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Skein relations for punctured surfaces

Esther Banaian*[1], Wonwoo Kang†[2], and Elizabeth Kelley‡[2]

[1] *Institute for Mathematics, Aarhus University*
[2] *Department of Mathematics, University of Illinois Urbana Champaign*

Abstract. We use a combinatorial expansion formula for cluster algebras of surface type via order ideals of posets to give explicit skein relations for elements of a cluster algebra arising from a punctured surfaces. An immediate corollary of this is that the bangles and bracelets of Musiker, Schiffler, and Williams, which are known to provide a basis in the unpunctured case, form a spanning set in the punctured case.

Keywords: skein relation, triangulated surfaces, cluster algebras

1 Introduction

Subsequent to the original introduction of cluster algebras by Fomin and Zelevinsky in 2002 [5], a significant amount of effort has been devoted to studying cluster algebras of *surface type*, as defined in [3, 4]. Such cluster algebras are particularly appealing objects of study because they admit constructions of a variety of combinatorial objects - including snake graphs, T-paths, and posets - that can be used to prove important structural results about positivity or the existence of bases. In this extended abstract, we use a cluster expansion formula from [11, 13] which expresses elements of a cluster algebra as generating functions of order ideals of certain posets. We use this expansion formula to prove *skein relations*, i.e. relations used to resolve intersections or incompatibilities of arcs. Topologically, a skein relation takes a pair of intersecting arcs or an arc with self-intersection and replaces this configuration with two sets of arcs which avoid the intersection in two different ways. This method gives a generalization and new perspective to *snake graph calculus*, as defined in [2]. Skein relations for unpunctured surfaces were given in [10, 1]. Skein relations on punctured surfaces in the coefficient-free case were discussed in [7] and specific forms of skein relations in the principal coefficient case (so called "tidy exchange relations") were given in [13]. Here, we give explicit formulae and show all skein relations on (potentially punctured) surfaces contain a term that is not divisible by any coefficient variable y_i. Consequently, we observe that the *bangles* and *bracelets* defined in [9] form spanning sets and are linearly independent.

*banaian@math.au.dk. Supported by Independent Research Fund Denmark (1026-00050B).
†wonwook2@illinois.edu
‡kelleye@illinois.edu. Supported by DMS-1937241.

2 Background

Cluster algebras are a type of recursively generated commutative ring with distinguished generators, called *cluster variables*, that appear in fixed-size subsets $\mathbf{x} = (x_1, \ldots, x_n)$ called *clusters*. Each cluster $\mathbf{x}$ has an associated set of *coefficients* $\mathbf{y} = (y_1, \ldots, y_n)$. Clusters can be obtained from each other via an involutive process called *mutation*. A single mutation μ_k uniquely exchanges a cluster variable $x_k \in \mathbf{x}$ for some $x'_k \notin \mathbf{x}$. The relation between x_k and x'_k is referred to as an *exchange relation*. Given a cluster $\mathbf{x}$, it is always possible to mutate at every $x_i \in \mathbf{x}$. A single cluster is sufficient to generate the entire cluster algebra.

Two of the most celebrated properties of cluster algebras are the *Laurent phenomenon* and *positivity*, which together state that every cluster algebra element can be written as a Laurent polynomial with positive integer coefficients in terms of any choice of cluster.

Triangulated surfaces provide a well-known geometric model for *ordinary cluster algebras of surface type* [3, 4]. Let S be a surface with (potentially empty) boundary and a non-empty set of *marked points* M, where there is at least one marked point on each boundary component. Marked points in the interior of S are referred to as *punctures*. Every such marked surface (S, M) has an associated cluster algebra $\mathcal{A}_S$. Clusters of $\mathcal{A}_S$ correspond to distinct triangulations of (S, M), with individual cluster variables corresponding to individual arcs (i.e., curves with endpoints in M and no self-intersections). Coefficients correspond to *laminations* [4], i.e. additional collections of curves on (S, M) that meet certain conditions. Following the restrictions in [8], we do not allow (S, M) to be a closed surface with exactly two punctures, a monogon with less than two punctures, an unpunctured bigon or triangle, or a sphere with less than four punctures.

In the surface model, mutation at x_k is represented by *flipping* the corresponding arc γ in a triangulation T - that is, by replacing γ with a different arc γ', which corresponds to x'_k, such that $T - \{\gamma\} \cup \{\gamma'\}$ is still a valid triangulation. To provide complete geometric models for cluster algebras from punctured surfaces [3] introduced the more general notion of *tagged arcs*. A *tagged arc* is an arc whose ends have been tagged either *plain* or *notched* such that: the arc does not cut out a once-punctured monogon, any end on ∂S is tagged plain, and both ends of a loop have the same tagging.

If η is a tagged arc with endpoints p and q, we write η^0 to denote the underlying plain arc. If we wish to emphasize the notching of η, we will write $\eta^{(p)}$ when η has a single notched end at p and $\eta^{(pq)}$ when η is notched at both endpoints. Two tagged arcs α and β are *compatible* if and only if the following properties hold: the isotopy classes of α^0 and β^0 contain non-intersecting representatives; if $\alpha^0 = \beta^0$ then at least one end of α has the same tagging as the corresponding end of β; and if $\alpha^0 \neq \beta^0$ have a shared endpoint, then α and β must have the same tagging at that endpoint. A *tagged triangulation* is a maximal collection of pairwise compatible tagged arcs. We will work with clusters associated to triangulations with only plain arcs.

