$ and $\boldsymbol{w}[k]$ are called *mutation sequences*, and n is the *rank*.

Let B be a $n \times n$ skew-symmetric matrix. Consider the $n \times 2n$ matrix $\begin{bmatrix} B & I \end{bmatrix}$ and a mutation sequence $\boldsymbol{w} = [i_1, \ldots, i_\ell]$. After the mutations at the indices $i_1, \ldots, i_\ell$ consecutively, we obtain $\begin{bmatrix} B^{\boldsymbol{w}} & C^{\boldsymbol{w}} \end{bmatrix}$. Write their entries as

$$B^{\boldsymbol{w}} = \left[b_{ij}^{\boldsymbol{w}} \right], \qquad C^{\boldsymbol{w}} = \left[c_{ij}^{\boldsymbol{w}} \right] = \begin{bmatrix} c_1^{\boldsymbol{w}} \\ \vdots \\ c_n^{\boldsymbol{w}} \end{bmatrix}, \tag{2.2}$$

where $c_i^{\boldsymbol{w}}$ are the row vectors.

Definition 2.3. The matrix $C^{\boldsymbol{w}}$ is called a C-matrix of B for any w [1]. The row vectors $c_i^{\boldsymbol{w}}$ are called *c-vectors* of B for any i and $\boldsymbol{w}$. Each non-zero entry of $c_i^{\boldsymbol{w}}$ will share the same sign [6], allowing us to define the *sign-vector* of $C^{\boldsymbol{w}}$, where the i-th entry is 1 if $c_i^{\boldsymbol{w}} > 0$ and -1 if $c_i^{\boldsymbol{w}} < 0$.

[1] This is slightly different from the original definition by Fomin and Zelevinsky

2.2 Reflections and L-matrices

In order to prove Theorem 1.6, we needed to study the L-matrices arising from reflections and a particular generalized intersection matrix associated to our exchange matrix.

Definition 2.4. A *generalized intersection matrix* (GIM) is a square matrix $A = [a_{ij}]$ with integral entries such that (1) for diagonal entries, $a_{ii} = 2$; (2) $a_{ij} > 0$ if and only if $a_{ji} > 0$; (3) $a_{ij} < 0$ if and only if $a_{ji} < 0$.

Let $\mathcal{A}$ be the (unital) $\mathbb{Z}$-algebra generated by s_i, e_i, $i = 1, 2, \ldots, n$, subject to the following relations:

$$s_i^2 = 1, \quad \sum_{i=1}^{n} e_i = 1, \quad s_i e_i = -e_i, \quad e_i s_j = \begin{cases} s_i + e_i - 1 & \text{if } i = j, \\ e_i & \text{if } i \neq j, \end{cases} \quad e_i e_j = \begin{cases} e_i & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Let $\mathcal{W}$ be the subgroup of the units of $\mathcal{A}$ generated by s_i, $i = 1, \ldots, n$. Note that $\mathcal{W}$ is (isomorphic to) the universal Coxeter group. An element $r \in \mathcal{W}$ is called a reflection if $r^2 = 1$. Let $\mathfrak{R} \subset \mathcal{W}$ be the set of reflections.

From now on, let $A = [a_{ij}]$ be an $n \times n$ symmetric GIM. Let $\Gamma = \sum_{i=1}^{n} \mathbb{Z}\alpha_i$ be the lattice generated by the formal symbols $\alpha_1, ..., \alpha_n$. Define a representation $\pi : \mathcal{A} \to \mathrm{End}(\Gamma)$ by

$$\pi(s_i)(\alpha_j) = \alpha_j - a_{ji}\alpha_i \quad \text{and} \quad \pi(e_i)(\alpha_j) = \delta_{ij}\alpha_i, \quad \text{for } i, j \in \{1, ..., n\}.$$

We suppress π when we write the action of an element of $\mathcal{A}$ on Γ.

