Séminaire Lotharingien de Combinatoire **91B** (2024)
Article #77, 12 pp.

Proceedings of the 36th Conference on Formal Power Series and Algebraic Combinatorics (Bochum)

Pattern-avoiding polytopes and Cambrian lattices

Esther Banaian*[1], Sunita Chepuri†[2], Emily Gunawan‡[3], and Jianping Pan§[4]

[1]*Department of Mathematics, Aarhus University, Aarhus, Denmark*
[2]*Department of Mathematics and Computer Science, University of Puget Sound, Tacoma, WA, USA*
[3]*Department of Mathematics and Statistics, University of Massachusetts Lowell, Lowell, USA*
[4]*Department of Mathematics, North Carolina State University, Raleigh, USA*

Abstract. For each Coxeter element c in the symmetric group, we define a pattern-avoiding Birkhoff subpolytope whose vertices are the c-singletons. We show that the normalized volume of our polytope is equal to the number of longest chains in a corresponding type A Cambrian lattice. Our work extends a result of Davis and Sagan which states that the normalized volume of the convex hull of the 132 and 312 avoiding permutation matrices is the number of longest chains in the Tamari lattice, a special case of a type A Cambrian lattice. Furthermore, we prove that each of our polytopes is unimodularly equivalent to the order polytope of the heap of the c-sorting word of the longest permutation. This gives an affirmative answer to a generalization of a question posed by Davis and Sagan.

Keywords: order polytopes, heap, Birkhoff polytopes, Cambrian lattices, permutations

1 Introduction

The sequence [6, A003121] counts shifted standard tableaux of staircase shape and longest chains in the Tamari lattice; it also counts the number of reduced words in a certain commutation class of the longest permutation. More recently, it was shown by Davis and Sagan in [2] that this sequence gives the normalized volume of a certain "pattern-avoiding polytope," a subpolytope of the Birkhoff polytope whose vertices are 132 and 312 avoiding permutations. Since these permutations form a distributive sublattice of the right weak order, Davis and Sagan asked whether their polytope might be unimodularly equivalent to the order polytope of the poset of join irreducibles of the 132 and 312 avoiding permutations.

*banaian@math.au.dk
†schepuri@pugetsound.edu
‡emily_gunawan@uml.edu
§jpan9@ncsu.edu

In the same paper, Davis and Sagan pointed out that the 132 and 312 avoiding permutations are known to be the c-singletons for the symmetric group for a specific "Tamari" Coxeter element c and proposed that it would be interesting to define similar pattern-avoiding polytopes for other Coxeter groups and study them from the perspective of c-singletons. The c-singletons are the spine of an important lattice called the c-Cambrian lattice [7], and they form a distributive sublattice of the right weak order [4].

In this article, we associate a pattern-avoiding polytope to each Coxeter element c in the symmetric group. We define this polytope to be the convex hull of the permutation matrices of the c-singletons (see Sections 3.2 and 4.1). We prove that our polytope is indeed unimodularly equivalent to the order polytope of the poset of join irreducibles of the c-singletons (see Section 5). In particular, for the Tamari Coxeter element, our result answers Davis and Sagan's question in the affirmative.

2 Background and notation

Denote the symmetric group on $n+1$ elements by A_n. We can represent a permutation $w \in A_n$ in *one-line notation* as $w = w(1)w(2)\cdots w(n+1)$. For each $i \in \{1, \ldots, n\}$, we write $s_i \in A_n$ to denote the *simple reflection* (or *adjacent transposition*) that swaps i and $i+1$ and fixes all other letters. Every permutation can be expressed as a product of simple reflections. Given $w \in A_n$, the minimum number of simple reflections among all such expressions for w is called the *(Coxeter) length* of w, and is denoted by $\ell(w)$. A *reduced decomposition* of w is an expression $w = s_{i_1} \cdots s_{i_{\ell(w)}}$ realizing the Coxeter length of w. To simplify notation, we refer to such a decomposition via its *reduced word* $\left[i_1 \cdots i_{\ell(w)}\right]$. For example, consider $w = 51342 \in A_4$. One of its reduced decompositions is $s_4s_2s_3s_2s_4s_1$ with $[423241]$ as the corresponding reduced word, and $\ell(w) = 6$.

A *Coxeter element* c in A_n is a product of all n simple reflections in any order, where each reflection appears exactly once. The longest permutation of A_n is the permutation $w_0 = (n+1)n\ldots 321$ and $\ell(w_0) = \binom{n+1}{2}$.

