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Ehrhart polynomials, Hecke series, and affine buildings

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Abstract. Given a lattice polytope P and a prime p, we define a function from the set of primitive symplectic p-adic lattices to the rationals that extracts the ℓth coefficient of the Ehrhart polynomial of P relative to the given lattice. Inspired by work of Gunnells and Rodriguez Villegas in type A, we show that these functions are eigenfunctions of a suitably defined action of the spherical symplectic Hecke algebra. Although they depend significantly on the polytope P, their eigenvalues are independent of P and expressed as polynomials in p. We define local zeta functions that enumerate the values of these Hecke eigenfunctions on the vertices of the affine Bruhat–Tits buildings associated with p-adic symplectic groups. We compute these zeta functions by enumerating p-adic lattices by their elementary divisors and, simultaneously, one Hermite parameter. We report on a general functional equation satisfied by these local zeta functions, confirming a conjecture of Vankov.

Keywords: Ehrhart polynomials, Hecke series, affine buildings, Satake isomorphism, symplectic lattices

1 Introduction

Let P be a fixed full-dimensional lattice polytope in $\mathbb{R}^n$, i.e. the convex hull of finitely many points $V(P)$ in $\Lambda_0 = \mathbb{Z}^n$. Given a lattice Λ such that $\Lambda_0 \subseteq \Lambda \subseteq \mathbb{Q}^n$, we denote the *Ehrhart polynomial of P with respect to Λ* by

$$E^{\Lambda}(P) = \sum_{\ell=0}^{n} c_{\ell}^{\Lambda}(P)T^{n} \in \mathbb{Q}[T]. \tag{1.1}$$

It is of interest to describe the variation of the coefficients $c_{\ell}^{\Lambda}(P)$ with Λ as compared to $c_{\ell}(P) = c_{\ell}^{\Lambda_0}(P)$; write $E(P)$ for $E^{\Lambda_0}(P)$. For $g \in \mathrm{GL}_n(\mathbb{Q}) \cap \mathrm{Mat}_n(\mathbb{Z})$ we define

$$g \cdot P = \mathrm{conv}\{g \cdot v \mid v \in V(P)\},$$

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which is again a lattice polytope. We write Λ_g for the lattice generated by the rows of $g \in \mathrm{GL}_n(\mathbb{Q})$. Thus, for every $g \in \mathrm{GL}_n(\mathbb{Q}) \cap \mathrm{Mat}_n(\mathbb{Z})$, we have

$$E(g \cdot P) = E^{\Lambda_{g^{-1}}}(P). \tag{1.2}$$

We note that $\Lambda_g \subseteq \mathbb{Z}^n \subseteq \Lambda_{g^{-1}} \subseteq \mathbb{Q}^n$ for $g \in \mathrm{GL}_n(\mathbb{Q}) \cap \mathrm{Mat}_n(\mathbb{Z})$ with $|\det(g)| > 1$.

Gunnells and Rodriguez Villegas [3] consider how the coefficients of $E^{\Lambda}(P)$ from Equation (1.1) relate to $E(P)$ for lattices Λ such that $\Lambda_0 \subseteq \Lambda \subseteq p^{-1}\Lambda_0 \subseteq \mathbb{Q}^n$. In Section 2.1 we revisit these results from our perspective. In addition, we consider a symplectic analogue of the work of Gunnells and Rodriguez Villegas.

1.1 Zeta functions of Ehrhart coefficients

For a prime p, we write $\mathbb{Z}_p$ for the ring of p-adic integers and $\mathbb{Q}_p$ for its field of fractions. Below we define, for each $n \in \mathbb{N} = \{1, 2, \dots\}$ and $\ell \in [2n]_0 = \{0, \dots, 2n\}$, local zeta functions which we call *Ehrhart–Hecke zeta functions*. These functions are Dirichlet series in a complex variable s encoding the ratio of ℓth coefficients of the Ehrhart polynomial of P, as the lattice Λ varies among symplectic lattices in $\mathbb{Q}_p^{2n}$.

Recall the group scheme GSp_{2n} of symplectic similitudes. For a ring K its K-rational points are, with $J = \left(\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right)$,

$$\mathrm{GSp}_{2n}(K) = \left\{ A \in \mathrm{GL}_{2n}(K) \;\middle|\; AJA^{\mathrm{t}} = \mu(A)J, \text{ for some } \mu(A) \in K^{\times} \right\}.$$

We set $G_n = \mathrm{GSp}_{2n}(\mathbb{Q}_p)$, $\Gamma_n = \mathrm{GSp}_{2n}(\mathbb{Z}_p)$, and $G_n^+ = \mathrm{GSp}_{2n}(\mathbb{Q}_p) \cap \mathrm{Mat}_{2n}(\mathbb{Z}_p)$. The set G_n^+/Γ_n is in bijection with the set of special vertices of the affine building associated with the group $\mathrm{GSp}_{2n}(\mathbb{Q}_p)$, which is of type $\widetilde{C}_n$.

We define the (*local*) *Ehrhart–Hecke zeta function* (of type C) as

$$\mathcal{Z}^{\mathsf{C}}_{n,\ell,p}(s) = \sum_{g \in G_n^+/\Gamma_n} \frac{c_\ell^{\Lambda_{g^{-1}}}(P)}{c_\ell(P)} |\Lambda_{g^{-1}} : \mathbb{Z}_p^n|^{-s}.$$

Informally speaking, the zeta function $\mathcal{Z}^{\mathsf{C}}_{n,\ell,p}(s)$ hence encodes the average ℓth coefficient of the Ehrhart polynomial of P across certain symplectic lattices.