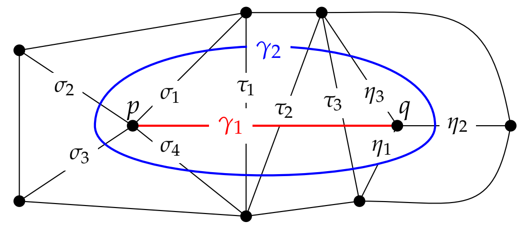


Figure 1: An example of an arc γ_1 and closed curve γ_2 on a triangulated surface.

3 Cluster expansion formula

3.1 The poset for an arc

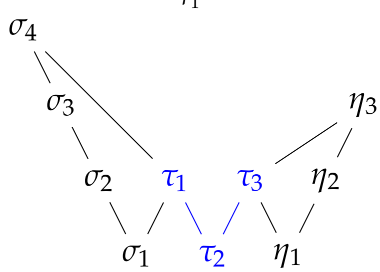
Let $T = \{\tau_1, \ldots, \tau_n\}$ be a triangulation of a surface (S, M). For any arc γ on (S, M), we construct a corresponding poset P_γ, following [11, 13]. We note that the posets P_γ will be exactly the poset of join-irreducibles in the lattice of perfect matchings of the snake graph $\mathcal{G}_\gamma$, as in [8, 14].

First, suppose that γ is an arc with both endpoints tagged plain. Fix an orientation for γ and let $\tau_{i_1}, \ldots \tau_{i_d}$ be the list of arcs of T crossed by γ, in the order determined by our choice of orientation of γ. We will place a poset structure on $[d]$ in the following way. Any two consecutive arcs crossed by γ, τ_{i_j} and $\tau_{i_{j+1}}$, border a triangle Δ_j that γ passes through between these crossings. Let s_j denote the shared endpoint of τ_{i_j} and $\tau_{i_{j+1}}$ which is a vertex of Δ_j. If s_j lies to the right of γ (with respect to the orientation placed on γ), then we set $j \gtrdot j+1$. Otherwise, we set $j \lessdot j+1$. The resulting poset is sometimes referred to as a *fence poset* since its Hasse diagram is a path graph. The process is the same if γ is a *generalized arc*, so that it has self-intersections.

Next, suppose that $\gamma^{(p)}$ is notched at its starting point $s(\gamma) = p$. Begin by drawing the fence poset for γ^0. Suppose the first triangle γ passes through is Δ_0. Necessarily, Δ_0 is bordered by τ_{i_1} and two spokes at p. Label these spokes σ_1, σ_m where σ_1 is the clockwise neighbor of τ_{i_1}. Label the remaining spokes in clockwise order. Then, we include elements $1^s, \ldots, m^s$ in the poset, and set $m^s \lessdot (m-1)^s \lessdot \cdots \lessdot 1^s$, $1^s \gtrdot 1$ and $m^s \lessdot 1$. If we have an arc which is instead notched at its terminal point, we repeat this process with elements $1^t, \ldots, m^t$, and we combine these processes for an arc tagged at both endpoints. We call the resulting posets *loop fence posets* as they correspond to the loop graphs given by Wilson in [14]. We say that the elements $1^s, \ldots, m^s$ are in a *loop*. If we wish to refer to a loop fence poset P with the loop portion removed, we will denote this P^0, so that $P_{\gamma^0} = P^0_\gamma$.

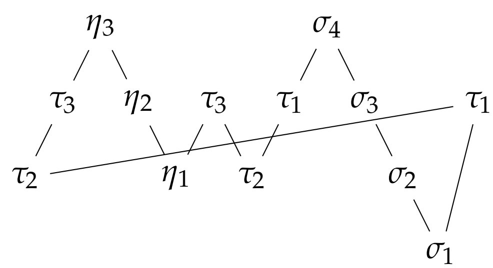
Finally, suppose that γ is a closed curve. Choose an point a of γ which is not a point of intersection between γ and T. Treat γ like an arc with $s(\gamma) = t(\gamma) = a$, choose an orientation of γ, and form the fence poset on $[d]$ associated to this arc. It must be that τ_{i_1} and τ_{i_d} share an endpoint which is an endpoint of the triangle containing a. If this

$P_{\gamma_1^{(pq)}}$



$$\mathbf{a}_{\gamma_1^{(pq)}} = \mathbf{e}_{\sigma_1} + \mathbf{e}_{\tau_2} + \mathbf{e}_{\eta_1}$$
$$\mathbf{b}_{\gamma_1^{(pq)}} = \mathbf{e}_{\tau_1} + \mathbf{e}_{\tau_3}$$
$$\mathbf{r}_{\gamma_1^{(pq)}} = 0$$

P_{γ_2}



$$\mathbf{a}_{\gamma_2} = \mathbf{e}_{\eta_1} + \mathbf{e}_{\tau_2} + \mathbf{e}_{\sigma_1}$$
$$\mathbf{b}_{\gamma_2} = \mathbf{e}_{\eta_3} + \mathbf{e}_{\tau_3} + \mathbf{e}_{\sigma_4}$$
$$\mathbf{r}_{\gamma_2} = 0$$

Table 1: The loop fence poset $P_{\gamma_1^{(pq)}}$ and circular fence poset P_{γ_2} for the arcs from Figure 1. Note that the fence poset $P_{\gamma_1} = P^0_{\gamma_1^{(pq)}}$ for the plain arc γ_1 appears as a subposet of $P_{\gamma_1^{(pq)}}$, indicated in blue, and has $\mathbf{a}_{\gamma_1} = \mathbf{e}_{\tau_2}$, $\mathbf{b}_{\gamma_1} = 0$, and $\mathbf{r}_{\gamma_1} = \mathbf{e}_{\sigma_1} + \mathbf{e}_{\eta_1}$.

endpoint is to the right of γ with the chosen orientation, we set $d \gtrdot 1$; otherwise we set $d \lessdot 1$. These posets are called *circular fence posets* since the underlying graph of such a Hasse diagram is a cycle. To improve readability, we will often refer to all of these types of posets as fence posets unless the specific type is relevant, in which case we use the specific term. See Table 1 for several examples; note here and for the remainder of the paper, we label the poset elements with the arcs they correspond to and we conflate these two notions when context is clear.