Given a skew-symmetric matrix B, for each linear ordering $\prec$ on $\{1, ..., n\}$, we define the associated GIM $A = [a_{ij}]$ by

$$a_{ij} = \begin{cases} b_{ij} & \text{if } i \prec j, \\ 2 & \text{if } i = j, \\ -b_{ij} & \text{if } i \succ j. \end{cases} \tag{2.5}$$

An ordering $\prec$ provides a certain way for us to regard the skew-symmetric matrix B as acyclic even when it is not.

Definition 2.6. When $\boldsymbol{w} = [\,]$, we let $r_i = s_i \in \mathfrak{R}$ for each $i \in \{1, ..., n\}$. For each mutation sequence $\boldsymbol{w}$ and each $i \in \{1, ..., n\}$, define $r_i^{\boldsymbol{w}} \in \mathfrak{R}$ inductively as follows:

$$r_i^{\boldsymbol{w}[k]} = \begin{cases} r_k^{\boldsymbol{w}} r_i^{\boldsymbol{w}} r_k^{\boldsymbol{w}} & \text{if } b_{ik}^{\boldsymbol{w}} c_k^{\boldsymbol{w}} > 0, \\ r_i^{\boldsymbol{w}} & \text{otherwise.} \end{cases} \tag{2.7}$$

Clearly, each $r_i^{\boldsymbol{w}}$ is written in the form

$$r_i^{\boldsymbol{w}} = g_i^{\boldsymbol{w}} s_i (g_i^{\boldsymbol{w}})^{-1}, \quad g_i^{\boldsymbol{w}} \in \mathcal{W}, \quad i \in \{1, ..., n\}.$$

Definition 2.8. Let sgn $= \{1, -1\}$ be the group of order 2, and consider the natural group action sgn $\times \mathbb{Z}^n \longrightarrow \mathbb{Z}^n$, where we identify Γ with $\mathbb{Z}^n$. Choose an ordering $\prec$ on $\{1, ..., n\}$ to fix a GIM A, and define

$$l_i^{\boldsymbol{w}} = g_i^{\boldsymbol{w}}(\alpha_i) \in \mathbb{Z}^n / \operatorname{sgn}, \qquad i \in \{1, ..., n\},$$

where we set $\alpha_1 = (1, 0, ..., 0), ..., \ \alpha_n = (0, ..., 0, 1)$. Then the *L-matrix* $L^{\boldsymbol{w}}$ associated to A is defined to be the $n \times n$ matrix whose i^{th} row is $l_i^{\boldsymbol{w}}$ for $i \in \{1, ..., n\}$, i.e., $L^{\boldsymbol{w}} = \begin{bmatrix} l_1^{\boldsymbol{w}} \\ \vdots \\ l_n^{\boldsymbol{w}} \end{bmatrix}$, and the vectors $l_i^{\boldsymbol{w}}$ are called the *l-vectors of A*. Note that the L-matrix and l-vectors associated to a GIM A implicitly depend on the representation π which is suppressed from the notation.

With the above machinery, we show the following, which further implies Theorem 1.6.

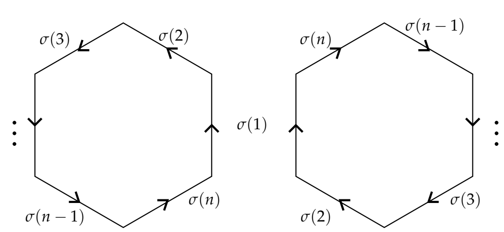
Theorem 2.9. *Let Q be a fork with n vertices, and let $\boldsymbol{w}$ be a fork-preserving mutation sequence. For each $i \in \{1, ..., n\}$, there exists a diagonal matrix $D_i^{\boldsymbol{w}}$ such that $(D_i^{\boldsymbol{w}})^2 = 1$ and $l_i^{\boldsymbol{w}} = c_i^{\boldsymbol{w}} D_i^{\boldsymbol{w}}$. In other words, the entries of l-vectors are equal to the entries of c-vectors up to sign.*

2.3 Geometry of reflections

Here we review the definition of admissible curves [18, 17].

