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Chromatic functions, interval orders, and increasing forests

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Abstract. The chromatic quasisymmetric functions (csf) of Shareshian and Wachs associated to unit interval orders have attracted a lot of interest since their introduction in 2016, both in combinatorics and geometry, because of their relation to the famous Stanley-Stembridge conjecture (1993) and to the topology of Hessenberg varieties, respectively. In the present work we study the csf associated to the larger class of interval orders with no restriction on the length of the intervals. Inspired by an article of Abreu and Nigro, we show that these csf are weighted sums of certain quasisymmetric functions associated to the increasing spanning forests of the associated incomparability graphs. Furthermore, we define quasisymmetric functions that include the unicellular LLT symmetric functions and generalize an identity due to Carlsson and Mellit. Finally we conjecture a formula giving their expansion in the type 1 power sum quasisymmetric functions which should extend a theorem of Athanasiadis.

Keywords: Chromatic quasisymmetric functions, LLT quasisymmetric functions, increasing spanning forests

1 Introduction

In [15] Shareshian and Wachs introduced the *chromatic quasisymmetric function* $\chi_G[X;q]$ associated to every graph G whose vertices are totally ordered, as a sum over proper colorings of G of suitable monomials. At $q = 1$ the series $\chi_G[X;q]$ reduces to the well-known chromatic symmetric function $\chi_G[X;1] = \chi_G(x)$ introduced by Stanley in [17]. A famous conjecture of Stanley and Stembridge ([17, Conjecture 5.1], [18, Conjecture 5.5]) states that if G is the incomparability graph of a $(\mathbf{3}+\mathbf{1})$-free poset, then $\chi_G[X;1]$ is e-positive, i.e. its expansion in the elementary symmetric functions has coefficients in $\mathbb{N}$. Shareshian and Wachs showed (cf. [15, Theorem 4.5]) that if G is the incomparability

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graph of a poset that is both $(\mathbf{3}+\mathbf{1})$-free and $(\mathbf{2}+\mathbf{2})$-free, then $\chi_G[X;q]$ is a symmetric function, and they conjecture that it is e-positive, i.e. its expansion in the elementary symmetric functions has coefficients in $\mathbb{N}[q]$. Thanks to a result of Guay-Paquet [9], it is known that the Shareshian-Wachs conjecture implies the Stanley-Stembridge conjecture. The former problem attracted a lot of attention recently: see e.g. [10, 2, 16, 7, 12, 8].

The posets that are $(\mathbf{3}+\mathbf{1})$-free and $(\mathbf{2}+\mathbf{2})$-free are precisely the *unit interval orders* (see [14]), whose elements are intervals in $\mathbb{R}$ of the same length, and an interval a is smaller than an interval b if all the points of a are strictly smaller than all the points of b. If in such a poset we order the intervals increasingly according to their left endpoints, then we get a total order on them, and now the incomparability graphs of these posets will inherit this total order on the vertices, giving the labelled graphs G involved in the Shareshian-Wachs conjecture. In our article we call these labelled graphs *Dyck graphs*, as they are in a natural bijection with Dyck paths.

If in the definition of unit interval orders we drop the condition on the intervals to have all the same length, then we get the *interval orders*. The incomparability graphs of these posets will be called *interval graphs* in our article, and their chromatic quasisymmetric functions $\chi_G[X;q]$ are the object of our study.

Inspired by the work of Abreu and Nigro [1], given an interval graph G, for every increasing spanning forest F of G we will define a quasisymmetric function $\mathcal{Q}_F^{(G)}$ so that the following formula holds (the statistic $\mathsf{wt}_G(F)$ is essentially the one in [1], while $\mathsf{ISF}(G)$ is the set of increasing spanning forests of G).

Theorem 4.1. *Given an interval graph G on n vertices, we have*

$$\chi_G[X;q] = \sum_{F \in \mathsf{ISF}(G)} q^{\mathsf{wt}_G(F)} \mathcal{Q}_F^{(G)}. \tag{1.1}$$

For every simple graph G with totally ordered vertices we introduce the quasisymmetric function $\mathrm{LLT}_G[X;q]$, analogous to $\chi_G[X;q]$ but defined as a sum over all (not necessarily proper) colorings of G of suitable monomials.

The main result of this article is the following theorem, stated in plethystic notation (ρ and ψ are well-known involutions of the algebra QSym of quasisymmetric functions).