1.2 Symplectic Hecke series

The zeta functions of Section 1.1 are closely connected to formal power series over the Hecke algebra associated with the pair (G_n^+, Γ_n). To explain this connection, we establish additional notation. For $m \in \mathbb{N}$ we define

$$D_n^{\mathsf{C}}(m) = \{A \in G_n^+ \mid AJA^{\mathrm{t}} = mJ\}.$$

Let $\mathcal{H}_p^{\mathsf{C}} = \mathcal{H}^{\mathsf{C}}(G_n^+, \Gamma_n)$ be the spherical Hecke algebra. The Hecke operator $T_n^{\mathsf{C}}(m)$ is

$$T_n^{\mathsf{C}}(m) = \sum_{g \in \Gamma_n \backslash D_n^{\mathsf{C}}(m)/\Gamma_n} \Gamma_n g \Gamma_n.$$

The (formal) symplectic Hecke series is defined as

$$\sum_{\alpha \geqslant 0} T_n^{\mathsf{C}}(p^\alpha) X^\alpha \in \mathcal{H}_p^{\mathsf{C}}[\![X]\!]. \tag{1.3}$$

Shimura's conjecture [6] that the series in (1.3) is a rational function in X was proved by Andrianov [1]. Explicit formulae, however, seem only to be known for $n \leqslant 4$; see [9].

We consider the image of the Hecke series in (1.3) under the Satake isomorphism $\Omega : \mathcal{H}_p^{\mathsf{C}} \to \mathbb{C}[x_0^{\pm 1}, \dots, x_n^{\pm 1}]^W$ mapping onto the ring of invariants of W, the Weyl group of G_n. For variables $\boldsymbol{x} = (x_0, \dots, x_n)$, we define the (*local*) *Satake generating function* as

$$R_{n,p}(\boldsymbol{x}, X) = \sum_{\alpha \geqslant 0} \Omega(T_n^{\mathsf{C}}(p^\alpha)) X^\alpha \in \mathbb{C}[\mathbf{x}^{\pm 1}][\![X]\!].$$

and the (*local*) *primitive local Satake generating function* as

$$R_{n,p}^{\mathrm{pr}}(\boldsymbol{x}, X) = (1 - x_0 X)\,(1 - x_0 x_1 \cdots x_n X)\, R_{n,p}(\boldsymbol{x}, X). \tag{1.4}$$

We write $V(\mathscr{X}_n)$ for the set of vertices of $\mathscr{X}_n$, the affine building $\mathscr{X}_n$ of type $\widetilde{\mathsf{A}}_{n-1}$ associated with the group $\mathrm{GL}_n(\mathbb{Q}_p)$, viz. homothety classes of full lattices in $\mathrm{GL}_n(\mathbb{Q}_p)$. In [2, Section 3.3] Andrianov shows, in essence, that $R_{n,p}^{\mathrm{pr}}$ can be interpreted as a sum over $V(\mathscr{X})$; see Theorem 1.1 below.

For a lattice $\Lambda \leqslant \mathbb{Z}_p^n$, set $\boldsymbol{\nu}(\Lambda) = (\nu_1 \leqslant \cdots \leqslant \nu_n) \in \mathbb{N}_0^n$ if $\mathbb{Z}_p^n/\Lambda \cong \mathbb{Z}/p^{\nu_1} \oplus \cdots \oplus \mathbb{Z}/p^{\nu_n}$. Setting $\nu_0 = 0$, we define

$$\boldsymbol{\mu}(\Lambda) = (\mu_1, \dots, \mu_n) = (\nu_n - \nu_{n-1},\ \dots,\ \nu_1 - \nu_0).$$

Having chosen a $\mathbb{Z}_p$-basis of $\mathbb{Z}_p^n$ we associate to each lattice $\Lambda \leqslant \mathbb{Z}_p^n$ a unique matrix

$$M_\Lambda = \begin{pmatrix} p^{\delta_1} & m_{12} & \cdots & m_{1n} \\ & p^{\delta_2} & \cdots & m_{2n} \\ & & \ddots & \vdots \\ & & & p^{\delta_n} \end{pmatrix} \in \mathrm{Mat}_n(\mathbb{Z}_p), \tag{1.5}$$

whose rows generate Λ and with $0 \leqslant v_p(m_{ij}) \leqslant \delta_j$ for all $1 \leqslant i < j \leqslant n$. The matrix M_Λ in (1.5) is said to be in Hermite normal form. We set $\delta(\Lambda) = (\delta_1, \dots, \delta_n)$. Clearly each homothety class $[\Lambda]$ contains a unique representative $\Lambda_{\mathrm{m}} \leqslant \mathbb{Z}_p^n$ such that $p^{-1}\Lambda_{\mathrm{m}} \not\leqslant \mathbb{Z}_p^n$.

Theorem 1.1 (Andrianov)**.** *Let $n \in \mathbb{N}$, $\boldsymbol{a} = (1, 2, \dots, n) \in \mathbb{N}^n$, $\boldsymbol{d} = (n, n-1, \dots, 1) \in \mathbb{N}^n$, and let $\langle , \rangle$ be the usual dot product. Then*

$$R_{n,p}^{\mathrm{pr}}(\boldsymbol{x}, X) = \sum_{[\Lambda] \in V(\mathscr{X}_n)} p^{\langle \boldsymbol{d}, \boldsymbol{\nu}(\Lambda_{\mathrm{m}})\rangle - \langle \boldsymbol{a}, \boldsymbol{\delta}(\Lambda_{\mathrm{m}})\rangle} x_1^{\delta_1(\Lambda_{\mathrm{m}})} \cdots x_n^{\delta_n(\Lambda_{\mathrm{m}})} (x_0 X)^{\nu_n(\Lambda_{\mathrm{m}})}.$$

