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A Consistent Sandpile Torsor Algorithm for Regular Matroids

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Abstract. Every regular matroid is associated with a *sandpile group*, which acts simply transitively on the set of bases in various ways. Ganguly and the second author introduced the notion of *consistency* to describe classes of actions that respect deletion-contraction in a precise sense, and proved the consistency of rotor-routing torsors (and uniqueness thereof) for plane graphs.

In this work, we prove that the class of actions introduced by Backman, Baker, and the fourth author, is consistent for regular matroids. This generalizes the above existence assertion, as well as makes progress on the goal of classifying all consistent actions.

Keywords: regular matroid, sandpile group, sandpile torsor, fourientation

1 Introduction

For over a century, mathematicians have been interested in enumerative properties of the *spanning trees* of graphs. A remarkable and relatively recent observation is that the set of spanning trees of a graph (and more generally, the *bases* of a *regular matroid*) admit interesting group actions, which bestow on these sets a group-like structure. We are curious about this mysterious algebraic structure, especially in cases where it is surprisingly canonical.

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To be more precise, the *sandpile group* (also called the critical group, Jacobian, etc.) $S(G)$ of a graph G is a finite abelian group whose size is equal to the number of spanning trees of G. The algebraic structure discussed in the previous paragraph is given by a *simply transitive action* of $S(G)$ on the spanning trees of G. Loosely speaking, we call such an action a *sandpile torsor*. To define sandpile torsors in a systematic way, it is necessary to work on graphs with some auxiliary data (see [19, Theorem 8.1]).

One possible setup is to work with *embedded* graphs (called *ribbon graphs* or *maps*). There are at least two known ways to associate each *rooted* embedded graph with a sandpile torsor: the *rotor-routing model* (see [13]) and the *Bernardi bijections* (see [5]). While these approaches have been shown to give different actions in general [8, 18], the situation changes dramatically when restricting to *plane graphs* (i.e., *planar* embedded graphs): both actions are independent of the root chosen ([7, Theorem 2], [5, Theorem 5.1]), and they in fact produce the same *sandpile torsor algorithm (on plane graphs)* [5, Theorem 7.1] (see [14] for an alternate definition of this action). This lead Klivans to conjecture that this algorithm was in some sense canonical, and all "nice" sandpile torsor algorithms on plane graphs must have the same structure [15, Conjecture 4.7.17]. This conjecture was made precise and proven by Ganguly and the second author [10].

The first challenge to resolve this conjecture was to give a suitable definition for a "nice" sandpile torsor algorithm. To do this, the authors introduce the notion of *consistency*. In general, sandpile groups do not behave well with respect to contraction and deletion: the additive relation $|S(G)| = |S(G \setminus e)| + |S(G/e)|$ implies that it is almost impossible to relate these groups directly in an algebraically meaningful way. Nevertheless, for specific combinations of group elements and spanning trees, there is a way to make sense of the contraction and deletion operations. A *consistent* sandpile torsor algorithm is essentially one which respects these operations.

A bit more precisely, fix a class of graphs G's (e.g., planar graphs) and a class of auxiliary structures α's for these graphs (e.g., planar embeddings) with a notion of deletion and contraction. Moreover, suppose both classes are minor closed. To any pair (G, α), a sandpile torsor algorithm associates a simply transitive action $\cdot$ of the sandpile group on the spanning trees. Note that the sandpile group is generated by equivalence classes corresponding to individual arcs (directed edges). On graphs, these arcs indicate a single chip on a vertex and a single negative chip on an adjacent vertex.

We say that the sandpile torsor algorithm is consistent if, given (G, α) that induces $\cdot$, and any arc f and spanning tree T of G such that $f \cdot T = T'$, we have:

- for any edge $e \notin T \cup T' \cup f$, the pair $(G \setminus e, \alpha \setminus e)$ induces $\cdot'$ with $f \cdot' T = T'$,
- for any edge $e \in T \cap T' \setminus f$, the pair $(G/e, \alpha/e)$ induces $\cdot''$ with $f \cdot'' (T \setminus e) = T' \setminus e$,
- the action of f does not modify the part of a spanning tree falling into a different biconnected component than f.

The two main results of [10] were proving the existence and the uniqueness of a consistent sandpile torsor algorithm on plane graphs.[1] The main theorem of our paper is generalizing the existence result to a regular matroid context, proving [10, Conjecture 6.11].