Let $\mathcal{Q}$ be a fork with n vertices labeled by $I := \{1, ..., n\}$ and point of return r. Let σ be the linear ordering given by $r \prec a_{n-1} \prec a_{n-2} \prec \cdots \prec a_1$, where $a_1, a_2, \ldots, a_{n-1}$ are the vertices of $Q \setminus \{r\}$ and $a_i \prec a_j$ if and only if there is an arrow from j to i.

We define a labeled Riemann surface Σ_σ[2] as follows. Let G_1 and G_2 be two identical copies of a regular n-gon. Label the edges of each of the two n-gons by $T_{\sigma(1)}, \ldots, T_{\sigma(n)}$ counter-clockwise. Fix the orientation of every edge of G_1 (resp. G_2) to be counter-clockwise (resp. clockwise) as in the following picture.



[2]The punctured discs appeared in Bessis' work [3]. For better visualization, here we prefer to use an alternative description using compact Riemann surfaces with one or two marked points.

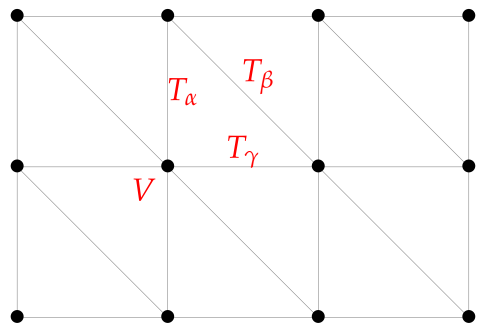


Figure 1: This picture illustrates a portion of the universal cover Σ_σ, and the three arcs T_α, T_β, and T_γ.

Let Σ_σ be the (compact) Riemann surface of genus $\lfloor \frac{n-1}{2} \rfloor$ obtained by gluing together the two n-gons with all the edges of the same label identified according to their orientations. The edges of the n-gons become N different curves in Σ_σ. If n is odd, all the vertices of the two n-gons are identified to become one point in Σ_σ and the curves obtained from the edges are loops. If n is even, two distinct vertices are shared by all curves. Let $\mathcal{T}$ be the set of all curves, i.e., $\mathcal{T} = T_1 \cup \cdots \cup T_n \subset \Sigma_\sigma$, and V be the set of the vertex (or vertices) on $\mathcal{T}$.

For simplicity, here we give a precise definition of an admissible curve for rank 3 quivers only, but it is straightforward to generalize to quivers of higher rank. For our geometric model on rank 3 quivers, we consider the (triangulated) torus with one marked point along with admissible curves (see Definition 2.10). The key point here is that there is a map from the set of admissible curves to $\mathfrak{R}$.

For each $\sigma \in \mathfrak{S}_3$, let Σ_σ be the closed Riemann surface of genus 1 with a single marked point V, and let $\widetilde{\Sigma_\sigma}$ be the universal cover of Σ_σ, which can be regarded as $\mathbf{R}^2$. Let $\alpha = \sigma(1)$, $\beta = \sigma(2)$, and $\gamma = \sigma(3)$. Fix three arcs T_α, T_β, and T_γ on Σ_σ and the projection $p : \widetilde{\Sigma_\sigma} \longrightarrow \Sigma_\sigma$ such that $p^{-1}(T_\alpha) = \mathbf{Z} \times \mathbf{R} \subset \mathbf{R}^2$, $p^{-1}(T_\beta) = \{(x,y) \,:\, x+y \in \mathbf{Z}\} \subset \mathbf{R}^2$, $p^{-1}(T_\gamma) = \mathbf{R} \times \mathbf{Z} \subset \mathbf{R}^2$, and $p^{-1}(V) = \mathbf{Z}^2 \subset \mathbf{R}^2$. Hence T_α is the vertical line segment, T_β is the diagonal, and T_γ is the horizontal line segment. Let $T = T_1 \cup T_2 \cup T_3$. See Figure 1.