Simple reflections satisfy *commutation relations* of the form $s_is_j = s_js_i$ for $|i-j| > 1$. An application of a commutation relation to a product of simple reflections is called a *commutation move*. When referring to reduced words, we will say adjacent letters i and j in a reduced word *commute* when $|i-j| > 1$. Given a reduced word $[u]$ of a permutation, the equivalence class consisting of all words that can be obtained from $[u]$ by a sequence of commutation moves is the *commutation class* of $[u]$.

2.1 Heaps

We review the classical theory of heaps, which was used in [12] to study fully commutative elements of a Coxeter group. Heaps also appeared as "the natural partial orders" in [3, Definition 6] and [5, Definition 1] and they were used to study certain acyclic domains. For a detailed list of attributions on the theory of heaps, see [10, Solutions to Exercise 3.123(ab)].

Definition 2.1. *Given a reduced word* $[u] = [u_1 \cdots u_\ell]$ *of a permutation, consider the partial order* $\preccurlyeq$ *on the set* $\{1, \ldots, \ell\}$ *obtained via the transitive closure of the relations*

$$x \prec y$$

for $x < y$ *such that* $|u_x - u_y| \leq 1$ *(that is,* u_x *and* u_y *do not commute). For each* $1 \leq x \leq \ell$*, the* label *of the poset element* x *is* u_x*. This labeled poset is called the* heap *for* $[u]$*, denoted* $\mathsf{Heap}([u])$*. The Hasse diagram for this poset with elements* $\{1, \ldots, \ell\}$ *replaced by their labels is called the* heap diagram *for* $[u]$*. The labels in the heap diagram are drawn in increasing order from left to right.*

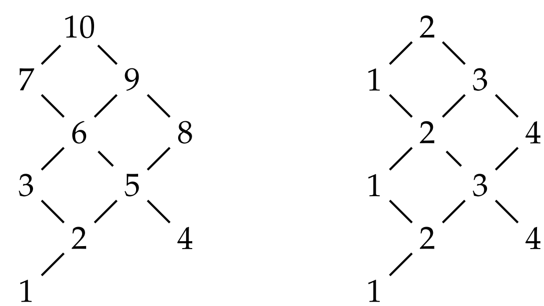

Figure 1: Hasse diagram of the underlying poset (left) and the heap diagram (right) of a commutation class of w_0 given in Example 2.2.

Example 2.2. *Consider a reduced word* $[u] = [u_1 \ldots u_{10}] = [1214321432]$ *of the longest element* w_0 *in* A_4*.*

1. *Figure 1 (left) shows a Hasse diagram of the underlying unlabeled poset* $\mathsf{Heap}([u])$*. Here* $\ell = 10$ *and so the the elements of the heap poset* $\mathsf{Heap}([u])$ *are* $\{1, 2, \ldots, 10\}$*.*

2. *Figure 1 (right) shows the heap diagram for* $\mathsf{Heap}([u])$*. The possible labels of the poset elements are* $\{1, 2, 3, 4\}$*.*

Linear extensions of $\mathsf{Heap}([u])$ relate to the commutation class of $[u]$.

Definition 2.3. *A* linear extension $\pi = \pi(1) \cdots \pi(\ell)$ *of a partial order* $\preccurlyeq$ *on* $\{1, \ldots, \ell\}$ *is a total order on the poset elements that is consistent with the structure of the poset. That is,* $x \prec y$ *implies* $\pi(x) < \pi(y)$*. A* labeled linear extension *of the heap of a reduced word* $[u] = [u_1 \cdots u_\ell]$ *is a word* $\left[u_{\pi(1)} \cdots u_{\pi(\ell)}\right]$*, where* $\pi = \pi(1) \cdots \pi(\ell)$ *is a linear extension of the heap.*

Proposition 2.4 ([12, Proof of Proposition 2.2] and [10, Solutions to Exercise 3.123(ab)])**.** *Given a reduced word $[u]$, the set of labeled linear extensions of the heap for $[u]$ is the commutation class of $[u]$.*

Example 2.5. *Three of labeled linear extensions of* $\mathsf{Heap}([u])$ *from Example 2.2 are* $[u]$ *itself,* $[1243124312]$*, and* $[4123412312]$*. Notice that these reduced words all belong to the same commutation class, due to Proposition 2.4.*

2.2 Order polytopes

In this section, we review *order polytopes*, following Stanley's paper [11]. Given a finite poset P, the order polytope of P is given by

$$\mathcal{O}(P) := \{x \in \mathbb{R}^P : 0 \leqslant x_t \leqslant 1 \text{ for all } t \in P \text{ and } x_t \leqslant x_s \text{ when } t \leqslant_P s\}.$$

Many basic properties of an order polytope are answered by the combinatorial structure of the poset. Below are some properties that are relevant to us.