1.3 The Hermite–Smith generating function

We define a generating function enumerating finite-index sublattices of $\mathbb{Z}_p^n$ simultaneously by their Hermite and Smith normal forms. For $n \in \mathbb{N}$, let $\boldsymbol{X} = (X_1, \dots, X_n)$ and $\boldsymbol{Y} = (Y_1, \dots, Y_n)$ be variables. The *Hermite–Smith generating function* is

$$\mathrm{HS}_{n,p}(\boldsymbol{X}, \boldsymbol{Y}) = \sum_{\Lambda \leqslant \mathbb{Z}_p^n} \boldsymbol{X}^{\boldsymbol{\mu}(\Lambda)} \boldsymbol{Y}^{\delta(\Lambda)} = \sum_{\Lambda \leqslant \mathbb{Z}_p^n} \prod_{i=1}^{n} X_i^{\mu_i(\Lambda)} Y_i^{\delta_i(\Lambda)} \in \mathbb{Z}[\![\boldsymbol{X}, \boldsymbol{Y}]\!]. \tag{1.6}$$

Clearly, if $\Lambda \leqslant \mathbb{Z}_p^n$ has finite index, then so does $p^m \Lambda$ for all $m \in \mathbb{N}_0$. This allows us to extract a "homothety factor" from the sum defining $\mathrm{HS}_{n,p}(\boldsymbol{X}, \boldsymbol{Y})$. The *primitive Hermite–Smith generating function* is

$$\mathrm{HS}_{n,p}^{\mathrm{pr}}(\boldsymbol{X}, \boldsymbol{Y}) = \sum_{[\Lambda] \in V(\mathscr{X}_n)} \boldsymbol{X}^{\boldsymbol{\mu}(\Lambda_{\mathrm{m}})} \boldsymbol{Y}^{\delta(\Lambda_{\mathrm{m}})} = (1 - X_n Y_1 \cdots Y_n)\, \mathrm{HS}_{n,p}(\boldsymbol{X}, \boldsymbol{Y}). \tag{1.7}$$

With this generating function we may obtain the primitive local Satake generating function of Section 1.2, as follows. We define a ring homomorphism

$$\begin{aligned} \Phi : \mathbb{Q}[\![X_1, X_2, \dots, Y_1, Y_2, \dots]\!] &\longrightarrow \mathbb{Q}[\![x_0, x_1, \dots, X]\!] \\ X_i &\longmapsto p^{\binom{i+1}{2}} x_0 X, \\ Y_i &\longmapsto p^{-i} x_i \end{aligned} \tag{1.8}$$

for all $i \in \mathbb{N}_0$. By design of Φ and Theorem 1.1 we have $\Phi(\mathrm{HS}_{n,p}^{\mathrm{pr}}) = R_{n,p}^{\mathrm{pr}}$.

Example 1.2. For $n = 2$, the Hermite–Smith generating function is

$$\begin{aligned} \mathrm{HS}_{2,p}(\boldsymbol{X}, \boldsymbol{Y}) &= \frac{1 - X_1^2 Y_1 Y_2}{(1 - X_1 Y_1)(1 - p X_1 Y_2)(1 - X_2 Y_1 Y_2)}, \\ R_{2,p}(\boldsymbol{x}, X) &= \frac{1 - p^{-1} x_0^2 x_1 x_2 X^2}{(1 - x_0 X)(1 - x_0 x_1 X)(1 - x_0 x_2 X)(1 - x_0 x_1 x_2 X)}. \end{aligned}$$

2 Main results

Interpreting the ℓ-th coefficients of the Ehrhart polynomial of the polytope P as a function on a set of (homothety classes of) p-adic lattices invites the definition of an action of the spherical Hecke algebra $\mathcal{H}_p^{\mathsf{C}}$. The latter is generated by a set of $n+1$ generators $T_n^{\mathsf{C}}(p, 0),\ T_n^{\mathsf{C}}(p^2, 1), \dots, T_n^{\mathsf{C}}(p^2, n)$. It suffices to explain how these generators act. For $k \in [n]$, define diagonal matrices in G_n^+ as follows:

$$D_0 = \mathrm{diag}(\underbrace{1, \dots, 1}_{n}, \underbrace{p, \dots, p}_{n}), \qquad D_k = \mathrm{diag}(\underbrace{1, \dots, 1}_{n-k}, \underbrace{p, \dots, p}_{k}, \underbrace{p^2, \dots, p^2}_{n-k}, \underbrace{p, \dots, p}_{k}).$$

Set $\mathscr{D}^{\mathsf{C}}_{n,k} = \Gamma_n D_k \Gamma_n/\Gamma_n$. The set $\mathscr{D}^{\mathsf{C}}_{n,k}$ can be interpreted as the set of symplectic lattices with symplectic elementary divisors equal to those of D_k. We define

$$T^{\mathsf{C}}_n(p,0)E(P) = \sum_{g\in\mathscr{D}^{\mathsf{C}}_{n,0}} E(g\cdot P), \qquad T^{\mathsf{C}}_n(p^2,k)E(P) = \sum_{g\in\mathscr{D}^{\mathsf{C}}_{n,k}} E(g\cdot P).$$

For $\ell \geqslant \mathbb{N}_0$, we define functions

$$\mathscr{E}_{n,p,\ell,P} : G^+_n/\Gamma_n \to \mathbb{C}, \quad \Gamma_n g \mapsto c_\ell(E^{\Lambda_{g^{-1}}}(P)).$$

Lastly, for all $T \in \mathcal{H}^{\mathsf{C}}_p$ set

$$T\mathscr{E}_{n,p,\ell,P}(\Gamma_n g) = c_\ell(TE^{\Lambda_{g^{-1}}}(P)).$$

Recall that P is full-dimensional; for $k \in [n]$, and $\ell \in [2n]_0$, we define

$$\nu^{\mathsf{C}}_{n,0,\ell}(p) = \frac{c_\ell(T^{\mathsf{C}}_n(p,0)E(P))}{c_\ell(E(P))}, \qquad \nu^{\mathsf{C}}_{n,k,\ell}(p) = \frac{c_\ell(T^{\mathsf{C}}_n(p^2,k)E(P))}{c_\ell(E(P))}.$$