Definition 2.10. An *admissible curve* is a pair consisting of a continuous function $\eta : [0,1] \longrightarrow \Sigma_\sigma$ and a sequence $\{i_\ell\}_{\ell=1}^{k}$ of entries with in $i_\ell \in \{1,2,3\}$ such that

1) $\eta(x) = V$ if and only if $x \in \{0,1\}$;

2) if $\eta(x) \in T \setminus \{V\}$ then $\eta([x-\epsilon, x+\epsilon])$ meets T transversally for sufficiently small $\epsilon > 0$;

3) $\eta(x_\ell) \in T_{i_\ell}$ and $\ell \in \{1,...,k\}$, where

$$\{x_1 < \cdots < x_k\} = \{x \in (0,1) \,:\, \eta(x) \in T\}$$

4) $v(\eta) \in \mathfrak{R}$, where $v(\eta) := r_{i_1} \cdots r_{i_k} \in \mathcal{W}$.

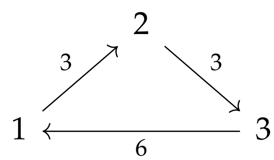
Example 2.11. In Example 3.5, when $\boldsymbol{w} = [1,2,3]$, the admissible curve η_2^w has

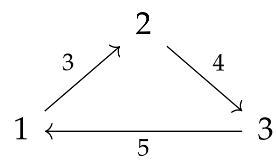
$$v(\eta_2^w) = r_2 r_1 r_3 r_1 r_2 r_1 r_3 r_1 r_2.$$

Note that η_2^w crosses T_2, T_1, T_3, T_1, T_2, T_1, T_3, T_1, T_2 in this order.

3 Examples

In this section, we will consider the following two quivers to demonstrate our theorems:

$P =$ 

and $Q =$ 

Both quivers are mutation-cyclic [2]. Also, P and Q are forks and are mutation-equivalent to only forks. In this section, we will consider the c-vectors of both P and Q under three mutation sequences, namely, $\boldsymbol{w} = [1]$, $\boldsymbol{w} = [1,2]$, and $\boldsymbol{w} = [1,2,3]$.

Example 3.1. An example of Theorem 1.8 is given in the table below:

Mutation Sequence	$[B^w \mid C^w]$-matrix for P	$[B^w \mid C^w]$-matrix for Q
$\boldsymbol{w} = [1]$	$\left[\begin{array}{ccc\|ccc} 0 & -3 & 6 & -1 & 0 & 0 \\ 3 & 0 & -15 & 0 & 1 & 0 \\ -6 & 15 & 0 & 6 & 0 & 1 \end{array}\right]$	$\left[\begin{array}{ccc\|ccc} 0 & -3 & 5 & -1 & 0 & 0 \\ 3 & 0 & -11 & 0 & 1 & 0 \\ -5 & 11 & 0 & 5 & 0 & 1 \end{array}\right]$
$\boldsymbol{w} = [1,2]$	$\left[\begin{array}{ccc\|ccc} 0 & 3 & -39 & -1 & 0 & 0 \\ -3 & 0 & 15 & 0 & -1 & 0 \\ 39 & -15 & 0 & 6 & 15 & 1 \end{array}\right]$	$\left[\begin{array}{ccc\|ccc} 0 & 3 & -28 & -1 & 0 & 0 \\ -3 & 0 & 11 & 0 & -1 & 0 \\ 28 & -11 & 0 & 5 & 11 & 1 \end{array}\right]$
$\boldsymbol{w} = [1,2,3]$	$\left[\begin{array}{ccc\|ccc} 0 & -582 & 39 & -1 & 0 & 0 \\ 582 & 0 & -15 & 90 & 224 & 15 \\ -39 & 15 & 0 & -6 & -15 & -1 \end{array}\right]$	$\left[\begin{array}{ccc\|ccc} 0 & -305 & 28 & -1 & 0 & 0 \\ 305 & 0 & -11 & 55 & 120 & 11 \\ -28 & 11 & 0 & -5 & -11 & -1 \end{array}\right]$

For each quiver, the sign vector of the C-matrix for $\boldsymbol{w} = [1]$, $\boldsymbol{w} = [1,2]$, and $\boldsymbol{w} = [1,2,3]$ is $(-1,1,1)$, $(-1,-1,1)$, and $(-1,1,-1)$).