Building off work from Bacher, de la Harpe, and Nagnibeda [1], Merino defined the sandpile group of a regular matroid [16]. For regular matroids, *bases* play the role that spanning trees played for graphs. In particular, the regular matroid version of a sandpile torsor algorithm is a map from any regular matroid M (with some auxiliary data) to a simply transitive action of the sandpile group of M on the set of bases of M.

Using the auxiliary data of *acyclic circuit-cocircuit signatures*, Backman, Baker, and the fourth author defined a sandpile torsor algorithm for regular matroids which was motivated from polyhedral geometry [2]. We call this the *BBY algorithm*.[2] Later, the first author [9], and the fourth author with Backman and Santos [4] independently showed that the same definition works also for the broader class of *triangulating circuit-cocircuit signatures*. We will continue to refer to this more general setting as the BBY algorithm.

The notion of consistency can be defined analogously for matroids. Moreover, deletion and contraction can be defined for triangulating signatures. Our main result is the following. (For a more formal statement, see Theorem 2.22.)

Theorem 1.1. *The BBY algorithm (that associates a sandpile action to a regular matroid equipped with a triangulating circuit-cocircuit signature) is consistent.*

We note that since rotor-routing torsors on plane graphs are special cases of BBY torsors, this theorem also implies the "existence" part from [10], i.e., that the rotor-routing algorithm is consistent, see Section 4.2. We conjecture that a converse also holds, namely, that for the auxiliary structure of triangulating signatures, the BBY action is the unique consistent sandpile action. This is an modified version of [10, Conjecture 6.14] for triangulating signatures instead of acyclic signatures.

The arguments to prove our theorem are fundamentally different from those of [10], as [10] frequently uses the vertices of the graph in its arguments, which do not have a matroidal analogue. Instead, we apply a framework introduced by the first author [9] that gives an alternate definition of the BBY algorithm using *fourientations*, an object that was first defined by Backman and Hopkins [3]. Our proof essentially comes down to classifying ways that consistency could be violated and then showing that each of these potential possibilities leads to a contradiction.

[1] More precisely, there is a unique collection of four sandpile torsor algorithms on plane graphs that are all closely related.

[2] The (implicit) original name of the corresponding bijections was *geometric bijections*, which the fourth author prefers.

2 Background and Notation

For a set X, we write $X \setminus x$ for $X \setminus \{x\}$, and use similar notation for other operations with a singleton. We call elements of $\mathbb{Z}^E$ *(integral) 1-chains*, where E is an index set. For a 1-chain $\overrightarrow{P}$, and some $e \in E$, we write $\overrightarrow{P}\langle e\rangle$ for the coefficient of e in $\overrightarrow{P}$. We also write $P := \{e \in E : \overrightarrow{P}\langle e\rangle \neq 0\}$ for the *support* of $\overrightarrow{P}$. For $e \in E$, denote by $\overrightarrow{P} \setminus e$ the 1-chain in $\mathbb{Z}^{E\setminus e}$ obtained by restricting $\overrightarrow{P}$ to $E \setminus e$. A 1-chain is *simple* if every coefficient is in $\{-1, 0, 1\}$. An *arc* is a simple 1-chain whose support has only one element. We write arcs in the form $\overrightarrow{e}$, where $e \in E$.

2.1 Oriented Matroids and Regular Matroids

We assume standard background on matroid and oriented matroid theory from the reader. Some standard references are [17] and [6]. Let A be an $r \times m$ *totally unimodular* matrix of full row rank, i.e., a matrix over the reals in which the determinant of every square submatrix is either $-1, 1$, or 0. Let E be a set that indexes the columns of A. Then A represents a *regular matroid* $M := M(A)$ whose ground set is E and of rank r. Denote by $\mathbf{B}(M), \mathbf{C}(M), \mathbf{C}^*(M)$ the set of bases, circuits, and cocircuits of M, respectively. We fix such A and M for the rest of this paper.

We call a simple 1-chain whose support is a circuit (resp. cocircuit) a *signed circuit* (resp. *signed cocircuit*) of M. The collections of signed circuits and signed cocircuits of M are denoted by $\overrightarrow{\mathbf{C}}(M)$ and $\overrightarrow{\mathbf{C}^*}(M)$, respectively. The sets $\overrightarrow{\mathbf{C}}(M)$ and $\overrightarrow{\mathbf{C}^*}(M)$ give M the structure of an oriented matroid, and it is shown in [6, Corollary 7.9.4] that all oriented matroid structures on M are equivalent up to reorientation.