The notation suggests that the value $\nu^{\mathsf{C}}_{n,k,\ell}(p)$ is independent of the polytope P, which is justified by Theorem A. General properties of the Ehrhart polynomial imply that

$$\nu^{\mathsf{C}}_{n,n,\ell}(p) = p^\ell, \qquad \nu^{\mathsf{C}}_{n,k,0}(p) = \#\mathscr{D}^{\mathsf{C}}_{n,k}.$$

Every $\mathbb{Q}$-linear homomorphism $\lambda : \mathcal{H}^{\mathsf{C}}_p \to \mathbb{C}$ is uniquely determined by parameters $(a_0,\ldots,a_n) \in \mathbb{C}^{n+1}$ such that if $\psi : \mathbb{C}[x_0^{\pm 1},\ldots,x_n^{\pm 1}] \to \mathbb{C}$ is given by $x_i = a_i$ then $\lambda = \psi\circ\Omega$; see [2, Proposition 3.3.36].

Theorem A. *The functions $\mathscr{E}_{n,p,\ell,P}$ are Hecke eigenfunctions under the action defined above; specifically, for all $k \in [n]$, we have*

$$T^{\mathsf{C}}_n(p,0)\mathscr{E}_{n,p,\ell,P} = \nu^{\mathsf{C}}_{n,0,\ell}(p)\mathscr{E}_{n,p,\ell,P}, \qquad T^{\mathsf{C}}_n(p^2,k)\mathscr{E}_{n,p,\ell,P} = \nu^{\mathsf{C}}_{n,k,\ell}(p)\mathscr{E}_{n,p,\ell,P},$$

where the $\nu^{\mathsf{C}}_{n,k,\ell}(p)$ are polynomials in p with integer coefficients which are independent of P. Moreover, the parameters associated to $\nu^{\mathsf{C}}_{n,k,\ell}(p)$ are $(p^\ell, p, p^2, \ldots, p^{n-1}, p^{n-\ell})$.

Table 1 lists the values of $\nu^{\mathsf{C}}_{n,k,\ell}(p)$ for small values of n and k.

Theorem A enables us to relate $\mathcal{Z}^{\mathsf{C}}_{n,\ell,p}(s)$ to $R_{n,p}(\boldsymbol{x}, X)$. Let $\psi_{n,\ell}$ be the ring homomorphism from $\mathbb{C}[\![x_0,x_1,\ldots,X]\!] \to \mathbb{C}[t]$ given by

$$X \mapsto t^n \qquad x_0 \mapsto p^\ell, \qquad x_n \mapsto p^{n-\ell}, \qquad x_i \mapsto p^i.$$

Corollary B. *For $n \in \mathbb{N}$ and $\ell \in [2n]_0$ we have, writing $t = p^{-s}$,*

$$(\psi_{n,\ell}\circ\Phi)(\mathrm{HS}^{\mathrm{pr}}_{n,p}(\boldsymbol{X},\boldsymbol{Y})) = \psi_{n,\ell}(R^{\mathrm{pr}}_{n,p}) = \mathcal{Z}^{\mathsf{C}}_{n,\ell,p}(s)\left(1-p^{\ell-s}\right)\left(1-p^{\binom{n+1}{2}-s}\right).$$

ℓ	$\nu^{\mathsf{C}}_{2,0,\ell}(p)$	$\nu^{\mathsf{C}}_{2,1,\ell}(p)$
4	$p^5+p^4+p^3+p^2$	$p^8+p^7+p^6+p^5$
3	$p^4+p^3+p^3+p^2$	$2p^6+p^5+2p^4-p^3$
2	$p^3+p^3+p^2+p^2$	$p^5+3p^4+p^3-p^2$
1	$p^3+p^2+p^2+p^1$	$2p^4+p^3+2p^2-p$
0	p^3+p^2+p+1	$p^4+p^3+p^2+p$

Table 1: The polynomials $\nu^{\mathsf{C}}_{2,k,\ell}(p)$ for $k \in \{0,1\}$ and $\ell \in [4]_0$.

Thanks to Corollary B, we can work with $\mathrm{HS}_{n,p}$ to prove that $R_{n,p}$ and $\mathcal{Z}^{\mathsf{C}}_{n,\ell,p}$ satisfy a self-reciprocity property, which proves the conjecture in [9, Remark 4].

Theorem C. *Let $n \in \mathbb{N}$. Then $\mathrm{HS}_{n,p}(\boldsymbol{X},\boldsymbol{Y})$ is a rational function in $\boldsymbol{X}$ and $\boldsymbol{Y}$. Furthermore, for $\boldsymbol{X}^{-1} = (X_1^{-1},\dots,X_n^{-1})$ and $\boldsymbol{Y}^{-1} = (Y_1^{-1},\dots,Y_n^{-1})$, we have*

$$\mathrm{HS}_{n,p}(\boldsymbol{X}^{-1},\boldsymbol{Y}^{-1})\Big|_{p\to p^{-1}} = (-1)^n p^{\binom{n}{2}} X_n Y_1 \cdots Y_n \cdot \mathrm{HS}_{n,p}(\boldsymbol{X},\boldsymbol{Y}).$$

We prove Theorem C by writing $\mathrm{HS}_{n,p}$ as a p-adic integral and applying results of [10], where the operation of inverting p is also explained.