Theorem 5.1. *Given G an interval graph on n vertices, we have*

$$(1-q)^n \rho\left(\psi\chi_G\left[X\frac{1}{1-q}\right]\right) = \mathrm{LLT}_G[X;q].$$

This result extends the identity in [6, Proposition 3.5] proved by Carlsson and Mellit when G is a Dyck graph.

In [5] the authors study a family of quasisymmetric functions that they call *type 1 quasisymmetric power sums*, denoted Ψ_α. Actually $\{\Psi_\alpha \mid \alpha \text{ composition}\}$ is a basis of QSym that refines the power symmetric function basis.

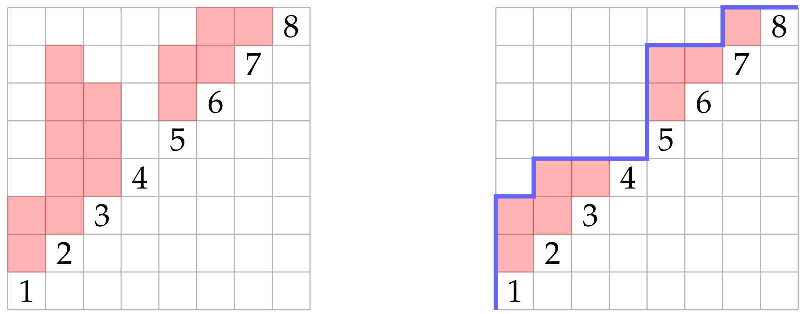

Figure 1: The interval graph $G = ([8], \{(1,2),(1,3),(2,3),(2,4),(2,5),(2,6),(2,7),(3,4),(3,5),(3,6),(5,6),(5,7),(6,7),(6,8),(7,8)\})$, on the left. On the right, the Dyck graph $G_2 = ([8], \{(1,2),(1,3),(2,3),(2,4),(3,4),(5,6),(5,7),(6,7),(7,8)\})$.

We state the following conjecture which is supposed to provide an extension of the formula proved by Athanasiadis in [4].

Conjecture 6.1. *For any interval graph G on n vertices we have*

$$\rho\psi\chi_G[X;q] = \sum_{\alpha \vDash n} \frac{\Psi_\alpha}{z_\alpha} \sum_{\sigma \in \mathcal{N}_{G,\alpha}} q^{\widetilde{\mathsf{inv}}_G(\sigma)}.$$

2 Preliminaries

For every $n \in \mathbb{Z}_{>0} := \{1,2,3,\dots\}$ we will use the notation $[n] := \{1,2,\dots,n\}$.

2.1 Interval graphs

In this abstract a graph will always be simple, i.e. no loops and no multiple edges.

In our work a (*labelled*) *graph* $G = ([n], E)$ will be called *interval* if whenever $\{i,j\} \in E$ and $i < j$, then $\{i,k\} \in E$ for every $i < k \leq j$. We will call $\mathcal{IG}_n$ the set of all interval graphs with vertex set $[n]$.

We can represent an interval graph $G = ([n], E)$ in the following way: in a $n \times n$ square grid we order the columns from left to right with numbers $1,2,\dots,n$ and similarly the rows from bottom to top; then we color the cells $\{i,j\} \in E$ with $i < j$. See Figure 1, on the left, for an example[1].

Notice that in these pictures we simply obtain a bunch of (possibly empty) colored columns, starting just above the diagonal cells. Hence clearly there are $n!$ interval graphs on n vertices.

[1]Sometimes we denote by (i,j) an edge $\{i,j\} \in E$ with $i < j$, like in the caption of Figure 1.

Given an interval graph G on n vertices, we can consider its *flipped*, obtained from G by replacing each edge $\{i,j\}$ with an edge $\{n+1-i, n+1-j\}$: in terms of pictures, this corresponds to flip the picture of G around the line $y=-x$.

An interval graph G on n vertices such that its flipped is still an interval graph is called a *Dyck graph*. The explanation of the name is obvious, since the picture of a Dyck graph determines a Dyck path: see the graph G_2 in Figure 1, on the right (the Dyck path is the thicker one).

It turns out that the interval graphs are the incomparability graphs of certain posets called *interval orders* (hence their name).