3.2 Minimal Terms

Let $\mathbf{a}_\gamma = (a_1, \ldots, a_n)$ where a_j is the number of times there is a minimal element $\tau_{i_k} \in P_\gamma$ such that $\tau_{i_k} = \tau_j$. Let $\mathbf{b}_\gamma = (b_1, \ldots, b_n)$ where b_j is the number of times there is an element $\tau_{i_k} \in P_\gamma$ which covers at least two elements and is not in a loop such that $\tau_{i_k} = \tau_j$. Note that one or both of the elements which τ_{i_k} covers can be in a loop.

Suppose γ is an plain arc and there exists $\tau_i, \tau_j \in T$ such that τ_i follows τ_1 in clockwise order in Δ_0, the first triangle γ passes through, and similarly τ_j follows τ_d in clockwise order in Δ_d, the last triangle γ passes through. Then we set $\mathbf{r}_\gamma = \mathbf{e}_i + \mathbf{e}_j$ where $\mathbf{e}_i$ is the i-th standard basis vector in $\mathbb{R}^n$. If γ is instead notched at an endpoint or the clockwise neighbor of τ_1 or τ_d is on the boundary of (S, M), then we omit its contribution.

Given any arc or closed curve γ, we define $\mathbf{g}_\gamma := -\mathbf{a}_\gamma + \mathbf{b}_\gamma + \mathbf{r}_\gamma$. We remark that this notation is inspired by the notation for the g-vector of a string module, as in [12].

Geiß, Labardini-Fragoso, and Schröer studied these **g**-vectors for plain arcs and closed curves in [6]. In particular, using Proposition 10.14 and Remark 11.1, they showed that $\mathbf{x}^{\mathbf{g}_\gamma}$ is the unique term in x_γ^T which is not divisible by any variable y_i. We show the same statement for a notched arc γ.

Lemma 1. *Let T be a triangulation of a surface without self-folded triangles. The monomial $\mathbf{x}^{\mathbf{g}_\gamma}$ is the unique term in the expansion of x_γ^T which is not divisible by any variable y_i.*

Given an arc $\tau_i \in T$, let $x_{CCW}(\tau_i) = x_{\tau_j} x_{\tau_k}$ if there are two arcs $\tau_j, \tau_k \in T$ that are counterclockwise neighbors of τ_i within the two triangles that it borders. If one or both of those neighbors is a boundary arc, then we ignore its contribution. The monomial $x_{CW}(\tau_i)$ is defined analogously using the clockwise neighbors of τ_i. We set $\widehat{y}_{\tau_i} := (x_{CCW}(\tau_i)/x_{CW}(\tau_i))\, y_{\tau_i}$. Let $J(P)$ denote the poset of lower order ideals of a poset P_γ. Each $I \in J(P)$ has an associated *weight* $w(I) = \prod_{j \in I} \widehat{y}_{\tau_{i_j}}$.

Proposition 1. *Let γ be an arc or closed curve on a marked surface (S, M) with triangulation T such that $\gamma \notin T$. Then, the associated element x_γ of the cluster algebra $\mathcal{A}(S, M)$ written with respect to the cluster corresponding to T can be expressed by*

$$x_\gamma^T = \mathbf{x}^{\mathbf{g}_\gamma} \sum_{I \in J(P_\gamma)} w(I).$$

Proof. If γ is not an arc such that $\gamma^0 \in T$, then this follows from combining Proposition 3.2 in [11] with Lemma 1. If $\gamma \neq \gamma^0$ and $\gamma^0 \in T$, we prove this expansion formula by using the algebraic identities that relate a singly-notched arc to plain arc and Theorem 12.9 in [8], which relates a doubly-notched arc to plain and singly-notched arcs. □

Example 1. Applying Proposition 1 to the arc $\gamma_1^{(pq)}$ from Table 1 produces

$$x_{\gamma_1^{(pq)}} = \frac{x_{\tau_1} x_{\tau_3}}{x_{\sigma_4} x_{\tau_2} x_{\eta_3}} \left[\frac{x_{\sigma_3} y_{\sigma_4} y_{\tau_2}}{x_{\sigma_1} x_{\tau_1}^2 x_{\tau_3}} + \frac{x_{\eta_1} y_{\eta_3}}{x_{\tau_1} x_{\tau_3}^2 x_{\eta_2}} + \frac{x_{\sigma_3} x_{\eta_1} y_{\sigma_4} y_{\eta_3}}{x_{\tau_1} x_{\sigma_1} x_{\tau_3} x_{\eta_2}} + \frac{y_{\eta_2} y_{\eta_3}}{x_{\tau_3}} + \frac{x_{\sigma_2} x_{\sigma_3} y_{\sigma_3}}{x_{\sigma_1} x_{\sigma_4}} + \cdots \right]$$