Corollary D. *For $n \in \mathbb{N}$ and $\ell \in [2n]_0$, we have*

$$\begin{aligned} \mathcal{Z}^{\mathsf{C}}_{n,\ell,p}(s)\Big|_{p\to p^{-1}} &= (-1)^{n+1} p^{n^2+\ell-2ns} \cdot \mathcal{Z}^{\mathsf{C}}_{n,\ell,p}(s), \\ R_{n,p}(\mathbf{x},X)\Big|_{p\to p^{-1}} &= (-1)^{n+1} p^{\binom{n}{2}} x_0^2 x_1 \dots x_n X^2 \cdot R_{n,p}(\mathbf{x},X). \end{aligned}$$

In the next theorem, we determine a formula for the specialization of $\mathrm{HS}^{\mathrm{pr}}_{n,p}$ which yields $\mathcal{Z}^{\mathsf{C}}_{n,\ell,p}$ by Corollary B. To this end we define

$$\overline{\mathrm{HS}}_{n,p}(\boldsymbol{X},Y) = \mathrm{HS}^{\mathrm{pr}}_{n,p}(\boldsymbol{X},1,\dots,1,Y).$$

We prove that $\overline{\mathrm{HS}}_{n,p}$ is a rational function in the $n+1$ variables $\boldsymbol{X}$ and Y and, in addition, the prime p. In order to describe the formula, we define additional notation. For $I = \{i_1 < \dots < i_\ell\} \subseteq [n-1]$, with $i_{\ell+1} = n$, $k \in [\ell+1]$, and a variable Z, we set

$$\begin{aligned} I^{(k)} &= \{i_j \mid j < k\} \cup \{i_j - 1 \mid j \geqslant k\} \\ \mathscr{G}_{n,I,k}(Z,\boldsymbol{X},Y) &= \left(\prod_{j=1}^{k-1} \frac{Z^{i_j(n-i_j-1)} X_{i_j}}{1 - Z^{i_j(n-i_j-1)} X_{i_j}}\right)\left(\prod_{j=k}^{\ell} \frac{Z^{i_j(n-i_j)} X_{i_j} Y}{1 - Z^{i_j(n-i_j)} X_{i_j} Y}\right). \end{aligned}$$

Theorem E. *Let $n \in \mathbb{N}$. For $I = \{i_1 < \cdots < i_\ell\}_< \subseteq [n-1]$, set*

$$\begin{aligned} W_{n,I}(Z, \boldsymbol{X}, Y) = &\sum_{k=1}^{\ell+1} Z^{-(n-i_k)} \binom{n-1}{I^{(k)}}_{Z^{-1}} \mathscr{G}_{n,I,k}(Z, \boldsymbol{X}, Y) \\ &+ \sum_{k=1}^{\ell} \frac{(1-Z^{-i_j})\mathscr{G}_{n,I,k}(Z, \boldsymbol{X}, Y)}{1 - Z^{i_j(n-i_j-1)} X_{i_j}} \left(\sum_{m=k+1}^{\ell+1} Z^{-(n-i_m)} \right) \binom{n-1}{I^{(k+1)}}_{Z^{-1}}. \end{aligned}$$

Then

$$\overline{\mathrm{HS}}_{n,p}(\boldsymbol{X}, Y) = \sum_{I \subseteq [n-1]} W_{n,I}(p, \boldsymbol{X}, Y) \in \mathbb{Z}(p, \boldsymbol{X}, Y).$$

Via the various substitutions given above, Theorem E yields explicit formulae for the functions $R_{n,p}$ and, specifically,

$$Z^{\mathsf{C}}_{n,\ell,p}(s) = (1 - p^{\ell - s})^{-1} (1 - p^{\binom{n+1}{2} - s})^{-1} \sum_{I \subseteq [n-1]} W_{n,I}\left(p, \left(p^{\binom{i+1}{2} + \ell - ns}\right)_{i=1}^{n}, p^{-\ell}\right).$$

In the next theorem we show that the primitive local Satake generating function can be viewed as a "p-analogue" of the fine Hilbert series of a Stanley–Reisner ring. Let V be a finite set. If $\Delta \subseteq 2^V$ is a simplicial complex on V, then the Stanley–Reisner ring of Δ over a ring K is

$$K[\Delta] = K[X_v \mid v \in V] / (\prod_{v \in \sigma} X_v \mid \sigma \in 2^V \setminus \Delta).$$

Theorem F. *For all $n \in \mathbb{N}$, let Δ_n be the n-simplex with vertices $[n]$ and $\Delta = \mathrm{sd}(\partial \Delta_n)$, the barycentric subdivision of boundary of Δ_n, with vertices given by the nonempty subsets of $[n]$. Let $\boldsymbol{y} = (y_I : \varnothing \neq I \subseteq [n])$ and $\varphi : \mathbb{Z}[\![\boldsymbol{y}]\!] \to \mathbb{Z}[\![\boldsymbol{x}, X]\!]$ via $y_I \mapsto x_0 X \prod_{i \in I} x_i$. Then*

$$R^{\mathrm{pr}}_{n,p}(\boldsymbol{x}, X)\big|_{p \to 1} = \varphi(\mathrm{Hilb}(\mathbb{Z}[\Delta]; \boldsymbol{y})) = \sum_{\sigma \in \Delta} \prod_{J \in \sigma} \frac{\varphi(y_J)}{1 - \varphi(y_J)}.$$

With Theorem F, we come full circle and relate the local Satake generating function $R_{n,p}$ to the Ehrhart series of the n-cube.

Corollary 2.1. *For all $n \in \mathbb{N}$, let P be the n-cube. Then*

$$R_{n,p}(\mathbf{1}, X)\big|_{p \to 1} = \mathrm{Ehr}_P(X) = \frac{\mathrm{E}_n(X)}{(1-X)^{n+1}},$$

where $\mathrm{E}_n(X) = \sum_{\sigma \in S_n} X^{\mathrm{des}(\sigma)}$ is the Eulerian polynomial.