Given a (naturally labelled) poset $P=([n],<_P)$, its *incomparability (labelled) graph* $\mathrm{Inc(P)}=([n],E_P)$ is defined by setting $\{i,j\}\in E_P$ if and only if i and j are incomparable in P.

Let $\mathcal{I}$ be the set of all bounded closed intervals of $\mathbb{R}$, and given $I=[a,b]$ and $J=[c,d]$ we set $I \prec J$ if and only if $b<c$. Clearly $(\mathcal{I},\prec)$ is a poset. Any subposet of $(\mathcal{I},\prec)$ is called an *interval order*.

2.2 Symmetric and quasisymmetric functions

In this section we recall a few basic facts of symmetric and quasisymmetric functions, mainly to fix the notation.

Given a composition $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_k)$ of $n\in\mathbb{N}$ (denoted $\alpha\vDash n$), we denote its *size* by $|\alpha|=\sum_i \alpha_i = n$ and its *length* by $\ell(\alpha)=k$. For brevity, sometimes we will use the *exponential notation*, so that for example we will write (1^4) for $(1,1,1,1)$, or $(1^3,2^2,1,3)$ for $(1,1,1,2,2,1,3)$.

To a composition $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_k)$ of n we associate a set $\mathsf{set}(\alpha)=\mathsf{set}_n(\alpha)\subseteq[n-1]$:

$$\mathsf{set}(\alpha)=\{\alpha_1,\alpha_1+\alpha_2,\dots,\alpha_1+\cdots+\alpha_{k-1}\}.$$

Viceversa, to a subset $S\subseteq[n-1]$ whose elements are $i_1<i_2<\cdots<i_k$ we associate the composition

$$\mathsf{comp}(S)=\mathsf{comp}_n(S)=(i_1,i_2-i_1,i_3-i_2,\dots,i_k-i_{k-1},n-i_k)\vDash n.$$

Notice that the functions set_n and comp_n are inverse of each others.

Given a composition $\alpha\vDash n$, $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_k)$, its *reversal* is $\alpha^r=(\alpha_k,\alpha_{k-1},\dots,\alpha_1)$, its *complement* is $\alpha^c=\mathsf{comp}([n-1]\setminus\mathsf{set}(\alpha))$, and its *transpose* is $\alpha^t=(\alpha^r)^c=(\alpha^c)^r$.

We denote by QSym the algebra of quasisymmetric functions in the variables $x_1,x_2,\dots$ and coefficients in $\mathbb{Q}(q)$, where q is a variable.

Given $n\in\mathbb{N}$ and $S\subseteq[n-1]$, we define the *fundamental (Gessel) quasisymmetric function* $L_{n,S}$ as

$$L_{n,S} := \sum_{\substack{i_1\leq i_2\leq\cdots\leq i_n\\ j\in S\Rightarrow i_j\neq i_{j+1}}} x_{i_1}x_{i_2}\cdots x_{i_n}$$

and for every $\alpha \vDash n$, we define $L_\alpha := L_{n,\mathsf{set}(\alpha)}$.

It is well known that $\{L_\alpha \mid \alpha \text{ composition}\}$ is a basis of QSym.

We have the following three involutions of QSym: ψ : QSym $\to$ QSym, defined by $\psi(L_\alpha) := L_{\alpha^c}$, ρ : QSym $\to$ QSym defined by $\rho(L_\alpha) = L_{\alpha^r}$, and ω : QSym $\to$ QSym defined by $\omega(L_\alpha) = L_{\alpha^t}$.

We will use the *plethysm* of quasisymmetric functions: cf. [11].

2.3 Colorings and (co)inversions

Given $n \in \mathbb{Z}_{>0}$, let $G = ([n], E)$ be a (simple) graph.

A *coloring* of G is simply a function $\kappa : [n] \to \mathbb{Z}_{>0}$. We call $\mathsf{C}(G)$ the set of colorings of G. We can and will identify a coloring $\kappa \in \mathsf{C}(G)$ with the word $\kappa(1)\kappa(2)\cdots\kappa(n)$ in the alphabet $\mathbb{Z}_{>0}$.

A *coloring* of G is called *proper* if $\{i, j\} \in E$ implies $\kappa(i) \neq \kappa(j)$. We call $\mathsf{PC}(G)$ the set of proper colorings of G. Notice that with the above identifications we always have that the symmetric group $\mathfrak{S}_n$ is a subset of $\mathsf{PC}(G)$.