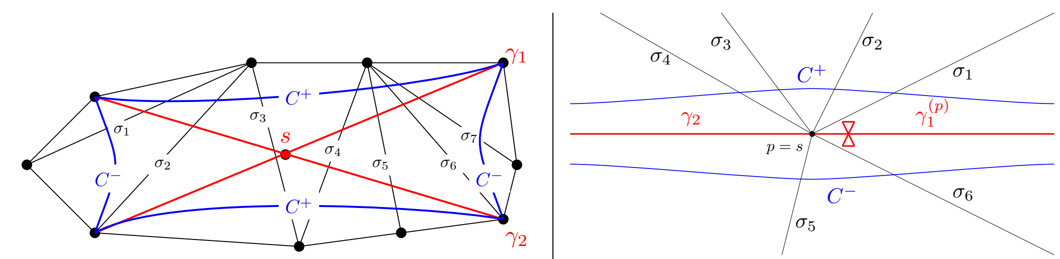
where we have explicitly shown only the terms arising from order ideals of size two.

4 Skein Relations

Let γ_1 and γ_2 be two curves with a point of incompatibility s; by this, we mean that either γ_1 and γ_2 intersect, or $\gamma_1^0 \neq \gamma_2^0$ share an endpoint and have opposite taggings at the endpoint. In some cases, γ_1 and γ_2 cross the same set of arcs before or after passing through s; if s is an intersection point, as we vary the representatives of γ_1 and γ_2 in their isotopy classes, the point s can lie on any of these arcs. We call such a configuration a *crossing overlap*. When the set-up is understood, we refer to this set of commonly crossed arcs as R.

If two arcs cross and this point of intersection is near the endpoint of one arc, then when we form some of the arcs in the resolution, these will *pivot* at this endpoint. For example, in the left diagram in Table 2, the left arc C^- pivots across σ_2 in counterclockwise direction and the right arc C^- pivots across σ_6 in clockwise direction. Some of these pivots will also affect the y-monomial in the resolution. For a pair of crossing curves, we define the *sweep set*, denoted Sw, to be the set of arcs that an arc in the resolution pivots past in in clockwise (resp counterclockwise) direction at a plain (resp notched) endpoint. Now suppose we instead have two arcs with incompatible taggings at a puncture p. Suppose $\gamma_1^{(p)}$ is tagged at p and γ_2 is not. Then, we define the sweep set to be the set of arcs from T which lie counterclockwise of γ_1 and clockwise of γ_2. See Table 2 for examples.

Given two arcs with an incompatibility and associated sets $R \cup Sw$, one can show that one of the sets of arcs in the resolution at the incompatibility will not cross any of the arcs in $R \cup Sw$. We will label the sets of arcs (called *multicurves*) in the resolution as C^+ and C^- where C^- is the set which does not cross any arcs in R and Sw.



$R = \{\sigma_3, \sigma_4, \sigma_5\}$	$R = \emptyset$
$Sw = \{\sigma_6\}$	$Sw = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$

Table 2: Examples of R and Sw for a transverse crossing (left) and an incompatibility at a puncture (right).

Theorem 1. *1. Let $\{\gamma_1, \gamma_2\}$ be a multicurve of arcs or closed curves which are incompatible. Choose one point of incompatibility and let C^+ and C^- be the resolution at this intersection. Then, $x_{\gamma_1}x_{\gamma_2} = x_{C^+} + Y_R Y_{Sw} x_{C^-}$.*

2. Let γ_1 be an arc or closed curve which is incompatible with itself. Choose one point of incompatibility and let C^+ and C^- be the resolution at this intersection. Then, $x_{\gamma_1} = x_{C^+} + Y_R Y_{Sw} x_{C^-}$.

We prove Theorem 1 in cases. In section 4.1, we will explain our proof method which can be used for all cases. In Sections 4.2 and 4.3, respectively, we will explicitly prove this

Theorem for a pair of arcs with incompatible taggings at a puncture and a pair of arcs with a transverse crossing. The relations that we explicitly discuss in these sections are helpful in unifying some of the cases outlined in [9]. For example, the relation discussed in 4.2 can be used to handle cases 6-11 from [9]. For the sake of brevity, we only include explicit proofs for these two examples.

4.1 General approach

Let γ_1 and γ_2 be two curves with a point of incompatibility and resolutions C^+ and C^-. Set $P_i := P_{\gamma_i}$ and $\mathbf{g}_i := \mathbf{g}_{\gamma_i}$. In light of Proposition 1, we can write $x_{\gamma_1}x_{\gamma_2}$ as

$$\mathbf{x}^{\mathbf{g}_1+\mathbf{g}_2} \sum_{(I_1,I_2)\in J(P_1)\times J(P_2)} w(I_1)w(I_2).$$