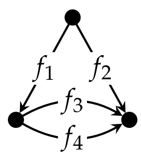
An *orientation* is a map from E to $\{-, +\}$. Adopting the usual convention, we write $\mathcal{O}\langle x\rangle$ for the value of $\mathcal{O}$ at x. Denote by $\mathbf{O}(M)$ the set of all orientations of M.

Definition 2.1. Let $\mathcal{O}$ be an orientation and P be a subset of E. Then we write ${}_P\mathcal{O}$ for the orientation obtained by reversing the elements of P. In other words, for $x \in E$, we have

$$ {}_P\mathcal{O}\langle x\rangle = \begin{cases} -\mathcal{O}\langle x\rangle & \text{if } x \in P, \\ \mathcal{O}\langle x\rangle & \text{if } x \notin P. \end{cases} $$

Definition 2.2. Let $\mathcal{O}$ be an orientation and $\overrightarrow{P}$ be a simple 1-chain. We say that $\overrightarrow{P}$ is *compatible with* $\mathcal{O}$ if for all $f \in P$, the sign of $\overrightarrow{P}\langle f\rangle$ matches the sign of $\mathcal{O}\langle f\rangle$. We denote compatibility by writing $\overrightarrow{P} \sim \mathcal{O}$.

Let $\mathcal{O} \in \mathbf{O}(M)$ and $\overrightarrow{C}$ be a signed circuit that is compatible with $\mathcal{O}$. We say that ${}_C\mathcal{O}$ is a *circuit reversal* of $\mathcal{O}$. Define *cocircuit reversals* analogously. Two orientations $\mathcal{O}_1$ and $\mathcal{O}_2$ differ by *circuit-cocircuit reversals* if $\mathcal{O}_1$ can be sent to $\mathcal{O}_2$ by a sequence of circuit and/or cocircuit reversals. It is easy to show that this is an equivalence relation on $\mathbf{O}(M)$.



$$\begin{bmatrix} 1 & 0 & -1 & -1 \\ -1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

Figure 1: A graph (graphic matroid) and its corresponding representing matrix.

Example 2.3. Take the graphic matroid in Figure 1, and take the orientation $\mathcal{O}$ with $\mathcal{O}\langle f_1\rangle = +$, $\mathcal{O}\langle f_2\rangle = -$, $\mathcal{O}\langle f_3\rangle = +$ and $\mathcal{O}\langle f_4\rangle = +$ (in short, $(+,-,+,+)$; we will use this shorthand throughout). The signed circuit $C = \overrightarrow{f_1} - \overrightarrow{f_2} + \overrightarrow{f_3}$ is compatible with $\mathcal{O}$. By reversing C, we get the orientation $(-,+,-,+)$.

Definition 2.4. The *circuit-cocircuit equivalence classes* of M are the orientations of M modulo the equivalence relation defined in the previous paragraph. The set of these equivalence classes is denoted $\mathbf{G}(M)$. For any element $\mathcal{O} \in \mathbf{O}(M)$, we write $[\mathcal{O}]$ for the equivalence class of $\mathbf{G}(M)$ containing $\mathcal{O}$.

The set $\mathbf{G}(M)$ was first explored by Gioan [11, 12], and it serves as an intermediate object to define the BBY action because of the natural torsor structure described in the next section; in particular, we have the following enumerative fact.

Theorem 2.5. *[12] For a regular matroid M, we have $|\mathbf{G}(M)| = |\mathbf{B}(M)|$.*

2.2 The sandpile group and its canonical action on $\mathbf{G}(M)$

Definition 2.6. Let $\Lambda(M) \subset \mathbb{Z}^E$ be the lattice generated $\overrightarrow{\mathbf{C}}(M)$ and $\Lambda^*(M) \subset \mathbb{Z}^E$ be the lattice generated by $\overrightarrow{\mathbf{C}^*}(M)$. The *sandpile group* of M is defined by:

$$S(M) := \frac{\mathbb{Z}^E}{\Lambda(M) \oplus \Lambda^*(M)}.$$

For a 1-chain $\overrightarrow{P}$, we write $[\overrightarrow{P}]$ for the equivalence class of $S(M)$ containing $\overrightarrow{P}$. Note that the sandpile group $S(M)$ is generated by elements $\{[\overrightarrow{f}] \mid \overrightarrow{f}$ is an arc of $M\}$. In [2], the authors define a natural group action of $S(M)$ on the set $\mathbf{G}(M)$, which generalizes the additive action in the more classical graphical case where elements of $S(M)$ and $\mathbf{G}(M)$ are represented as "chip configurations". For details on the "chip" perspective, see [15]. This natural action is called the canonical action.