Given $\kappa \in \mathsf{C}(G)$ a *G-inversion* of κ is a pair (i, j) with $\{i, j\} \in E$, $i < j$ and $\kappa(i) > \kappa(j)$. Similarly, a *G-coinversion* of κ is a pair (i, j) with $\{i, j\} \in E$, $i < j$ and $\kappa(i) < \kappa(j)$. We denote by $\mathsf{Inv}_G(\kappa)$, respectively $\mathsf{CoInv}_G(\kappa)$, the set of G-inversions, respectively G-coinversions, of κ. Let us denote by $\mathsf{Inv}(G)$ the (finite) set of possible sets of G-inversions of a coloring of G: in other words $\mathsf{Inv}(G) := \{\mathsf{Inv}_G(\sigma) \mid \sigma \in \mathfrak{S}_n\}$. Similarly, we set $\mathsf{CoInv}(G) := \{\mathsf{CoInv}_G(\sigma) \mid \sigma \in \mathfrak{S}_n\}$.

We can now set for every $\kappa \in \mathsf{C}(G)$

$$\mathsf{inv}_G(\kappa) := |\mathsf{Inv}_G(\kappa)| \quad \text{and} \quad \mathsf{coinv}_G(\kappa) := |\mathsf{CoInv}_G(\kappa)|.$$

Example 2.1. Consider the graph G in Figure 1, and $\sigma = 31852647 \in \mathfrak{S}_8 \subseteq \mathsf{PC}(G)$. Then

$$\begin{aligned}\mathsf{Inv}_G(\sigma) &= \{(1,2),(3,4),(3,5),(3,6),(6,7)\} \text{ and}\\ \mathsf{CoInv}_G(\sigma) &= \{(1,3),(2,3),(2,4),(2,5),(2,6),(2,7),(5,6),(5,7),(6,8),(7,8)\},\end{aligned}$$

so that $\mathsf{inv}_G(\sigma) = 5$ and $\mathsf{coinv}_G(\sigma) = 10$.

Let $\phi : \mathsf{C}(G) \to \mathfrak{S}_n$ be the *standardization* from left to right: given $\kappa(1)\kappa(2)\cdots\kappa(n)$, if $c_1 < c_2 < \cdots < c_k$ is the ordered set of values $\kappa(i)$, then $\phi(\kappa)$ is the permutation obtained by replacing the d_1 occurrences of c_1 with the numbers $1, 2, \ldots, d_1$ from left to right, then the d_2 occurrences of c_2 with the numbers $d_1 + 1, d_1 + 2, \ldots, d_1 + d_2$ from left to right, and so on. For example $\phi(3253353) = 2163475$.

Remark 2.2. Observe that for any $\kappa \in \mathsf{C}(G)$,

$$\mathsf{CoInv}_G(\kappa) \subseteq \mathsf{CoInv}_G(\phi(\kappa)) \quad \text{and} \quad \mathsf{Inv}_G(\kappa) = \mathsf{Inv}_G(\phi(\kappa)).$$

The asymmetry is due to the fact that the standardization ϕ is from left to right. But observe that if $\kappa \in \mathsf{PC}(G)$, then in fact $\mathsf{CoInv}_G(\kappa) = \mathsf{CoInv}_G(\phi(\kappa))$ as well.

2.4 Interval graphs and (co)inversions

Given $n \in \mathbb{Z}_{>0}$, let $G = ([n], E)$ be an interval graph.

Given $\tau \in \mathfrak{S}_n$, set

$$\mathsf{Des}_G(\tau) := \{i \in [n-1] \mid \tau(i) > \tau(i+1) \text{ or } \{\tau(i), \tau(i+1)\} \in E\} \subseteq [n-1].$$

The next proposition is sort of implicit in the work of Shareshian and Wachs [15].