1. The dimension of $\mathcal{O}(P)$ is given by the number of elements in P.

2. The volume of $\mathcal{O}(P)$ can be computed from the number of linear extensions of P.

3. The vertices of $\mathcal{O}(P)$ are exactly the indicator vectors of order ideals of P.

In this paper, we will be focusing on order polytopes for certain heap posets.

3 c-singletons

3.1 c-sorting words and c-sortable permutations

In this section, we review c-sorting words and c-sortable elements, which were introduced in [8]. Given a Coxeter element c and reduced word $[a_1a_2\ldots a_n]$, define an infinite word

$$c^\infty := a_1a_2\ldots a_n \mid a_1a_2\ldots a_n \mid \cdots$$

consisting of repeated copies of the given reduced word for c. The symbols "|" are "dividers" which facilitate the definition of sortable elements. The c-sorting word of $w \in A_n$ is the lexicographically first (as a sequence of positions in c^∞) subword of c^∞ that is a reduced word for w. We denote this word by $\mathsf{sort_c}(w)$.

We say that the identity permutation is *c-sortable*. If w is not the identity permutation, we can think of $\mathsf{sort_c}(w)$ as a sequence of nonempty subsets of $\{a_1, \ldots, a_n\}$. The subsets $K_1, K_2, \ldots, K_p$ in this sequence are the sets of letters of c that occur between two adjacent dividers, so we have $x \in K_j$ if x is in the jth copy of c inside c^∞. We say that a permutation w is *c-sortable* if $K_1 \supseteq K_2 \supseteq \cdots \supseteq K_p$. The set of c-sortable permutations does not depend on the choice of reduced word for c.

Example 3.1. *Consider the Coxeter element* $c = s_1 s_2 s_3 s_4 = [1234]$ *of* A_4*. Then the c-sorting word of the permutation* 42351 *is* $[1234 \mid 2 \mid 1]$*. Our subsets are* $K_1 = \{1,2,3,4\}$*,* $K_2 = \{2\}$*, and* $K_3 = \{1\}$*. Since* $K_2 \not\supseteq K_3$*, these sets do not form a nested sequence and therefore* 42351 *is not c-sortable. On the other hand, the permutation* 43215 *has c-sorting word* $[123 \mid 12 \mid 1]$ *and is c-sortable.*

Reading showed in [9] that the restriction of the right weak order to c-sortable elements is a lattice which is isomorphic to an important quotient of the right weak order called the c-Cambrian lattice [7]. For the Coxeter element $c = s_1 s_2 \ldots s_n$, the c-sortable elements form the Tamari lattice. For this reason, we refer to this Coxeter element as the "Tamari" Coxeter element of A_n. Cambrian lattices and c-sortable elements have strong connections to cluster algebras, representation theory, and many areas of combinatorics, and they are widely studied. We will be interested in a subclass of c-sortable elements, called c-singletons, which we describe next.

3.2 c-singleton permutations

There is an order-preserving projection $\pi_\downarrow^c : A_n \to A_n$ which sends an element w to the largest c-sortable element that is weakly below w in the right weak order [9, Proposition 3.2]. In [4], Hohlweg, Lange, and Thomas used this map to introduce an important subclass of c-sortable elements: A c-sortable w is called a *c-singleton* if the preimage of $\{w\}$ under $\pi_\downarrow^c$ is the singleton $\{w\}$ itself. We will use the following characterization of c-singletons.

Theorem 3.2 ([4, Theorem 2.2])**.** *A permutation w is a c-singleton if and only if some reduced word of w is a prefix of a word in the commutation class of* $\mathsf{sort_c}(w_0)$*, the c-sorting word of the longest permutation* w_0*.*

The set of c-singletons form a distributive sublattice of the right weak order due to [4, Proposition 2.5]. We denote this lattice by $\mathcal{L}(\text{c-singletons})$. By [5, Proposition 3], $\mathcal{L}(\text{c-singletons})$ is isomorphic to the lattice of order ideals of $\mathsf{Heap}(\mathsf{sort_c}(w_0))$, which we denote by $J(\mathsf{Heap}(\mathsf{sort_c}(w_0)))$.