Definition 2.7. [2] The *canonical action* of $S(M)$ on $\mathbf{G}(M)$ is defined by linearly extending the following action of each generator $[\overrightarrow{f}]$ on circuit-cocircuit reversal classes. Given $[\mathcal{O}]$, one can prove that there exists some orientation $\mathcal{O}' \in \mathbf{O}(M)$ such that $-\overrightarrow{f} \sim \mathcal{O}'$ and $[\mathcal{O}'] = [\mathcal{O}]$. Define the action by $[\overrightarrow{f}] \cdot [\mathcal{O}] = [{}_f\mathcal{O}']$.

Lemma 2.8. *[2, Theorem 4.3.1.] The canonical action is well-defined and simply transitive.*

Example 2.9. Take the graphic matroid on Figure 1, and take the orientation $\mathcal{O} = (-,-,+,+)$. Since $-\overrightarrow{f_1} \sim \mathcal{O}$, we have $[\overrightarrow{f_1}] \cdot [\mathcal{O}] = [(+,-,+,+)]$.

As a more interesting example, take $[\overrightarrow{f_3}] \cdot [\mathcal{O}]$. Since $\overrightarrow{f_3} \sim \mathcal{O}$, we need to reverse a signed circuit or cocircuit containing f_3, and then reverse f_3 again. $-\overrightarrow{f_1} + \overrightarrow{f_3} + \overrightarrow{f_4}$ is a signed cocircuit containing f_3. Hence $[(+,-,+,-)]$ is the circuit-cocircuit equivalence class of $[\overrightarrow{f_3}] \cdot [\mathcal{O}]$.

As such, any bijection between $\mathbf{G}(M)$ and $\mathbf{B}(M)$ yields a simply transitive group action of $S(M)$ on $\mathbf{B}(M)$ via composing with the canonical action.

2.3 Fourientations

The notion of fourientations was introduced and systematically studied by Backman and Hopkins [3]. The first author applied this notion in [9] to study the connection between the BBY bijections and Lawrence polytopes. We find the language of fourientations also helpful in the proof of our main theorem.

Definition 2.10. Given a set E, a *fourientation* $\mathcal{F}$ is a map from E to the set $\{\emptyset, -, +, \pm\}$. We denote the set of fourientations on the ground set of a matroid M by $\mathbf{F}(M)$.

As with orientations, for $x \in E$, we write $\mathcal{F}\langle x \rangle$ for the output of the map at x. Intuitively, each element of the ground set can be oriented in either direction, bi-oriented, or unoriented.

For $\mathcal{F}_1, \mathcal{F}_2 \in \mathbf{F}(M)$, we write $\mathcal{F}_1 \cup \mathcal{F}_2$ and $\mathcal{F}_1 \cap \mathcal{F}_2$ for the fourientations obtained by taking pointwise union or intersection (treating $-, +, \pm$ as $\{-\}, \{+\}, \{-,+\}$). Furthermore, we define $-\mathcal{F}_1$ to be $\mathcal{F}_1$ with $-$ and $+$ swapped, and $\mathcal{F}_1^c$ to be $-\mathcal{F}_1$ with $\emptyset$ and $\pm$ swapped.

2.4 Triangulating Signatures and the Backman-Baker-Yuen Bijection

In [2], Backman, Baker, and the fourth author defined a family of explicit bijections between $\mathbf{G}(M)$ and $\mathbf{B}(M)$. These maps were generalized in [4] and [9] to the context we use in this paper. Below, we give their constructions in the language of fourientations.

Definition 2.11. A *circuit signature* $\sigma \subset \overrightarrow{\mathbf{C}}(M)$ is a collection of signed circuits of M such that for each circuit $C \in \mathbf{C}(M)$, exactly one of the two signed circuits supported on C is contained in σ. We write $\sigma(C)$ for the circuit supported on C that is contained in σ.