Proof. It follows from Theorem F that

$$(1-X)^2 \, R_{n,p}(\mathbf{1}, X)\big|_{p \to 1} = \sum_{\sigma \in \Delta} \prod_{J \in \sigma} \frac{X}{1-X}, \tag{2.1}$$

where Δ is the barycentric subdivision of the boundary of the n-simplex. From [5, Theore. 9.1] and Equation (2.1) it follows that

$$R_{n,p}(\mathbf{1},X)\big|_{p\to 1}=\frac{\mathrm{E}_n(X)}{(1-X)^{n+1}}=\sum_{k\geqslant 0}(k+1)^nX^k=\mathrm{Ehr}_P(X). \qquad \square$$

2.1 The type–A story

Our work was inspired by Gunnells and Rodriguez Villegas. In [3] they considered type-A versions of some of the questions outlined above. We paraphrase parts of [3] from the perspective of our work in type C. For a prime p we define the (*local*) *Ehrhart–Hecke zeta function* (of type A) as

$$\mathcal{Z}^{\mathsf{A}}_{n,\ell,p}(s)=\sum_{\substack{\mathbb{Z}_p^n\leqslant\Lambda\leqslant\mathbb{Q}_p^n\\ |\Lambda:\mathbb{Z}_p^n|<\infty}}\frac{c_\ell^\Lambda(P)}{c_\ell(P)}|\Lambda:\mathbb{Z}_p^n|^{-s}. \tag{2.2}$$

Let $\Gamma_n^{\mathsf{A}}=\mathrm{GL}_n(\mathbb{Z})$ and $G_n^{\mathsf{A}}=\mathrm{Mat}_n(\mathbb{Z})\cap\mathrm{GL}_n(\mathbb{Q})$. For $m\in\mathbb{N}$, let

$$D_n^{\mathsf{A}}(m)=\{g\in G_n^{\mathsf{A}}\mid |\det(g)|=m\},$$

so $D_n^{\mathsf{A}}(m)$ is a finite union of double cosets relative to Γ_n^{A}. We define

$$T_n^{\mathsf{A}}(m)=\sum_{g\in\Gamma_n^{\mathsf{A}}\backslash D_n^{\mathsf{A}}(m)/\Gamma_n^{\mathsf{A}}}\Gamma_n^{\mathsf{A}}g\Gamma_n^{\mathsf{A}},$$

where the sum runs over a set of representatives of the double cosets, which is an element of the Hecke algebra determined by $(\Gamma_n^{\mathsf{A}},G_n^{\mathsf{A}})$. Moreover, if $\gcd(m,m')=1$, then

$$T_n^{\mathsf{A}}(m)T_n^{\mathsf{A}}(m')=T_n^{\mathsf{A}}(mm').$$

For $k\in[n]_0$ define $\pi_k(p)=\mathrm{diag}(1,\dots,1,\overbrace{p,\dots,p}^{k})$ and $T_n^{\mathsf{A}}(p,k)=\Gamma_n^{\mathsf{A}}\pi_k(p)\Gamma_n^{\mathsf{A}}$, which decomposes into a finite (disjoint) union of right cosets relative to Γ_n^{A}.

Gunnells and Rodriguez Villegas [3] considered the following action of the Hecke algebra on the Ehrhart polynomial $E(P)=E^{\Lambda_0}(P)$ of P:

$$T_n^{\mathsf{A}}(p,k)E(P)=\sum_{g\in\Gamma_n^{\mathsf{A}}\pi_k(p)\Gamma_n^{\mathsf{A}}/\Gamma_n^{\mathsf{A}}}E(g\cdot P), \tag{2.3}$$

where the sum runs over a set of right coset representatives. The action in (2.3) is independent of the chosen representatives since Γ_n^{A} comprises bijections of $\mathbb{Z}^n$. Our definition in (2.3) differs from [3] only cosmetically via (1.2).

Denote by $\mathrm{Gr}(\ell, n, p)$ the set of ℓ-dimensional subspaces in $\mathbb{F}_p^n$. For $n \in \mathbb{N}$, $\ell, k \in [n]_0$, and $U \in \mathrm{Gr}(\ell, n, p)$, define

$$\nu^{\mathsf{A}}_{n,k,\ell}(p) = \sum_{W \in \mathrm{Gr}(k,n,p)} \#(U \cap W).$$

Let $\psi^{\mathsf{A}}_{n,\ell} : \mathbb{Q}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \to \mathbb{Q}$ be given by $x_n \mapsto p^\ell$ and $x_i \mapsto p^i$ for all $i \in [n-1]$. Let further ω denote the Satake isomorphism from the p-primary part of the Hecke algebra associated with $(\Gamma^{\mathsf{A}}_n, G^{\mathsf{A}}_n)$, written $\mathcal{H}^{\mathsf{A}}_p$, to the symmetric subring of $\mathbb{Q}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$.

Let $s_{n,k}(x_1, \ldots, x_n)$ be the (homogeneous) elementary symmetric polynomial of degree k, and set $s_{n,-1} = 0$.

Theorem 2.2 ([3]). *For $n \in \mathbb{N}$, $k, \ell \in [n]_0$, and a prime p, we have*

$$\nu^{\mathsf{A}}_{n,k,\ell}(p) = p^k \binom{n-1}{k}_p + p^\ell \binom{n-1}{k-1}_p = \psi^{\mathsf{A}}_{n,\ell}(\omega(T^{\mathsf{A}}_n(p,k))).$$

Moreover,

$$\mathcal{Z}^{\mathsf{A}}_{n,\ell,p}(s) = (1 - p^{\ell - s})^{-1} \prod_{k=1}^{n-1} (1 - p^{k-s})^{-1}.$$

Proof. First we prove the claims concerning $\nu^{\mathsf{A}}_{n,k,\ell}(p)$. Therefore,

$$\begin{aligned}
\nu^{\mathsf{A}}_{n,k,\ell}(p) &= \binom{n}{k}_p - \binom{n-1}{k-1}_p + p^\ell \binom{n-1}{k-1}_p && \text{([3, Lem. 3.3])} \\
&= p^k \binom{n-1}{k}_p + p^\ell \binom{n-1}{k-1}_p && \text{(Pascal identity)} \\
&= p^k s_{n-1,k}(1, p, \ldots, p^{n-2}) + p^\ell s_{n-1,k-1}(1, p, \ldots, p^{n-2}) && \text{([4, Ex. I.2.3])} \\
&= p^{-\binom{k}{2}} \psi^{\mathsf{A}}_{n,\ell}(s_{n,k}) \\
&= \psi^{\mathsf{A}}_{n,\ell}(\omega(T^{\mathsf{A}}_n(p,k))). && \text{([2, Lem. 3.2.21])}
\end{aligned}$$