Proposition 3.3. *The following map is a poset isomorphism*

$$\begin{aligned} f : \mathcal{L}(\textit{c-singletons}) &\to J(\mathsf{Heap}(\mathsf{sort_c}(w_0))) \\ w &\mapsto \mathsf{Heap}(\mathsf{sort_c}(w)) \end{aligned}$$

between the c-singletons and the order ideals of the heap poset $\mathsf{Heap}(\mathsf{sort_c}(w_0))$*.*

As we noted in Section 2.2, the vertices of the order polytope $\mathcal{O}(P)$ of a poset P correspond to the order ideals of P. As a consequence, the c-singletons are in bijection with the vertices of $\mathcal{O}(\mathsf{Heap}(\mathsf{sort_c}(w_0)))$.

4 c-Birkhoff polytopes

The *Birkhoff polytope* is the convex hull of all permutation matrices. Davis and Sagan [2] studied a "pattern-avoiding" subpolytope of the Birkhoff polytope whose vertices correspond to the permutations avoiding the pattern 132 and 312. As noted in [2, Remark 3.6], for the "Tamari" Coxeter element $c = s_1 s_2 \dots s_n$, the c-singletons are precisely the permutations which avoid these same patterns 132 and 312.

4.1 A pattern-avoidance criterion for c-singletons

There is a similar classification of c-singletons for other c (see Proposition 4.1). In this section, we generalize Davis and Sagan's pattern-avoiding polytope coming from the "Tamari" Coxeter element c to all Coxeter elements c in A_n.

Let c be a Coxeter element in A_n. There is exactly one commutation class for c, so every reduced word for c has the same heap. By abuse of notation we write $\mathsf{Heap}(c)$ to denote this heap. Then $\mathsf{Heap}(c)$ is the partial order on $\{1, \dots, n\}$ obtained via the transitive closure of the cover relations $i - 1 \prec i$ if $i - 1$ appears to the left of i in every reduced word of c, and $i - 1 \succ i$ otherwise.

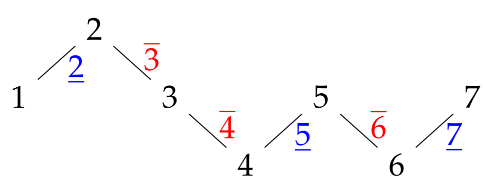

Figure 2: The heap diagram for the Coxeter element $c = s_1 s_4 s_3 s_2 s_6 s_5 s_7$ which corresponds to lower-barred numbers $\underline{2}$, $\underline{5}$, $\underline{7}$ and upper-barred numbers $\overline{3}$, $\overline{4}$, $\overline{6}$.

As described in [7, Chapter 6], we partition the integers in $[2, n]$ into lower-barred and upper-barred numbers $\underline{[2, n]}$ and $\overline{[2, n]}$, respectively. If $i - 1 \prec i$, define i to be a lower-barred number $\underline{i} \in \underline{[2, n]}$; If $i - 1 \succ i$, define i to be a upper-barred number $\bar{i} \in \overline{[2, n]}$. For example, see Figure 2.

We say that a permutation w avoids the pattern $31\underline{2}$ if w contains no 312-pattern such that the last entry "2" in the pattern is a lower-barred number. Similarly, a permutation w avoids the pattern $\overline{2}31$ if the one-line notation w contains no 231-pattern such that the first entry "2" in the pattern is an upper-barred number.

The following result of [7, Proposition 5.7] characterizes c-sortable and c-singleton permutations using pattern-avoidance.

Proposition 4.1. *A permutation $w \in A_n$ is c-sortable if and only if the one-line notation of w avoids the patterns $31\underline{2}$ and $\overline{2}31$. Furthermore, a c-sortable permutation w is a c-singleton if and only if w avoids the patterns $13\underline{2}$ and $\overline{2}13$.*

Definition 4.2. *For a Coxeter element c in A_n, we define the* c-singleton Birkhoff polytope*, or* c-Birkhoff polytope *for short, to be the convex hull of the permutation matrices corresponding to the c-singletons, that is, the permutations avoiding the four patterns listed in Proposition 4.1. We denote the c-Birkhoff polytope as* $\mathsf{Birk}(c)$.

Note that our convention is that the permutation matrix of the permutation $w = w(1)\ldots w(n+1)$ has 1's in entries $(i, w(i))$.