Define a *cocircuit signature* $\sigma^* \subset \overrightarrow{\mathbf{C}^*}(M)$ analogously. For a cocircuit C^*, we write $\sigma^*(C^*)$ for the signed cocircuit supported on C^* that is contained in σ^*.

Figure 2: The circuit signature of Example 2.12 (left panels) and the cocircuit signature of Example 2.12 (right panels).

By a *circuit-cocircuit signature* we mean pair consisting of a circuit signature and a cocircuit signature.

Example 2.12. For the graph of Figure 1, the signed circuits $\overrightarrow{f_1} - \overrightarrow{f_2} + \overrightarrow{f_3}$, $\overrightarrow{f_1} - \overrightarrow{f_2} + \overrightarrow{f_4}$ and $-\overrightarrow{f_3} + \overrightarrow{f_4}$ form a circuit signature. The signed cocircuits $-\overrightarrow{f_1} + \overrightarrow{f_3} + \overrightarrow{f_4}$, $-\overrightarrow{f_1} - \overrightarrow{f_2}$, and $\overrightarrow{f_2} + \overrightarrow{f_3} + \overrightarrow{f_4}$ form a cocircuit signature. See also Figure 2.

Definition 2.13. Fix a circuit-cocircuit signature (σ, σ^*) and a basis B. For each $e \notin B$, let C_e be the unique circuit contained in $B \cup \{e\}$ (known as the *fundamental circuit* of e with respect to B). For each $e \in B$, let C_e^* be the unique cocircuit contained in $(E \setminus B) \cup \{e\}$ (known as the *fundamental cocircuit* of e with respect to B).

We denote by $\mathcal{F}(B, \sigma)$ the fourientation where all $e \in B$ are bi-oriented and all the $e \in E \setminus B$ are oriented according to $\sigma(C_e)$. Similarly, we denote by $\mathcal{F}(B, \sigma^*)$ the fourientation where all $e \in E \setminus B$ are bi-oriented and all $e \in B$ are oriented according to $\sigma^*(C_e^*)$.

Example 2.14. For the graph of Figure 1, take the spanning tree T consisting of edges f_1 and f_3. Take the circuit-cocircuit signature (σ, σ^*) of Example 2.12. Then $\mathcal{F}(T, \sigma) = (\pm, -, \pm, +)$ and $\mathcal{F}(T, \sigma^*) = (-, \pm, +, \pm)$ respectively for edges f_1, f_2, f_3 and f_4.

The BBY bijection will depend on a circuit-cocircuit signature, but in order to obtain a bijection, we need some "niceness" for the signatures, notably, the following.

Definition 2.15. [9] A circuit signature σ (resp. cocircuit signature σ^*) is called *triangulating* if for any distinct $B_1, B_2 \in \mathbf{B}(M)$, the fourientation $\mathcal{F}(B_1, \sigma) \cap -\mathcal{F}(B_2, \sigma)$ is not compatible with any $\overrightarrow{C} \in \overrightarrow{\mathbf{C}}(M)$ (resp. $\mathcal{F}(B_1, \sigma^*) \cap -\mathcal{F}(B_2, \sigma^*)$ is not compatible with any $\overrightarrow{C}^* \in \overrightarrow{\mathbf{C}^*}(M)$). A circuit-cocircuit signature (σ, σ^*) is *triangulating* if σ and σ^* are both triangulating.

Here a simple 1-chain $\overrightarrow{P}$ (for example $\overrightarrow{C}$ or $\overrightarrow{C}^*$) is *compatible* with a fourientation $\mathcal{F}$ if for all $f \in P$, either $\mathcal{F}\langle f \rangle = \pm$ or the sign of $\overrightarrow{P}\langle f \rangle$ matches the sign of $\mathcal{F}\langle f \rangle$.

Definition 2.16. Let M be a regular matroid and (σ, σ^*) be a circuit-cocircuit signature. An orientation $\mathcal{O}$ is *σ-compatible* (resp. *σ^*-compatible*) if every signed circuit (resp. cocircuit) compatible with $\mathcal{O}$ is in σ (resp. σ^*). An orientation is *(σ, σ^*)-compatible* if it is both σ-compatible and σ^*-compatible.

Fix a triangulating signature (σ, σ^*). By [9, Proposition 1.21(1)] there is a unique (σ, σ^*)-compatible orientation in each equivalence class in $\mathbf{G}(M)$. Hence the following notion is well-defined.