Proposition 2.3. *Given $G = ([n], E)$ an interval graph, for every $S \in \mathsf{Inv}(G)$ we have*

$$\sum_{\substack{\kappa \in \mathsf{PC}(G) \\ \mathsf{Inv}_G(\kappa) = S}} q^{\mathsf{inv}_G(\kappa)} x_\kappa = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \mathsf{Inv}_G(\sigma) = S}} q^{\mathsf{inv}_G(\sigma)} L_{n, \mathsf{Des}_G(\sigma^{-1})} = q^{|S|} \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \mathsf{Inv}_G(\sigma) = S}} L_{n, \mathsf{Des}_G(\sigma^{-1})},$$

and for every $S \in \mathsf{CoInv}(G)$ we have

$$\sum_{\substack{\kappa \in \mathsf{PC}(G) \\ \mathsf{CoInv}_G(\kappa) = S}} q^{\mathsf{coinv}_G(\kappa)} x_\kappa = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \mathsf{CoInv}_G(\sigma) = S}} q^{\mathsf{coinv}_G(\sigma)} L_{n, \mathsf{Des}_G(\sigma^{-1})} = q^{|S|} \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \mathsf{CoInv}_G(\sigma) = S}} L_{n, \mathsf{Des}_G(\sigma^{-1})}.$$

3 Increasing spanning forests and quasisymmetric functions

Given a graph $G = ([n], E)$, we say that a subgraph $F \subseteq G$ is a *spanning forest* if F is a forest on the vertices $[n]$. In this case, the connected components are labelled trees, with the vertex set contained in $[n]$. Given such a tree T, we call $root(T)$ its minimal vertex. Then T is called *increasing* if in the paths stemming from $root(T)$ the other vertices appear in increasing order.

A spanning forest F of a graph $G = ([n], E)$ is called *increasing* if all its connected components are increasing trees. In this case, we think of F as the ordered collection $F = (T_1, T_2, \dots, T_k)$, where the T_i are its connected components, ordered so that

$$root(T_1) < root(T_2) < \cdots < root(T_k).$$

E.g. the forest $F = (T_1, T_2)$ in Figure 2, with[1] $T_1 = (V(T_1), E(T_1)) = (\{1,3\}, \{(1,3)\})$ and $T_2 = (V(T_2), E(T_2)) = (\{2,4,5,6,7,8\}, \{(2,4), (2,5), (2,6), (5,7), (6,8)\})$, is an increasing spanning forest of the graph G in Figure 1, .

We denote by $\mathsf{ISF}(G)$ the set of increasing spanning forests of G.

Given a graph $G = ([n], E)$ and an $F \in \mathsf{ISF}(G)$, $F = (T_1, T_2, \dots, T_k)$, we say that a pair (u, v) with $u, v \in [n]$ is a *G-inversion* of F if $u \in V(T_i)$, $v \in V(T_j)$, $i > j$ and $(u, v) \in E$ (so that $u < v$). Given an edge $(u, v) \in E(T_i)$ of T_i we define its *weight* in G,

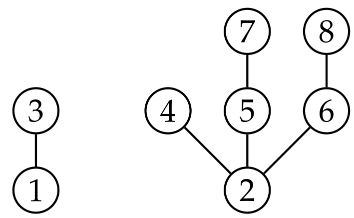

Figure 2: An example of increasing spanning forest of the graph G in Figure 1.

denoted $\mathsf{wt}_G((u,v))$, to be the number of $w \in V(T_i)$ vertex of T_i such that $u \leq w < v$ and $(w,v) \in E(G)$. So for every tree T_i we define its *weight* in G as

$$\mathsf{wt}_G(T_i) = \sum_{(u,v)\in E(T_i)} \mathsf{wt}_G((u,v))$$

and finally the *weight* of F (in G) as

$$\mathsf{wt}_G(F) := \#\{G\text{-inversions of } F\} + \sum_{i=1}^{k} \mathsf{wt}_G(T_i).$$

Example 3.1. The forest $F = (T_1, T_2)$ in Figure 2 is an increasing spanning forest of the graph G in Figure 1: we observe that its only G-inversion is $(2,3)$ (as 3 occurs in T_1, 2 occurs in T_2 and $(2,3) \in E$), $\mathsf{wt}_G(T_1) = \mathsf{wt}_G((1,3)) = 1$, and

$$\begin{aligned}\mathsf{wt}_G(T_2) &= \mathsf{wt}_G((2,4)) + \mathsf{wt}_G((2,5)) + \mathsf{wt}_G((2,6)) + \mathsf{wt}_G((5,7)) + \mathsf{wt}_G((6,8)) \\ &= 1 + 1 + 2 + 2 + 2 = 8,\end{aligned}$$

so that $\mathsf{wt}_G(F) = 1 + 1 + 8 = 10$.

Let $G = ([n], E)$ be an interval graph, i.e. $G \in \mathcal{IG}_n$. We define a function $\Phi_G : \mathsf{PC}(G) \to \mathsf{ISF}(G)$ via Algorithm 1 and Algorithm 2.