Remark 4.3. *Davis and Sagan suggested in [2, Remark 3.6] that it would be interesting to define pattern-avoiding polytopes for other Coxeter groups. Theorem 4.12 of [8] characterizes type B and D c-sortable elements as signed permutations satisfying certain pattern avoidance conditions. This can be used to give us an analog of Proposition 4.1 for type B and D c-singletons.*

4.2 Relations for c-Birkhoff polytopes

The classical Birkhoff polytope of A_n lives in a $(n+1)^2$-dimensional ambient space. Since each of its row and column sum up to one, it is a n^2-dimensional polytope. The row and column relations for the classical Birkhoff polytope also hold in the c-Birkhoff polytope, since the vertices still come from permutation matrices. Our goal in this section is to exhibit $\binom{n}{2}$ additional relations which any point in $\mathsf{Birk}(c)$ satisfies, so that it is in fact an $\binom{n+1}{2}$-dimensional polytope.

From the pattern avoidance criteria in Proposition 4.1, we see that if w is a c-singleton and $i \in \underline{[2,n]}$, all numbers less than i or all numbers greater than i must appear after i in the one-line notation w and similarly for upper-barred numbers. In particular, if $m = \max(i+1, n-i+2)$, then i cannot appear in any of the last m spots of w.

We will consider points in $\mathsf{Birk}(c)$ as matrices $(x_{i,j})$ for $1 \leq i,j \leq n+1$.

Proposition 4.4. *Let c be a Coxeter element in A_n, and let $(x_{i,j})$ be a point in the c-Birkhoff polytope. For $2 \leq i \leq n$, if $i \in \underline{[2,n]}$ and $m = \max(i+1, n+2-i)$, we have*

$$x_{m,i} = x_{m+1,i} = \cdots = x_{n+1,i} = 0.$$

Otherwise, $i \in \overline{[2,n]}$ and if $r = \min(i-1, n+2-i)$, then

$$x_{1,i} = x_{2,i} = \cdots = x_{r,i} = 0.$$

Using the pattern avoidance repeatedly puts restrictions on which numbers can appear together in the first u spots of a c-singleton for some values of u.

Theorem 4.5. *Let c be a Coxeter element in A_n. For each $1 \leq i \leq \frac{n-1}{2}$ and $i+1 \leq u \leq n-i$, there exists a sequence $i = v_0 < v_1 < \cdots < v_d$, where $d \geq 1$, such that*

$$\sum_{j=0}^{d} \sum_{i=1}^{u} x_{i,v_j}$$

is equal to either 1 or d (depending on i and u) for all points in the c-Birkhoff polytope.

Example 4.6. *Let $c = s_1 s_2 s_4 s_3$. Then the c-Birkhoff polytope has $\binom{4}{2} = 6$ additional relations. Proposition 4.4 gives us four relations*

$$x_{5,2} = x_{4,3} = x_{5,3} = x_{1,4} = 0.$$

Theorem 4.5 gives us two more relations. For $i = 1$, $u = 2$, the sequence is $v_0 = 1, v_1 = 3, v_2 = 4$, and we have the relation

$$\sum_{j=0}^{2} \sum_{i=1}^{2} x_{i,v_j} = \sum_{j \in \{1,3,4\}} \sum_{i=1}^{2} x_{i,j} = x_{1,1} + x_{2,1} + x_{1,3} + x_{2,3} + x_{1,4} + x_{2,4} = 1.$$

For $i = 1$, $u = 3$, the sequence is $v_0 = 1$, $v_1 = 5$ and we have the relation

$$\sum_{j=0}^{1} \sum_{i=1}^{3} x_{i,v_j} = \sum_{j \in \{1,5\}} \sum_{i=1}^{3} x_{i,j} = x_{1,1} + x_{2,1} + x_{3,1} + x_{1,5} + x_{2,5} + x_{3,5} = 1.$$

5 $\mathsf{Birk}(c)$ and $\mathcal{O}(\mathsf{Heap}(\mathsf{sort}_c(w_0)))$

In this section, we will prove $\mathsf{Birk}(c)$ is unimodularly equivalent to the order polytope $\mathcal{O}(\mathsf{Heap}(\mathsf{sort}_c(w_0)))$. We will achieve this by first explicitly constructing a lattice-preserving projection Π_c on $\mathsf{Birk}(c)$, and then show the existence of a unimodular transformation $\mathcal{U}_c$.

5.1 A lattice-preserving projection

Let c be a Coxeter element of A_n. We define a projection Π_c on $(n+1) \times (n+1)$-matrices which reads $\binom{n+1}{2}$ of the entries in a specific order. We describe the reverse order by reading entries in the matrix.