We set $w(I_1, I_2)$ to be the product of the weights of the components $w(I_1)w(I_2)$. If $C^+ = \{\gamma_3, \gamma_4\}$, then we set $J(C^+) = J(P_3) \times J(P_4)$; otherwise, C^+ is a singleton $\{\gamma_3\}$ and we set $J(C^+) = J(P_3)$. We define $J(C^-)$ similarly. Our method of proof centers on finding a partition of $J(\mathcal{P}_1) \times J(\mathcal{P}_2) = A \sqcup B$ such that $(\varnothing,\varnothing) \in A$, and bijections Φ_A between A and $J(C^+)$ and Φ_B between B and $J(C^-)$. Moreover, we require that the bijection between A and $J(C^+)$ is weight-preserving, so that $w(I_1)w(I_2) = w(\Phi_A(I_1, I_2))$ and that the bijection between B and $J(C^-)$ is weight preserving up to a unique monomial, so that for some monomial Z in x and y variables, $w(I_1)w(I_2) = Zw(\Phi_B(I_1, I_2))$. Let $Z = XY$ be the decomposition of Z into x and y variables. The final step of each proof is to show that $\mathbf{g}_1 + \mathbf{g}_2$ is equal to the sum of the $\mathbf{g}$-vectors for the posets in C^+ (denoted $\mathbf{g}_{C^+}$) and $\mathbf{g}_1 + \mathbf{g}_2 + \deg(X)$ is equal to the the sum of the $\mathbf{g}$-vectors for the posets in C^- (denoted $\mathbf{g}_{C^-}$) where $\deg(X) = (\deg_{x_{\tau_1}}(X), \ldots, \deg_{x_{\tau_n}}(X))$. Then, we can rewrite $x_{\gamma_1}x_{\gamma_2}$ as

$$\begin{aligned}
&= \mathbf{x}^{\mathbf{g}_1+\mathbf{g}_2} \sum_{(I_1,I_2)\in A} w(\Phi_A(I_1, I_2)) + \mathbf{x}^{\mathbf{g}_1+\mathbf{g}_2} \sum_{(I_1,I_2)\in B} Zw(\Phi_B(I_1, I_2)) \\
&= \mathbf{x}^{\mathbf{g}_{C^+}} \sum_{\boldsymbol{I}\in J(C^+)} w(\boldsymbol{I}) + \mathbf{x}^{\mathbf{g}_{C^-}} Y \sum_{\boldsymbol{I}\in J(C^-)} w(\boldsymbol{I}) = x_{C^+} + Yx_{C^-},
\end{aligned}$$

where x_{C^+} is the product of x variables associated to the arcs in C^+ In each example, Z will be a product of $\hat{y}$-variables that corresponds to the preimage of a tuple of emptysets in $J(C^-)$. For part (2) of Theorem 1, we have similar statements with just one poset I_1. When resolving a self-intersection, it is possible for one arc to have a contractible kink, in which case we remove the kink and multiply the associated expression by -1; in this case, the bijections are adjusted to account for the difference in sign.

4.2 Incompatibility at punctures

Consider two arcs, $\gamma_1^{(p)}$ and γ_2 which are incompatible at a puncture p as on the right hand side of Table 2. Recall from Section 2 that this means $\gamma_1^{(p)}$ and γ_2 have opposite taggings at p and $\gamma_1^0 \neq \gamma_2^0$. Orient $\gamma_1^{(p)}$ and γ_2 to both begin at p. Let the spokes at p from T be $\sigma_1, \dots, \sigma_m$, labeled in counterclockwise order such that the first triangle that $\gamma_1^{(p)}$ passes through is bounded by σ_1 and σ_m. If $\gamma_2 \notin T$, let $1 \leq k \leq m$ be such that the first triangle γ_2 passes through is bounded by σ_k and σ_{k+1}, where we interpret σ_{m+1} as σ_1. If $\gamma_2 \in T$, then we let k be such that $\gamma_2 = \sigma_k$.

Draw a small circle h that encompasses p and does not cross any arcs of T except the spokes at p. We define $\gamma_1^{-1} \circ_{CCW} \gamma_2$ as the arc which results from following γ_1 from $t(\gamma_1)$ with reverse orientation until its intersection with h, following h counterclockwise until its intersection with γ_2, and then following γ_2 until $t(\gamma_2)$. We define $\gamma_1^{-1} \circ_{CW} \gamma_2$ similarly. Set $\gamma_3 := \gamma_1^{-1} \circ_{CCW} \gamma_2$ and $\gamma_4 := \gamma_1^{-1} \circ_{CW} \gamma_2$ and note that γ_3 crosses $\sigma_1, \dots, \sigma_k$ and γ_4 crosses $\sigma_{k+1}, \dots, \sigma_m$. On the right hand side of Table 2, $k = 4$, γ_3 is the arc denoted C^+ and γ_4 is the arc denoted C^-.

When $k = m$ and $\gamma_2 \notin T$, so that the first triangles $\gamma_1^{(p)}$ and γ_2 pass through are the same, then we have two additional cases based on whether $\gamma_1^{(p)}$ is clockwise or counterclockwise of γ_2 at p. Since these cases produce different sets Sw, we differentiate them. We refer to the case where $\gamma_1^{(p)}$ lies clockwise from γ_2 as the $k = 0$ case.

It is only in the $k = 0$ and $k = m$ cases when $\gamma_2 \notin T$ that we will have a crossing overlap. If $\tau_{i_1}, \dots, \tau_{i_{d_1}}$ and $\tau_{j_1}, \dots, \tau_{j_{d_2}}$ are the ordered sequences of arcs from T crossed by γ_1 and γ_2 respectively, and $w \geq 1$ is the largest number such that $\tau_{i_r} = \tau_{j_r}$ for all $1 \leq r \leq w$, then $R = \{\tau_{i_1}, \dots, \tau_{i_w}\}$, regarded as a multiset. When $\gamma_2 \in T$, then there is no possible case for $k = 0$, and $R = \emptyset$ in the $k = m$ case.