Definition 2.17. Given an orientation $\mathcal{O}$, let $\mathcal{O}^\circ$ be the (unique) (σ, σ^*)-compatible orientation in the same reversal class as $\mathcal{O}$. Likewise, given $[\mathcal{O}] \in \mathbf{G}(M)$, let $[\mathcal{O}]^\circ = \mathcal{O}^\circ$. Furthermore, let $\mathbf{O}^\circ(M)$ be the set of all (σ, σ^*)-compatible orientations.

Note that $\mathcal{O}^\circ$ and $\mathbf{O}^\circ(M)$ both depend on the choice of circuit-cocircuit signature. We omit a reference to this signature in the notation for readability.

We can also directly define the canonical action on the set $\mathbf{O}^\circ(M)$, namely, for $g \in S(M)$ and $\mathcal{O} \in \mathbf{O}^\circ(M)$, we define $g \cdot \mathcal{O} := (g \cdot [\mathcal{O}])^\circ$.

Example 2.18. Let us return to Example 2.9, and take the signature of Example 2.12. It can be checked that this is triangulating. With this, $[\overrightarrow{f_1}] \cdot (-,-,+,+) = (+,-,+,+)$ and $[\overrightarrow{f_3}] \cdot (-,-,+,+) = (+,-,-,+)$.

Now we are in the position to introduce the BBY bijection.

Definition 2.19 (BBY bijection)**.** Fix a regular matroid M and a pair (σ, σ^*) of triangulating signatures. The map $\beta_{(M,\sigma,\sigma^*)} : \mathbf{B}(M) \to \mathbf{O}(M)$ is given by $B \mapsto \mathcal{F}(B, \sigma) \cap \mathcal{F}(B^*, \sigma^*)$.

Theorem 2.20. *[4, 9] For a regular matroid M and a pair (σ, σ^*) of triangulating signatures. The BBY map $\beta_{(M,\sigma,\sigma^*)}$ is a bijection between $\mathbf{B}(M)$ and $\mathbf{O}^\circ(M)$. In particular, this map induces a bijection between $\mathbf{B}(M)$ and $\mathbf{G}(M)$.*

The bijection $\beta_{(M,\sigma,\sigma^*)}$ together with the canonical action in Section 2.2 induces a simply transitive group action of $S(M)$ on $\mathbf{B}(M)$ that we call the *BBY action*.

Example 2.21. Take the graphic matroid M of Figure 1 with the circuit-cocircuit signature (σ, σ^*) of Example 2.12. Let $T = \{f_1, f_3\}$. Then $\beta_{(M,\sigma,\sigma^*)}(T) = (-,-,+,+)$.

Let us compute the BBY action of $\overrightarrow{f_1}$ on T. We have $[\overrightarrow{f_1}] \cdot (-,-,+,+) = (+,-,+,+)$ by Example 2.18. One can check that for $T' = \{f_2, f_3\}$ we have $\beta_{M,\sigma,\sigma^*}(T') = (+,-,+,+)$. Hence $[\overrightarrow{f_1}] \cdot \beta_{(M,\sigma,\sigma^*)}(T) = \beta_{(M,\sigma,\sigma^*)}(T')$.

2.5 The Main Theorem

Before stating the main theorem, we remark that any triangulating circuit-cocircuit signature (σ, σ^*), and any $e \in E$ that is not a loop or coloop naturally yields triangulating circuit-cocircuit signatures $(\sigma \setminus e, \sigma^* \setminus e)$ and $(\sigma / e, \sigma^* / e)$ on $M \setminus e$ and M/e respectively. The following theorem says that the BBY algorithm is consistent.

(1)

(2)

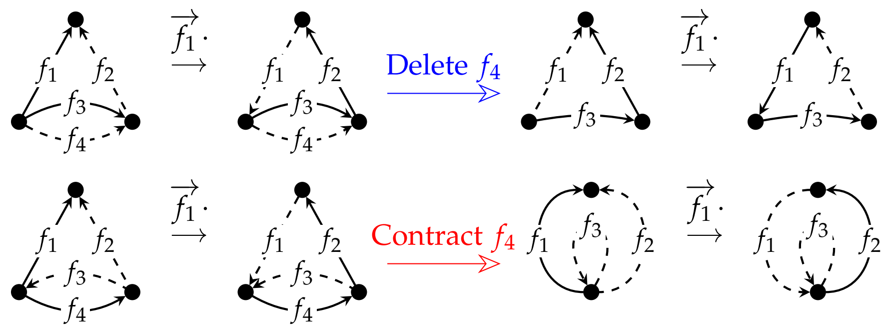


Figure 3: Above are illustrations for the first two parts of Theorem 2.22. See Example 2.23 for details.