Let $1 < \underline{p_1} < \cdots < \underline{p_r} < n+1$ be the set of lower-barred numbers and $\overline{q_1} < \ldots \overline{q_s} < n+1$ be the set of upper-barred numbers for c. Let σ be the permutation

$(n+1)\, p_r\, p_{r-1} \ldots p_1\, 1\, q_1 q_2 \ldots q_s$ written in one-line notation. The first entries we will read are

$$\begin{aligned} &(p_1-1, p_1), (p_1-2, p_1), \ldots, (1, p_1),\\ &\ldots\\ &(p_r-1, p_r), (p_r-2, p_r), \ldots, (1, p_r),\\ &(n, n+1), (n-1, n+1), \ldots, (1, n+1). \end{aligned}$$

The remaining entries come from $q_s, \ldots, q_1$. For each q_i, take $q_i - 1$ entries as follows:

- Let $m = \min(q_i - 1, n+1-q_i)$. Let $\sigma_1^i, \sigma_2^i, \ldots, \sigma_m^i$ be the m numbers of σ (in one-line notation) immediately before q_i, from right to left.
- First take the m entries $(n+1, \sigma_1^i), (n, \sigma_2^i), \ldots, (n+2-m, \sigma_i^m)$.
- Then take the additional $q_i - 1 - m$ entries $(q_i - 1, q_i), (q_i - 2, q_i), \ldots, (m+1, q_i)$.

Example 5.1. *Let $c = s_1 s_4 s_3 s_2 s_6 s_5 s_7$ be the Coxeter element whose Heap diagram and corresponding upper- and lower-barred numbers are illustrated in Figure 2. Then $\underline{p_1}, \underline{p_2}, \underline{p_3} = \underline{2}, \underline{5}, \underline{7}$ and $\overline{q_1}, \overline{q_2}, \overline{q_3} = \overline{3}, \overline{4}, \overline{6}$. We have $\sigma = 87521346$. We compute the projection Π_c in Figure 3 (left).*

	28	X	X	24	X	18	11
		X	X	25	X	19	12
			X	26	6	20	13
				27	7	21	14
					8	22	15
	3			X		23	16
4	1	9		X			17
2	X	5	10	X		X	

0	①	0	0	⓪	0	⓪	⓪
0	0	0	0	①	0	⓪	⓪
1	0	0	0	⓪	⓪	⓪	⓪
0	0	1	0	⓪	⓪	⓪	⓪
0	0	0	1	0	⓪	⓪	⓪
0	⓪	0	0	0	1	⓪	⓪
⓪	⓪	⓪	0	0	0	1	⓪
⓪	0	⓪	⓪	0	0	0	1

Figure 3: Left: The projection Π_c of Example 5.1. Right: permutation matrix for b_4 of Example 5.5.

Theorem 5.2. *Π_c is a lattice-preserving projection on the c-Birkhoff polytope.*

5.2 A diagonal reading word

Let c be a Coxeter element of A_n and let $\mathcal{R}_c$ denote the labeled linear extension of $\mathsf{Heap}(\mathsf{sort}_c(w_0))$ which is lexicographically first, in the sense of lexicographic order on heap labels. Observe that $\mathcal{R}_c$ is the word formed by concatenating the diagonals of

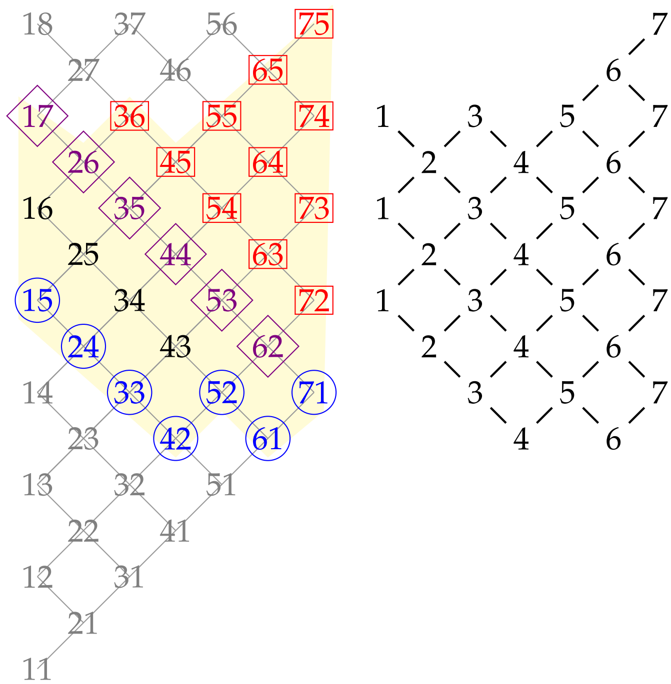

Figure 4: Left: Algorithm for constructing the heap $\mathsf{Heap}(\mathsf{sort}_c(w_0))$ for $[4321657]$-sorting word of the longest element w_0 in A_7. Right: The heap diagram for $\mathsf{Heap}(\mathsf{sort}_c(w_0))$.