Proposition 2. *Let $\gamma_1^{(p)}$ and γ_2 be arcs which are incompatible at a puncture p. For k and R as defined above, set*

$$Y_R = \prod_{\tau \in R} y_\tau \quad \text{and} \quad Y_{Sw} = \prod_{\sigma_i \in Sw} y_{\sigma_i} = \prod_{i=1}^{k} y_{\sigma_i}.$$

Then, we have $x_{\gamma_1^{(p)}} x_{\gamma_2} = C^+ + C^-$ where C^+ and C^- are defined as follows:

	C^+	C^-
$k \neq 0, m$	x_{γ_3}	$Y_{Sw} x_{\gamma_4}$
$k = 0$	x_{γ_4}	$Y_R x_{\gamma_3}$
$k = m$	x_{γ_3}	$Y_{Sw} Y_R x_{\gamma_4}$

Proof. We detail the $k \neq 0, m$ and $\gamma_2 \notin T$ case; the special cases follow from various modifications to these overarching ideas. The posets $P_{\gamma_1^{(p)}}, P_{\gamma_3}$ and P_{γ_4} are provided in Table 3; we suppress the poset P_{γ_2} as its structure is not important for the proof.

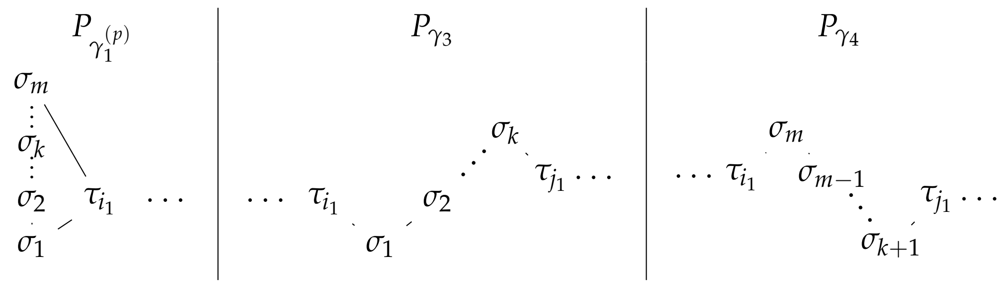


Table 3: Posets for a resolution of an incompatibility for the $k \neq 0, m$ cases. Recall that τ_{i_1} is the first arc crossed by γ_1 and γ_2 is the first arc crossed by γ_2.

Let $A_1 \subseteq J(P_{\gamma_1^{(p)}}) \times J(P_{\gamma_2})$ consist of all pairs (I_1, I_2) such that $\sigma_k \notin I_1$ and let A_2 consist of all pairs such that $\sigma_k \in I_1, \sigma_{k+1} \notin I_1$, and $\tau_{j_1} \in I_2$. Let B be the complement of $A_1 \sqcup A_2$; in other words, B consists of pairs (I_1, I_2) such that $\tau_{j_1} \in I_2$ only if $\sigma_{k+1} \in I_1$.

It is clear that A_1 is in bijection with $\{I_3 \in J(P_3) : \sigma_k \notin I_3\}$ and A_2 is in bijection with $\{I_3 \in J(P_3) : \sigma_k \in I_3\}$, where this bijection sends each element to its image in P_{γ_3}. Similarly, we have a bijection $B \cong P_{\gamma_4}$ which sends $(I_1, I_2) \in B$ to $(I_1 \backslash \langle \sigma_k \rangle) \cup I_2$. The description of B ensures that this set is an order ideal so that this map is well-defined.

We now compare the g-vectors. Let $\delta_{\tau_{i_1} > \tau_{i_2}} = 1$ if τ_{i_2} exists and $\tau_{i_1} > \tau_{i_2}$. We have that $\mathbf{g}_{\gamma_1^{(p)}} = -\mathbf{e}_{\sigma_1} + \delta_{\tau_{i_1} > \tau_{i_2}} \mathbf{e}_{\tau_{i_1}} + \mathbf{g}_1'$ where $\mathbf{g}_1'$ involves contributions from τ_{i_ℓ} for $\ell > 1$. For simplicity, suppose $\tau_{j_1} < \tau_{j_2}$. Then, $\mathbf{g}_{\gamma_2} = \mathbf{e}_{\sigma_k} - \mathbf{e}_{\tau_{j_1}} + \mathbf{g}_2'$ for similarly defined $\mathbf{g}_2'$. We see immediately that $\mathbf{g}_{\gamma_3} = -\mathbf{e}_{\sigma_1} + \mathbf{e}_{\sigma_k} + \delta_{\tau_{i_1} > \tau_{i_2}} \mathbf{e}_{\tau_{i_1}} + \mathbf{g}_1' - \mathbf{e}_{\tau_{j_1}} + \mathbf{g}_2' = \mathbf{g}_1 + \mathbf{g}_2$. Now, we compute $\mathbf{g}_{\gamma_4} = -\mathbf{e}_{\sigma_{k+1}} + \mathbf{e}_{\sigma_m} - (1 - \delta_{\tau_{i_1} > \tau_{i_2}}) \mathbf{e}_{\tau_{i_1}} + \mathbf{g}_1' + \mathbf{g}_2'$, so that $\mathbf{g}_{\gamma_4} - (\mathbf{g}_{\gamma_1^{(p)}} + \mathbf{g}_{\gamma_2}) = \mathbf{e}_{\sigma_m} + \mathbf{e}_{\sigma_1} + \mathbf{e}_{\tau_{j_1}} - \mathbf{e}_{\sigma_k} - \mathbf{e}_{\sigma_{k+1}} - \mathbf{e}_{\tau_{i_1}}$. Let $\sigma_{[i]}$ denote the third arc in the triangle formed by σ_i and σ_{i+1}. Then, from the definition, we have $\hat{y}_{\sigma_i} = y_{\sigma_i} \frac{x_{\sigma_{i-1}} x_{\sigma_{[i]}}}{x_{\sigma_{i+1}} x_{\sigma_{[i-1]}}}$. One can see that $\hat{y}_{\sigma_1} \cdots \hat{y}_{\sigma_k} = (y_{\sigma_1} \cdots y_{\sigma_k}) \frac{x_{\sigma_m} x_{\sigma_1} x_{\sigma_{[k]}}}{x_{\sigma_k} x_{\sigma_{k+1}} x_{\sigma_{[0]}}}$, and the claim follows after noting that $\sigma_{[k]} = \tau_{j_1}$ and $\sigma_{[0]} = \sigma_{[m]} = \tau_{i_1}$. One can repeat similar calculations if $\tau_{j_1} > \tau_{j_2}$.