Example 3.2. The quadratic equation for the quiver P is given by

$$x^2 + y^2 + z^2 - 3xy - 6xz + 3yz = 1.$$

and the quadratic equation for Q is given by

$$x^2 + y^2 + z^2 - 3xy - 5xz + 4yz = 1.$$

It is easy to verify that the c-vectors

$$(x, y, z) = (90, 224, 15) \text{ and } (x, y, z) = (-6, -15, -1)$$

both satisfy the quadratic equation for P and that the c-vectors

$$(x, y, z) = (55, 120, 11) \text{ and } (x, y, z) = (-5, -11, -1)$$

both satisfy the quadratic equation for Q.

Example 3.3. In this example, we demonstrate Corollary 1.9. If we mutate the reflections for both of P and Q with $\boldsymbol{w} = [1]$, then we arrive at

$$r_1^{\boldsymbol{w}} = r_1, \quad r_2^{\boldsymbol{w}} = r_2, \quad r_3^{\boldsymbol{w}} = r_1 r_3 r_1.$$

If we mutate both of them with $\boldsymbol{w} = [1, 2]$, then we arrive at

$$r_1^{\boldsymbol{w}} = r_1, \quad r_2^{\boldsymbol{w}} = r_2, \quad r_3^{\boldsymbol{w}} = r_2 r_1 r_3 r_1 r_2.$$

If we mutate both of them with $\boldsymbol{w} = [1, 2, 3]$, then we arrive at

$$r_1^{\boldsymbol{w}} = r_1, \quad r_2^{\boldsymbol{w}} = r_2 r_1 r_3 r_1 r_2 r_1 r_3 r_1 r_2, \quad r_3^{\boldsymbol{w}} = r_2 r_1 r_3 r_1 r_2.$$

We can see that both of these are fork-preserving mutation sequences with the same initial orientation for the B matrix.

Example 3.4. In this example, we demonstrate Theorem 1.10. If we take the three mutated reflections from Example 3.3 for $\boldsymbol{w} = [1]$, then

$$r_1^{\boldsymbol{w}} r_3^{\boldsymbol{w}} r_2^{\boldsymbol{w}} = r_3 r_1 r_2.$$

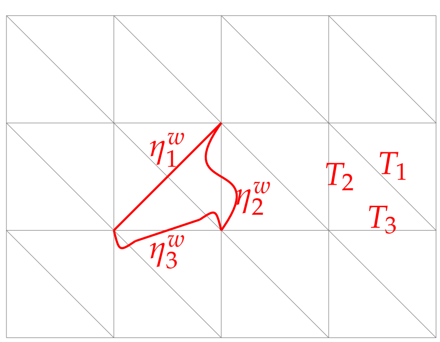
For $\boldsymbol{w} = [1, 2]$, we have

$$r_1^{\boldsymbol{w}} r_2^{\boldsymbol{w}} r_3^{\boldsymbol{w}} = r_3 r_1 r_2.$$

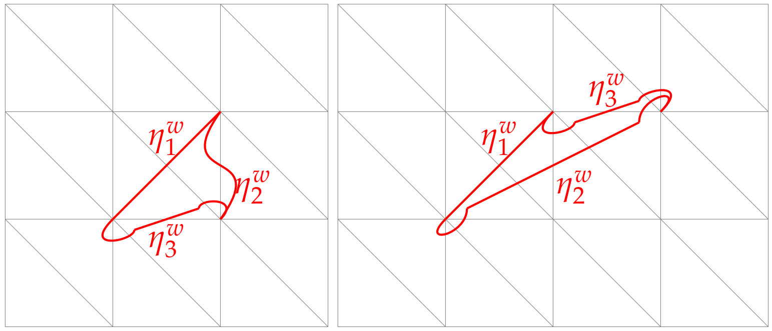
Finally, for $\boldsymbol{w} = [1, 2, 3]$, we have

$$r_1^{\boldsymbol{w}} r_3^{\boldsymbol{w}} r_2^{\boldsymbol{w}} = r_3 r_1 r_2.$$

Example 3.5. In this example, we demonstrate Corollary 1.11. If we take the three mutated reflections from Example 3.3 for $\boldsymbol{w} = [1]$, then we get the following admissible curves:



For $\boldsymbol{w} = [1,2]$ and $\boldsymbol{w} = [1,2,3]$, we get the following admissible curves respectively:



Note that these curves are pairwise non-crossing as well as non-self-crossing. Also, using the labeling of the 3 arcs from the picture for the first set of non-crossing curves, we can recover the sequence of reflections from the curves in each picture and confirm the correspondence.