$\mathsf{Heap}(\mathsf{sort}_c(w_0))$ from left to right; within each diagonal, read from southeast to northwest. For this reason, we refer to the reduced word $\mathcal{R}_c$ as the *diagonal reading word* of $\mathsf{Heap}(\mathsf{sort}_c(w_0))$.

Thanks to [1, 5], we can give a nice algorithmic construction of $\mathsf{Heap}(\mathsf{sort}_c(w_0))$ using the upper- and lower-barred numbers corresponding to c. For example, Figure 4 shows this construction for $\mathsf{Heap}(\mathsf{sort}_c(w_0))$ where $c = [4321657]$ with lower-barred numbers $\underline{5}, \underline{7}$ and upper-barred numbers $\overline{2}, \overline{3}, \overline{4}, \overline{6}$. This algorithm gives us the following lemma.

Lemma 5.3. *Let c be any Coxeter element in A_n with s upper-barred numbers $1 < \overline{q_1} < \cdots < \overline{q_s} < n+1$ and r lower-barred numbers $1 < \underline{p_1} < \cdots < \underline{p_r} < n+1$. Then*

$$\mathcal{R}_c = \Big[(\underline{p_1} - 1)...1\Big] \dots \Big[(\underline{p_r} - 1)...1\Big]\,[n...1]\,[n...(n - \overline{q_s} + 2)] \dots [n...(n - \overline{q_1} + 2)]$$

is a concatenation of n factors, where each factor is a decreasing sequence of consecutive integers.

Example 5.4. *For example, the reduced word $[u] = [1]\,[21]\,[4321]\,[432]$ given in Example 2.2 is the diagonal reading word $\mathcal{R}_{[1243]}$ of the heap diagram in Figure 1. The diagonal reading word of the heap diagram given in Figure 4 is*

$$\mathcal{R}_{[4321657]} = [4321]\,[654321]\,[7654321]\,[76543]\,[765]\,[76]\,[7]\,.$$

5.3 Unimodular equivalence

Note that $\mathcal{R}_c$ is of length $\binom{n+1}{2} = \ell(w_0)$, and write $\mathcal{R}_c = \left[r_1 \dots r_{\ell(w_0)}\right]$. For each $1 \leqslant i \leqslant \ell(w_0)$, define b_i to be the length-i prefix of $\mathcal{R}_c$, that is, $b_i = [r_1 \dots r_i]$. Since $\mathcal{R}_c$ is a labeled linear extension of $\mathsf{Heap}(\mathsf{sort_c}(w_0))$, Proposition 2.4 tells us that it is in the commutation class of $\mathsf{sort_c}(w_0)$, and thus Theorem 3.2 tells us that each b_i is a c-singleton.

Given a c-singleton w, let $f(w)$ be the corresponding order ideal of $\mathsf{Heap}(\mathsf{sort_c}(w_0))$, where f is as defined in Proposition 3.3. Consider the vector in $\mathbb{R}^{\ell(w_0)}$ defined by the indicator function of $f(w)$, following the linear extension given by $\mathcal{R}_c$. Let $o(w)$ denote this vector in reverse order. In particular, note that $o(b_i)$ is the vector whose last i entries are 1s and whose all other entries are 0s.

Example 5.5. *Let c be as in Example 5.1 with $\underline{p_1}, \underline{p_2}, \underline{p_3} = \underline{2}, \underline{5}, \underline{7}$ and $\overline{q_1}, \overline{q_2}, \overline{q_3} = \overline{3}, \overline{4}, \overline{6}$. We have $\mathcal{R}_c = [1\ 4321\ 654321\ 7654321\ 76543\ 765\ 76]$. Therefore $b_4 = s_1 s_4 s_3 s_2 = 25134678$ and its permutation matrix is in Figure 3 (right). We can then compute*

$$\Pi_c(b_4) = (0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,1)\,, \text{ and}$$
$$o(b_4) = (0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,1,1)\,.$$