□

4.3 Transverse Crossings

Here, we consider two arcs, γ_1 and γ_2 that have a point of intersection. For brevity, here we will assume these arcs have a crossing overlap, so that $R \neq \emptyset$. If not, we have two more cases based on the fact that the point of intersection must occur in the first or last triangle of one or both of the arcs.

We orient γ_1 and γ_2 so that they pass through the arcs in R in the same direction. With our fixed point of intersection s, let $\gamma_1 \circ \gamma_2$ denote the arc given by following γ_1 along its orientation until s and then following γ_2. Let $\gamma_3 = \gamma_1 \circ \gamma_2, \gamma_4 = \gamma_2 \circ \gamma_1, \gamma_5 = \gamma_1 \circ \gamma_2^{-1}$, and $\gamma_6 = \gamma_2^{-1} \circ \gamma_1$, where -1 denotes using the reverse orientation. Note that γ_3 and γ_4 both pass through R, though they do not have a crossing overlap here, while γ_5 and γ_6 avoid the intersections with the arcs in R. Therefore, $C^+ = \{\gamma_3, \gamma_4\}$ and $C^- = \{\gamma_5, \gamma_6\}$.

Proposition 3. *Let γ_1 and γ_2 be two arcs which intersect in a crossing overlap R. Let the resolution be $\{\gamma_3, \gamma_4\} \cup \{\gamma_5, \gamma_6\}$. Then,*

$$x_{\gamma_1} x_{\gamma_2} = x_{\gamma_3} x_{\gamma_4} + Y_R Y_{Sw} x_{\gamma_5} x_{\gamma_6}$$

where $Y_R = \prod_{\tau \in R} y_\tau$ and $Y_{Sw} = \prod_{\tau \in Sw} y_\tau$.

In the proof, we will use a poset-theoretic version of a tool from [1]. Let posets P_1 and P_2 have a crossing overlap in a region R. Index the elements in $P_1 \cap R$ as $P_1(1), \ldots, P_1(m)$ such that $P_1(i)$ only has cover relations with $P_1(i-1)$ and $P_1(i+1)$, when these exist, and index the elements in $P_2 \cap R$ analogously such that $P_1(i)$ and $P_2(i)$ are equivalent for each i. Given $I_1 \in J(P_1)$ and $I_2 \in J(P_2)$, let the *switching position* be the smallest value j such that $P_1(j) \in I_1$ if and only if $P_2(j) \in I_2$. One can show that a switching position exists unless $R \subseteq I_1$ and $R \cap I_2 = \emptyset$ or vice versa.

Proof. We say that a subset R of a poset P is on *top* if there is no $j \in P \backslash R$ such that j is larger than an element in R and define a subset being on *bottom* similarly. One can show that, when γ_1 and γ_2 have a crossing overlap, up to relabeling, R_1 is on top and R_2 is on bottom. In the following, suppose that γ_1 crosses arcs $\alpha_1, \ldots, \alpha_{d_1}$ in T and γ_2 crosses $\eta_1, \ldots, \eta_{d_2}$. We assume that these arcs have a crossing overlap in regions $R_1 \subseteq P_1$ and $R_2 \subseteq P_2$. Let $1 \leq s \leq t \leq d_1$ and $1 \leq s' \leq t' \leq d_2$ be such that $R_1 = \{\alpha_s, \ldots, \alpha_t\}$ and $R_2 = \{\eta_{s'}, \ldots, \eta_{t'}\}$.

We focus on one case which includes a nonempty set Sw as an illustrative proof. We will omit discussion of **g**-vectors as the previous proof already illustrated all relevant ideas. Suppose $s' = 1$ and $s(\gamma_2)$ is notched. It must be that $s > 1$ in order for γ_1 and γ_2 to have an intersection. Necessarily, the arc α_{s-1} is a spoke incident to the puncture $s(\gamma_2)$. Index this set of spokes as $\sigma_1, \ldots, \sigma_m$ in counterclockwise order such that $\alpha_{s-1} = \sigma_1$. Suppose that γ_1 crosses $\sigma_1, \ldots, \sigma_k$ and let β be the arc which γ_1 crosses right before crossing σ_k, if it exists. We will assume $t < d_1$ and $t' < d_2$; we can repeat these arguments two times if we also have one of these cases. Table 4 provides the posets P_1, P_2, P_5, and P_6. If β does not exist, then P_5 is the chain between σ_{k+1} and σ_{k-1}, with order as in the Table. The poset P_3 is obtained by taking P_1 and replacing $R_1 > \alpha_{t+1}$ with $R_3 < \eta_{t'+1}$ and P_4 is obtained dually from P_2.