Theorem 2.22. *Let M be a regular matroid and (σ, σ^*) be a triangulating circuit-cocircuit signature. Suppose that $\overrightarrow{f}$ is an arc and $B_1, B_2 \in \mathbf{B}(M)$ such that*

$$[\overrightarrow{f}] \cdot \beta_{(M,\sigma,\sigma^*)}(B_1) = \beta_{(M,\sigma,\sigma^*)}(B_2).$$

1. *For any $e \in (B_1^c \cap B_2^c) \setminus f$, we have*

$$[\overrightarrow{f}] \cdot \beta_{(M\setminus e,\sigma\setminus e,\sigma^*\setminus e)}(B_1) = \beta_{(M\setminus e,\sigma\setminus e,\sigma^*\setminus e)}(B_2).$$

2. *For any $e \in (B_1 \cap B_2) \setminus f$, we have*

$$[\overrightarrow{f}] \cdot \beta_{(M/e,\sigma/e,\sigma^*/e)}(B_1 \setminus e) = \beta_{(M/e,\sigma/e,\sigma^*/e)}(B_2 \setminus e).$$

3. *If e and f are in different connected components of M, then $e \in B_1 \iff e \in B_2$.*

Theorem 2.22 is a generalization of [10, Theorem 4.6] from plane graphs to regular matroids.

Example 2.23. Take the graphic matroid M from Figure 1 with the cycle-cocycle signature (σ, σ^*) from Example 2.12. The first row of Figure 3 demonstrates Theorem 2.22(1) and the second row demonstrates Theorem 2.22(2). The depicted orientations are the circuit-cocircuit minimal orientations assigned to the spanning trees by the BBY bijection.

Let us explain the first row. The upper left panel shows action of $\overrightarrow{f_1}$ on the basis $\{f_1, f_3\}$: The action produces $\{f_2, f_3\}$ as explained in Example 2.18. We have $\sigma \setminus f_4 = \{\overrightarrow{f_1} - \overrightarrow{f_2} + \overrightarrow{f_3}\}$, and $\sigma^* \setminus f_4 = \{-\overrightarrow{f_1} + \overrightarrow{f_3}, -\overrightarrow{f_1} - \overrightarrow{f_2}, \overrightarrow{f_2} + \overrightarrow{f_3}\}$. One can check that indeed $[\overrightarrow{f_1}] \cdot \beta_{(M\setminus f_4,\sigma\setminus f_4,\sigma^*\setminus f_4)}(\{f_1, f_3\}) = \beta_{(M\setminus f_4,\sigma\setminus f_4,\sigma^*\setminus f_4)}(\{f_2, f_3\})$.

3 Some Proof Ingredients

We now introduce one of the main tools that we used to prove our main result, Theorem 2.22. The purpose of this theorem is to localize the changes in the BBY action that we need to analyze. We include the statement here for it may be of independent interest.

Theorem 3.1. *Let $\overrightarrow{f}$ be an arc and $\mathcal{O}_1, \mathcal{O}_2 \in \mathbf{O}^\circ(M)$ such that $\overrightarrow{f} \cdot \mathcal{O}_1 = \mathcal{O}_2$. Then $\mathcal{O}_1$ can be transformed to $\mathcal{O}_2$ by the following (at most) three step process.*

1. *Reverse at most one signed circuit or cocircuit containing f that is compatible with $\mathcal{O}_1$.*
2. *Reverse $\overrightarrow{f}$.*
3. *Reverse at most one signed circuit or cocircuit containing f that is compatible with the new orientation.*

Furthermore, the following conditions hold.

a. *A reversal occurs during step 1 (respectively, step 3) if and only if $\overrightarrow{f} \sim \mathcal{O}_1$ (respectively, $-\overrightarrow{f} \sim \mathcal{O}_2$).*

b. *If reversals occur during both step 1 and step 3, one of these is a circuit reversal while the other is a cocircuit reversal.*

Theorem 2.22(3) follows immediately from Theorem 3.1. For the rest, by duality, it suffices to focus on Theorem 2.22(1). The deletion of an edge e does not affect the cocircuit reversals that occur in Theorem 3.1 in an essential way. The case that the edge e appears in the circuits reversed in Theorem 3.1 is the main obstacle. However, we prove that this cannot happen using fourientations.