Lemma 5.6. *Let c be a Coxeter element of A_n. Then the $\left(\binom{n+1}{2} - i + 1\right)^{th}$ entry of the vector $\Pi_c(b_i)$ is 1, and all earlier entries of $\Pi_c(b_i)$ are zero. That is, the matrix whose columns are $\Pi_c(b_i)$ is a lower antidiagonal triangular matrix with 1's along the antidiagonal.*

Theorem 5.7. *Fix a Coxeter element c in A_n. There exists a $\binom{n+1}{2} \times \binom{n+1}{2}$ lower-triangular matrix $\mathcal{U}_c$ with 1's on the main diagonal such that $\mathcal{U}_c \circ \Pi_c(b_i) = o(b_i)$ for all $1 \leqslant i \leqslant \binom{n+1}{2}$. Furthermore, we have $\mathcal{U}_c \circ \Pi_c(w) = o(w)$ for any c-singleton w.*

Corollary 5.8. *The c-Birkhoff polytope $\mathsf{Birk}(c)$ is unimodularly equivalent to the order polytope $\mathcal{O}(\mathsf{Heap}(\mathsf{sort_c}(w_0)))$.*

Proof. This follows from the facts that the projection Π_c preserves lattice points (Theorem 5.2) and that the linear transformation $\mathcal{U}_c$ has determinant 1 (Theorem 5.7). □

Corollary 5.9. *The normalized volume of the c-Birkhoff polyope is equal to the number of longest chains in the corresponding Cambrian lattice.*

Corollary 5.9 recovers, and generalizes, a result of Davis and Sagan in [2].

Remark 5.10. *One might ask whether our result generalizes as follows: If $w \in A_n$ and $[u]$ is a reduced word for w then the order polytope of $\mathsf{Heap}[u]$ is unimodularly equivalent to the convex hull of the permutations corresponding to order ideals of $\mathsf{Heap}[u]$. This is not true in general; for A_4 the reduced words [2123243212] and [3432312343] are counterexamples. It would be interesting to determine when the above statement holds.*

Acknowledgements

We thank the Minnesota Research Workshop in Algebraic Combinatorics 2022 (MRWAC 2022) organizers for bringing us together. This project benefited from the input and feedback of many people including Emily Barnard, Rob Davis, Jesus De Loera, Galen Dorpalen-Barry, Elizabeth Kelley, Jean-Philippe Labbé, Fu Liu, Emily Meehan, Alejandro Morales, Sasha Pevzner, Nathan Reading, Bruce Sagan, Frank Sottile, Jessica Striker, and Sylvester Zhang. We would also like to thank the anonymous reviewers for their feedback and suggestions.

References

[1] R. Bédard. "On commutation classes of reduced words in Weyl groups". *European J. Combin.* **20**.6 (1999), pp. 483–505. DOI.

[2] R. Davis and B. Sagan. "Pattern-avoiding polytopes". *European J. Combin.* **74** (2018), pp. 48–84. DOI.

[3] A. Galambos and V. Reiner. "Acyclic sets of linear orders via the Bruhat orders". *Soc. Choice Welf.* **30**.2 (2008), pp. 245–264. DOI.

[4] C. Hohlweg, C. Lange, and H. Thomas. "Permutahedra and generalized associahedra". *Adv. Math.* **226**.1 (2011), pp. 608–640. DOI.

[5] J.-P. Labbé and C. Lange. "Cambrian acyclic domains: counting c-singletons". *Order* **37**.3 (2020), pp. 571–603. DOI.

[6] OEIS Foundation Inc. (2022). "The On-Line Encyclopedia of Integer Sequences". `http://oeis.org`.

[7] N. Reading. "Cambrian lattices". *Adv. Math.* **205**.2 (2006), pp. 313–353. DOI.

[8] N. Reading. "Clusters, Coxeter-sortable elements and noncrossing partitions". *Trans. Amer. Math. Soc.* **359**.12 (2007), pp. 5931–5958. DOI.

[9] N. Reading. "Sortable elements and Cambrian lattices". *Algebra Universalis* **56**.3-4 (2007), pp. 411–437. DOI.

[10] R. Stanley. *Enumerative combinatorics. Volume 1*. Second. Vol. 49. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2012, pp. xiv+626.

[11] R. P. Stanley. "Two poset polytopes". *Discrete Comput. Geom.* **1**.1 (1986), pp. 9–23. DOI.

[12] J. R. Stembridge. "On the fully commutative elements of Coxeter groups". *J. Algebraic Combin.* **5**.4 (1996), pp. 353–385. DOI.