We set A to be the union of pairs (I_1, I_2) such that one of the following holds: (1) there is a switching position between R_1 and R_2, (2) $R_1 \subseteq I_1$ and $R_2 \cap I_2 = \emptyset$, (3) $R_2 \subseteq I_2, R_1 \cap I_1 = \emptyset, \alpha_{t+1} \in I_1$ and $\eta_{t'+1} \notin I_2$, or (4) $R_2 \subseteq I_2, R_1 \cap I_1 = \emptyset, \alpha_{t+1} \notin I_1$, $\sigma_k \in I_2$ only if $\beta \in I_1$ and $\sigma_{k+1} \notin I_2$ if the highest element σ_m is in I_1. If β does not exist,

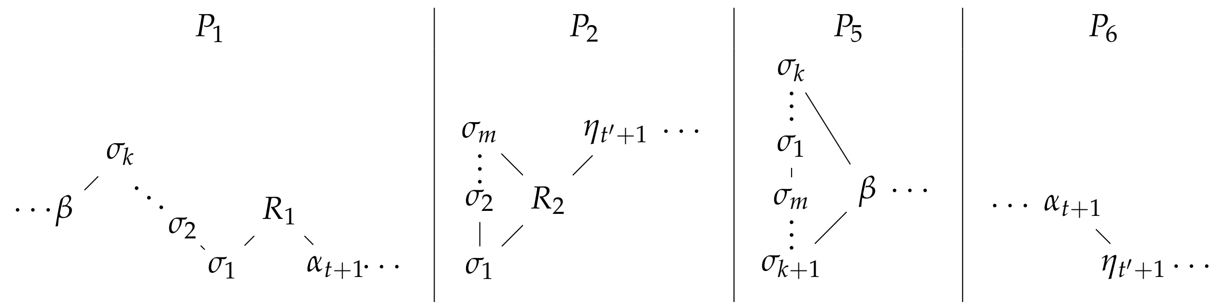


Table 4: Some of the posets for a resolution of a transverse crossing between γ_1 and γ_2

the condition involving β is removed. We define Φ_A as follows. If (I_1, I_2) has a switching position, which is j in R_1 and j' in R_2, then we set $\Phi_A(I_1, I_2) = (I_3, I_4)$ where I_3 is the result of taking all elements of I_1 up to α_j and all elements of I_2 after $\eta_{j'}$ and I_4 is the result of taking all elements of I_2 up to $\eta_{j'}$ and all elements of I_1 after α_j. Since $\alpha_j \in I_1$ if and only if $\eta_{j'} \in I_2$, these form order ideals. If a pair (I_1, I_2) is from item (2) we send R_1 to R_3, if from item (3) we send R_2 to R_4, and if from item (4) we send R_2 to R_3. Some of the elements σ_i do not have one clear image in $P_3 \times P_4$, so care is taken in these latter items to send them to appropriate places so that the resulting sets are still order ideals.

We let B be the complement of A in $J(P_1) \times J(P_2)$; explicitly, B is the set of tuples such that $R_1 \cap I_1 = \emptyset$, $R_2 \cup \langle \sigma_k \rangle \subseteq I_2$, $\alpha_{t+1} \in I_1$ only if $\eta_{t'+1} \in I_2$, and $\beta \in I_1$ only if $\sigma_{k+1} \in I_2$. Our definition of B implies that the restrictions of $I_1 \sqcup (I_2 \backslash (R_2 \cup \langle \sigma_k \rangle))$ to P_5 and P_6 are order ideals. This defines our bijection Φ_B. □

5 Implications

In [9], given a surface (S, M), Musiker, Schiffler, and Williams define two sets of arcs, bangles $\mathcal{C}^\circ$ and bracelets $\mathcal{C}$, and show that the set of elements of $\mathcal{A}_S$ arising from each ($\mathcal{B}^\circ$ and $\mathcal{B}$ respectively) forms a basis of $\mathcal{A}_S$. They leave as a question whether these sets could also give the basis of $\mathcal{A}_S$ when (S, M) has punctures; the lack of skein relations in the punctured setting is a large reason why they did not extend their basis to this case.

Our skein relations show that a product $x_{\gamma_1} x_{\gamma_2}$ of incompatible arcs can be written in terms of $\mathcal{B}^\circ$ and of $\mathcal{B}$, which shows that these sets are still spanning in the punctured case. Moreover, because our relations are always of the form $x_{\gamma_1} x_{\gamma_2} = x_{C^+} + Y x_{C^-}$, we know that Lemma 6.3 from [9] remains true. As explained in Section 8.5 of the same article, this will show that these sets are also linearly-independent.

Lemma 2. *Let γ_1 and γ_2 be multicurves with at least one point of incompatibility on (S, M). Then the expansion*

$$x_{\gamma_1} x_{\gamma_2} = \sum_i Y_i M_i,$$

where $M_i \in \mathcal{B}^\circ$ and the Y_i represent monomials in the coefficient variables, has a unique index j such that $Y_j = 1$.

As future work, it remains for us to verify that $\mathcal{B}$ and $\mathcal{B}^\circ$ are still subsets of $\mathcal{A}_S$. Although we expect this to be true, it is non-trivial to prove and will, as a consequence, complete the proof that $\mathcal{B}$ and $\mathcal{B}^0$ remain bases in the punctured setting.

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