Here is a short illustration of how the fourientations help with the proof. Under the assumption of Theorem 2.22, denote

$$\begin{aligned} \mathcal{O}_1 &= \mathcal{F}(B_1,\sigma) \cap \mathcal{F}(B_1,\sigma^*), & \mathcal{O}_2 &= \mathcal{F}(B_2,\sigma) \cap \mathcal{F}(B_2,\sigma^*), \\ \mathcal{F} &= \mathcal{F}(B_1,\sigma) \cap -\mathcal{F}(B_2,\sigma), \quad \text{and} & \mathcal{F}^* &= \mathcal{F}(B_1,\sigma^*) \cap -\mathcal{F}(B_2,\sigma^*). \end{aligned}$$

Theorem 3.1 describes the difference between $\mathcal{O}_1$ and $\mathcal{O}_2$. For the edges where $\mathcal{O}_1$ and $\mathcal{O}_2$ coincide, the following lemma transfers the information to fourientations.

Lemma 3.2. *[9, Lemma 2.8] For any $x \in E$, if $\mathcal{O}_1\langle x\rangle = \mathcal{O}_2\langle x\rangle$, then $\mathcal{F}\langle x\rangle = \mathcal{F}^*\langle x\rangle$.*

We have a similar lemma when $\mathcal{O}_1$ and $\mathcal{O}_2$ differ, which is more technical and omitted here. We also know that $\mathcal{F}$ (resp. $\mathcal{F}^*$) is not compatible with any $\overrightarrow{C} \in \overrightarrow{\mathbf{C}}(M)$ (resp. $\overrightarrow{C}^* \in \overrightarrow{\mathbf{C}^*}(M)$) from Definition 2.15. Combining all this information on the two fourientations, we are able to prove the desired result.

We show the power of the fourientation language by giving a short proof [9, Remark 2.10] of the following result in [2], which was first proven using a geometric argument.

Corollary 3.3. $\beta_{(M,\sigma,\sigma^*)}$ *is injective.*

Proof. If $\mathcal{O}_1 = \mathcal{O}_2$ comes from two distinct bases via $\beta_{(M,\sigma,\sigma^*)}$, then Lemma 3.2 and the triangulating assumption of (σ, σ^*) imply that $\mathcal{F} = \mathcal{F}^*$ is not compatible with any signed circuit/cocircuit, which contradicts the *3-painting axiom* [6, Theorem 3.4.4] in oriented matroid theory. See [9, Lemma 2.3] for a fourientation version of the 3-painting axiom. ∎

4 Special Instances of Consistency

4.1 Acyclic signatures

The notion of acyclic signatures was introduced in [20, 2]. A circuit signature σ is *acyclic* if the only set of nonnegative λ_C values satisfying $\sum_{\overrightarrow{C}\in\sigma} \lambda_C \overrightarrow{C} = \mathbf{0}$ is where every λ_C is zero. Acyclic cocircuit signatures are defined analogously.

The seemingly technical definition arrives naturally in the context of polyhedral geometry. By [9, Lemma 3.4], acyclic signatures are triangulating, and it can be proven that the property of being acyclic is preserved under deletion or contraction of signatures. Hence we have the following corollary, which was Conjecture 6.11 of [10].

Corollary 4.1. *The BBY actions with respect to acyclic signatures are consistent.*

4.2 The Planar Case

For a plane graph, circuits oriented counterclockwise form a triangulating circuit signature [2]. Also, for any graph, the signed cocircuits "oriented away" from a fixed vertex v form a triangulating cocircuit signature. Notice that the circuit and cocircuit signature given in Example 2.12 fall into the above cases.

Combining [5, Theorem 7.1] and [2, Example 1.1.3], the rotor-routing torsor action of a plane graph is equal to the BBY action with respect to this circuit-cocircuit signature. Moreover, it is apparent (and can be proven rigorously) that an embedding-preserving deletion (respectively, contraction) of a plane graph induces the deletion (respectively, contraction) of the circuit-cocircuit signature. As a final corollary of all these, we have:

Corollary 4.2. *[10, Theorem 4.6] The rotor-routing torsor algorithm is consistent.*

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