We now tend to the last claim. Tamagawa [7] established the identity

$$\sum_{m \geqslant 0} T^{\mathsf{A}}_n(p^m) X^m = \left(\sum_{k=0}^{n} (-1)^k p^{\binom{k}{2}} T^{\mathsf{A}}_n(p,k) X^k \right)^{-1} \in \mathcal{H}^{\mathsf{A}}_p[\![X]\!]. \tag{2.4}$$

Applying $\psi^{\mathsf{A}}_{n,\ell} \circ \omega$ to (2.4) and setting $X = p^{-s}$, we have

$$\sum_{m \geqslant 0} \psi^{\mathsf{A}}_{n,\ell}(\omega(T^{\mathsf{A}}_n(p^m))) p^{-ms} = \left(\sum_{k=0}^{n} \psi^{\mathsf{A}}_{n,\ell}(s_{n,k})(-p)^{-ks} \right)^{-1} = (1 - p^{\ell - s})^{-1} \prod_{k=1}^{n-1} (1 - p^{k-s})^{-1}.$$

Since $\nu^{\mathsf{A}}_{n,k,\ell}(p)$ is an eigenvalue for $T_n(p,k)$, it follows that

$$\mathcal{Z}^{\mathsf{A}}_{n,\ell,p}(s) = \sum_{m \geqslant 0} \psi^{\mathsf{A}}_{n,\ell}(\omega(T^{\mathsf{A}}_n(p^m))) p^{-ms}. \qquad \square$$

Corollary 2.3. *Let $\zeta(s)$ be the Riemann zeta function. For $n \in \mathbb{N}$ and $\ell \in [n]_0$, we have*

$$\prod_{\text{prime } p} \mathcal{Z}^{\mathsf{A}}_{n,\ell,p}(s) = \zeta(s-\ell)\prod_{k=1}^{n-1}\zeta(s-k).$$

3 Examples

3.1 Hecke eigenfunctions

We give some explicit examples, showing in Figure 3.1 that the eigenfunctions of Theorem A depend significantly on the polytope. We do this by displaying a graph whose vertices correspond to homothety classes of lattices. We evaluate the functions $\mathscr{E}_{n,p\ell,P}$ on Λ_{m} for each homothety class $[\Lambda]$.

3.2 Local Ehrhart–Hecke zeta functions

For $n \in [3]$ and $\ell \in [2n]_0$, we record the rational functions $W_{n,\ell}(X,Y) \in \mathbb{Q}(X,Y)$ where, for all primes, $\mathcal{Z}^{\mathsf{C}}_{n,\ell,p}(s) = W_{n,\ell}(p, p^{-ns})$. We computed these with SageMath [8].

$$\begin{aligned}
W_{1,\ell}(X,Y) &= \frac{1}{(1-XY)(1-X^{\ell}Y)} \\
W_{2,\ell}(X,Y) &= \frac{1-X^{2+\ell}Y^2}{(1-X^2Y)(1-X^3Y)(1-X^{\ell}Y)(1-X^{\ell+1}Y)} \\
W_{3,\ell}(X,Y) &= \frac{1+(X^{1+\ell}+X^4)Y - A_{\ell}(X)Y^2 + (X^{6+2\ell}+X^{9+\ell})Y^3 + X^{10+2\ell}Y^4}{(1-X^3Y)(1-X^5Y)(1-X^6Y)(1-X^{\ell}Y)(1-X^{2+\ell}Y)(1-X^{3+\ell}Y)} \\
W_{4,\ell}(X,Y) &= \frac{N_{4,\ell}(X,Y)}{D_{4,\ell}(X,Y)},
\end{aligned}$$

where $A_{\ell}(X) = X^{7+\ell} + 2X^{6+\ell} + 2X^{4+\ell} + X^{3+\ell}$,

$$\begin{aligned}
N_{4,\ell}(X,Y) = 1 &+ (X^5+X^6+X^7+X^8+X^{1+\ell}+X^{2+\ell}+X^{3+\ell}+X^{4+\ell})Y + (X^{13} \\
&- X^{4+\ell} - 2X^{5+\ell} - 2X^{6+\ell} - 2X^{7+\ell} - 2X^{8+\ell} - 2X^{9+\ell} - 3X^{10+\ell} \\
&- 2X^{11+\ell} - 2X^{12+\ell} - 2X^{13+\ell} - X^{14+\ell} + X^{5+2\ell})Y^2 + (X^{14+\ell} \\
&- X^{18+\ell} + X^{10+2\ell} - X^{14+2\ell})Y^3 - (X^{23+\ell} - X^{14+2\ell} - 2X^{15+2\ell} \\
&- 2X^{16+2\ell} - 2X^{17+2\ell} - 3X^{18+2\ell} - 2X^{19+2\ell} - 2X^{20+2\ell} - 2X^{21+2\ell} \\
&- 2X^{22+2\ell} - 2X^{23+2\ell} - X^{24+2\ell} + X^{15+3\ell})Y^4 - (X^{24+2\ell} + X^{25+2\ell} \\
&+ X^{26+2\ell} + X^{27+2\ell} + X^{20+3\ell} + X^{21+3\ell} + X^{22+3\ell} + X^{23+3\ell})Y^5 \\
&- X^{28+3\ell}Y^6,
\end{aligned}$$

$$D_{4,\ell}(X,Y) = (1-X^4Y)(1-X^7Y)(1-X^9Y)(1-X^{10}Y) \\ \times (1-X^{\ell}Y)(1-X^{3+\ell}Y)(1-X^{5+\ell}Y)(1-X^{6+\ell}Y).$$