Example 3.6. To demonstrate how to calculate l-vectors, we consider $l_2^{\boldsymbol{w}}$ for the quiver Q with $\boldsymbol{w} = [1,2,3]$ and linear ordering $2 \prec 1 \prec 3$. First, construct the GIM

$$A = \begin{bmatrix} 2 & -3 & -5 \\ -3 & 2 & 4 \\ -5 & 4 & 2 \end{bmatrix}.$$

Then consider the following matrices in $M_{3\times 3}(\mathbb{Z})$.

$$S_1 = \begin{bmatrix} -1 & 0 & 0 \\ 3 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix}, \qquad S_2 = \begin{bmatrix} 1 & 3 & 0 \\ 0 & -1 & 0 \\ 0 & -4 & 1 \end{bmatrix}, \qquad S_3 = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -4 \\ 0 & 0 & -1 \end{bmatrix}.$$

Using the sequence of reflections from Example 3.3 and the definition of l-vectors, we know that

$$\begin{aligned} l_2^{\boldsymbol{w}} &= s_2 s_1 s_3 s_1(\alpha_2) \\ &= (S_2^T S_1^T S_3^T S_1^T(\alpha_2^T))^T \\ &= \alpha_2 S_1 S_3 S_1 S_2 \\ &= (\alpha_2) S_1 S_3 S_1 S_2 \\ &= (3\alpha_1 + \alpha_2) S_3 S_1 S_2 \\ &= (3\alpha_1 + \alpha_2 + 11\alpha_3) S_1 S_2 \\ &= (55\alpha_1 + \alpha_2 + 11\alpha_3) S_2 \\ &= 55\alpha_1 + 120\alpha_2 + 11\alpha_3. \end{aligned}$$

These calculations can then be used to demonstrate Theorem 2.9. Compare the table below with the one given in Example 3.1.

Mutation Sequence	L-matrix for P	L-matrix for Q
$w = [1]$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 6 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix}$
$w = [1, 2]$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 6 & 15 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & 11 & 1 \end{bmatrix}$
$w = [1, 2, 3]$	$\begin{bmatrix} 1 & 0 & 0 \\ 90 & 224 & 15 \\ 6 & 15 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 55 & 120 & 11 \\ 5 & 11 & 1 \end{bmatrix}$

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References

[1] B. Baumeister, M. Dyer, C. Stump, and P. Wegener. "A note on the transitive Hurwitz action on decompositions of parabolic Coxeter elements". *Proceedings of the American Mathematical Society, Series B* **1**.13 (Dec. 2014), pp. 149–154. DOI.

[2] A. Beineke, T. Brüstle, and L. Hille. "Cluster-Cyclic Quivers with Three Vertices and the Markov Equation". *Algebras and Representation Theory* **14**.1 (Feb. 2011), pp. 97–112. DOI.

[3] D. Bessis. "A dual braid monoid for the free group". *Journal of Algebra* **302**.1 (Aug. 2006), pp. 55–69. DOI.

[4] P. Caldero and B. Keller. "From triangulated categories to cluster algebras II". *Annales Scientifiques de l'Ecole Normale Superieure* **39**.6 (Nov. 2006), pp. 983–1009. DOI.

[5] A. N. Chávez. "On the c-Vectors of an Acyclic Cluster Algebra". *International Mathematics Research Notices* **2015**.6 (Jan. 2015), pp. 1590–1600. DOI.

[6] H. Derksen, J. Weyman, and A. Zelevinsky. "Quivers with potentials and their representations I: Mutations". *Selecta Mathematica* **14**.1 (Oct. 2008), pp. 59–119. DOI.