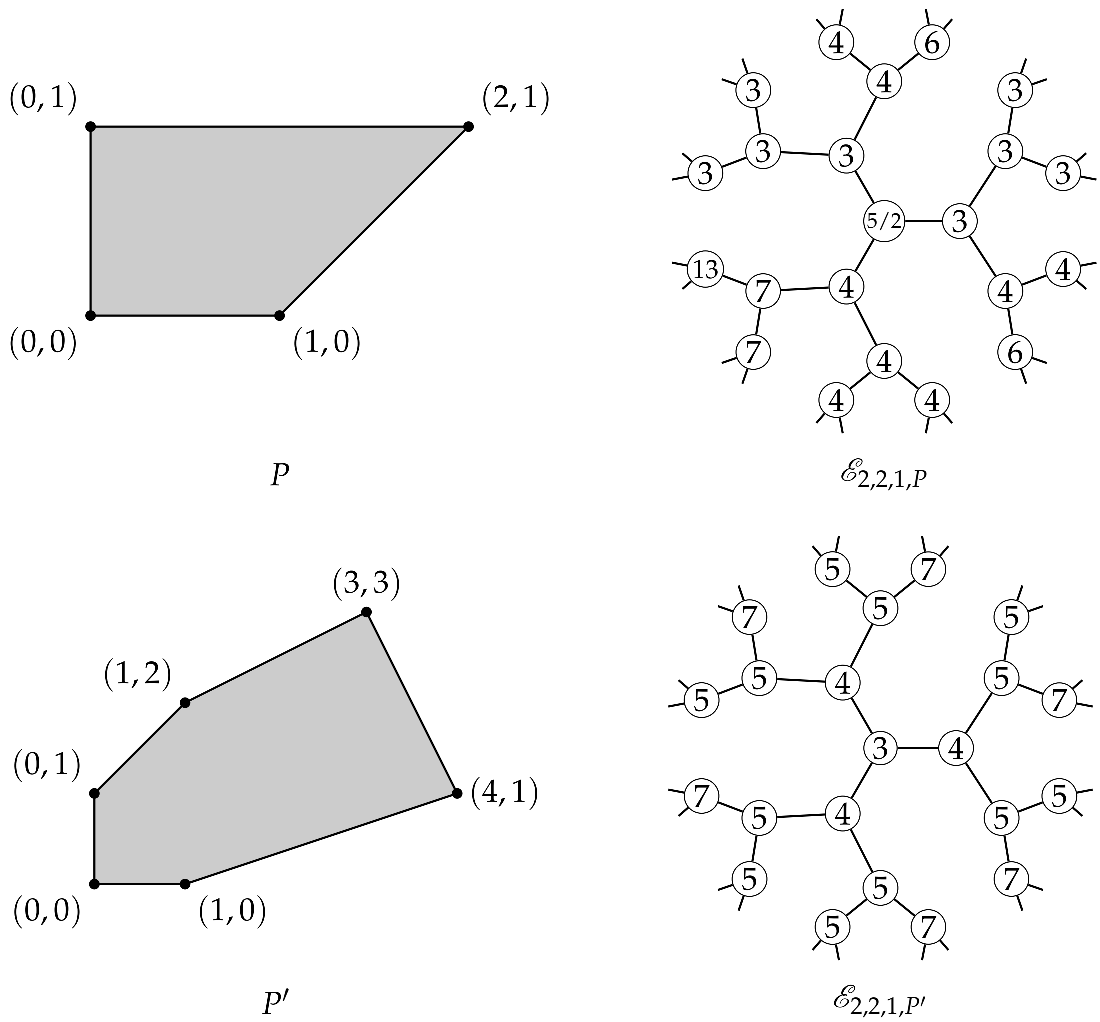

Figure 3.1: Polytopes and some values of $\mathscr{E}_{2,2,1,P}$ displayed on lattices in the affine building of type $\widetilde{\mathsf{A}}_1$ associated with the group $\mathrm{GSp}_2(\mathbb{Q}_p) \cong \mathrm{GL}_2(\mathbb{Q}_p)$. The center vertex corresponds to the homothety class of the identity, and the values are the linear coefficients of the Ehrhart polynomials with respect to the corresponding lattices.

References

[1] A. N. Andrianov. "Rationality theorems for Hecke series and Zeta functions of the groups GL_n and Sp_n over local fields". *Izv. Akad. Nauk SSSR Ser. Mat.* (1969), pp. 466–505.

[2] A. N. Andrianov. *Quadratic forms and Hecke operators*. Vol. 286. Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 1987, pp. xii+374. DOI.

[3] P. E. Gunnells and F. Rodriguez Villegas. "Lattice polytopes, Hecke operators, and the Ehrhart polynomial". *Selecta Math. (N.S.)* **13**.2 (2007), pp. 253–276.

[4] I. G. Macdonald. *Symmetric functions and Hall polynomials*. Second. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 1995, pp. x+475.

[5] T. K. Petersen. *Eulerian numbers*. Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser/Springer, New York, 2015, pp. xviii+456. DOI.

[6] G. Shimura. "On modular correspondences for $Sp(n, Z)$ and their congruence relations". *Proc. Nat. Acad. Sci. U.S.A.* **49** (1963), pp. 824–828. DOI.

[7] T. Tamagawa. "On the ζ-functions of a division algebra". *Ann. of Math. (2)* **77** (1963), pp. 387–405. DOI.

[8] The Sage Developers. *SageMath, the Sage Mathematics Software System (Version 9.2)*. `https://www.sagemath.org`. 2020.

[9] K. Vankov. "Explicit Hecke series for symplectic group of genus 4". *J. Théor. Nombres Bordeaux* **23**.1 (2011), pp. 279–298.

[10] C. Voll. "Functional equations for zeta functions of groups and rings". *Ann. of Math. (2)* **172**.2 (2010), pp. 1181–1218. DOI.