[7] H. Derksen, J. Weyman, and A. Zelevinsky. "Quivers with potentials and their representations II: Applications to cluster algebras". *Journal of the American Mathematical Society* **23**.3 (July 2010), pp. 749–790. DOI.

[8] T. J. Ervin, B. Jacskon, K. Lee, and S. D. Nguyen. "Geometry of c-vectors and C-Matrices for Mutation-Infinite Quivers". In Preparation.

[9] A. Felikson and P. Tumarkin. "Acyclic cluster algebras, reflection groups, and curves on a punctured disc". *Advances in Mathematics* **340** (Dec. 2018), pp. 855–882. DOI.

[10] B. Feng, A. Hanany, Y.-H. He, and A. M. Uranga. "Toric duality as Seiberg duality and brane diamonds". *Journal of High Energy Physics* **2001**.12 (Jan. 2002), p. 035. DOI.

[11] S. Fomin and S. Neville. "Long mutation cycles". Apr. 2023. arXiv:2304.11505.

[12] S. Fomin and A. Zelevinsky. "Cluster algebras I: Foundations". *Journal of the American Mathematical Society* **15**.2 (Apr. 2002), pp. 497–529. DOI.

[13] S. Fomin and A. Zelevinsky. "Cluster algebras IV: Coefficients". *Compositio Mathematica* **143**.1 (Jan. 2007). Publisher: London Mathematical Society, pp. 112–164. DOI.

[14] M. Gross, P. Hacking, S. Keel, and M. Kontsevich. "Canonical bases for cluster algebras". *Journal of the American Mathematical Society* **31**.2 (Apr. 2018), pp. 497–608. DOI.

[15] A. Hubery and H. Krause. "A categorification of non-crossing partitions". *Journal of the European Mathematical Society* **18**.10 (Sept. 2016), pp. 2273–2313. DOI.

[16] K. Igusa and R. Schiffler. "Exceptional sequences and clusters". *Journal of Algebra* **323**.8 (Apr. 2010), pp. 2183–2202. DOI.

[17] K.-H. Lee and K. Lee. "A Correspondence between Rigid Modules Over Path Algebras and Simple Curves on Riemann Surfaces". *Experimental Mathematics* **30**.3 (July 2021), pp. 315–331. DOI.

[18] K.-H. Lee, K. Lee, and M. R. Mills. "Geometric description of C-vectors and real Lösungen". *Mathematische Zeitschrift* **303**.2 (Jan. 2023), p. 44. DOI.

[19] F. Lin, G. Musiker, and T. Nakanishi. "Two Formulas for F-Polynomials". *International Mathematics Research Notices* (Apr. 2023), rnad074. DOI.

[20] T. Nakanishi and A. Zelevinsky. "On tropical dualities in cluster algebras". *Algebraic groups and quantum groups*. Vol. 565. Contemp. Math. Amer. Math. Soc., Providence, RI, 2012, pp. 217–226. DOI.

[21] S. D. Nguyen. "A proof of Lee-Lee's conjecture about geometry of rigid modules". *Journal of Algebra* **611** (Dec. 2022), pp. 422–434. DOI.

[22] P.-G. Plamondon. "Cluster algebras via cluster categories with infinite-dimensional morphism spaces". *Compositio Mathematica* **147**.6 (Nov. 2011). Publisher: London Mathematical Society, pp. 1921–1954. DOI.

[23] A. Schofield. "General Representations of Quivers". *Proceedings of the London Mathematical Society* **s3-65**.1 (1992), pp. 46–64. DOI.

[24] A. I. Seven. "Cluster algebras and symmetric matrices". *Proceedings of the American Mathematical Society* **143**.2 (Oct. 2014), pp. 469–478. DOI.

[25] D. Speyer and H. Thomas. "Acyclic Cluster Algebras Revisited". *Algebras, Quivers and Representations: The Abel Symposium 2011*. Abel Symposia. Berlin, Heidelberg: Springer, 2013, pp. 275–298. DOI.

[26] M. Warkentin. "Exchange Graphs via Quiver Mutation". *Dissertation* (Jan. 2014), p. 103. Link.