Algorithm 1 Algorithm defining the function $\mathsf{getW}(G, v, S, \kappa)$

Input: A graph $G = ([n], E)$, $S \subset [n]$, $v \in [n] \setminus S$, and $\kappa \in \mathsf{PC}(G)$

Output: W $\triangleright$ It will be $W \subseteq S \cup \{v\}$

```
W ← {v}
for w ∈ S do
    if {u ∈ W | u < w, (u,w) ∈ E and κ(u) < κ(w)} ≠ ∅ then
        W ← W ∪ {w}
    end if
end for
```

Algorithm 2 The algorithm defining the function $\Phi_G(\kappa)$

Input: A graph $G = ([n], E)$ and $\kappa \in \mathsf{PC}(G)$
Output: $F = (T_1, T_2, \dots)$ ▷ It will be $F \in \mathsf{ISF}(G)$

```
S ← [n]
F ← ( )                                              ▷ Empty list
while S ≠ ∅ do
    v ← min(S)
    S ← S \ {v}
    T = (V(T), E(T)) ← ({v}, ∅)                       ▷ The tree we are going to build
    W ← getW(G, v, S, κ)                              ▷ Defined in Algorithm 1
    for i ∈ {2, ..., #W} do
        L ← {u ∈ V(T) | u < W_i and (u, W_i) ∈ E}      ▷ W = {W_1 < W_2 < ··· < W_#W}
        r ← #{u ∈ L | (u, W_i) ∈ E and κ(u) < κ(W_i)}
        T ← (V(T) ∪ {W_i}, E(T) ∪ {(L_{#L−r+1}, W_i)}) ▷ L = {L_1 < L_2 < ··· < L_#L}
        S ← S \ {W_i}
    end for
    Append T to the right of F
end while
```

Proposition 3.2. *Given a graph $G = ([n], E)$, the Algorithm 2 defines a function $\Phi_G : \mathsf{PC}(G) \to \mathsf{ISF}(G)$.*

The first nontrivial property of the function Φ_G is its surjectivity.

Theorem 3.3. *Let $G = ([n], E)$ a graph. There exists an explicit function $f_G : \mathsf{ISF}(G) \to \mathfrak{S}_n \subset \mathsf{PC}(G)$ such that $\Phi_G \circ f_G(F) = F$ for every $F \in \mathsf{ISF}(G)$. In particular Φ_G is surjective, f_G is injective, and $\mathsf{ISF}(G) = \{\Phi_G(\sigma) \mid \sigma \in \mathfrak{S}_n\} = \{\Phi_G(\sigma) \mid \sigma \in f_G(\mathsf{ISF}(G))\}$.*

When $G = ([n], E)$ is an interval graph, Φ_G has also the following property.

Proposition 3.4. *Given an interval graph $G = ([n], E)$, the function $\Phi_G : \mathsf{PC}(G) \to \mathsf{ISF}(G)$ defined by Algorithm 2 is such that for every $\kappa, \kappa' \in \mathsf{PC}(G)$, $\Phi_G(\kappa) = \Phi_G(\kappa')$ if and only if $\mathsf{CoInv}_G(\kappa) = \mathsf{CoInv}_G(\kappa')$. Moreover $\mathsf{wt}_G(\Phi_G(\kappa)) = \mathsf{coinv}_G(\kappa)$ for every $\kappa \in \mathsf{PC}(G)$.*

We are now ready to define quasisymmetric functions associated to increasing spanning forests of interval graphs.

Definition 3.5. Given an interval graph $G = ([n], E)$, and given $F \in \mathsf{ISF}(G)$, we define the formal power series

$$\mathcal{Q}_F^{(G)} = \mathcal{Q}_F^{(G)}[X] := \sum_{\substack{\kappa \in \mathsf{PC}(G) \\ \Phi_G(\kappa) = F}} x_\kappa.$$

We have the following fundamental formula.

Theorem 3.6. *Given an interval graph $G = ([n], E) \in \mathcal{IG}_n$, and given $F \in \mathsf{ISF}(G)$, we have*

$$\mathcal{Q}_F^{(G)} = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \mathsf{CoInv}_G(\sigma) = \mathsf{CoInv}_G(F)}} L_{n, \mathsf{Des}_G(\sigma^{-1})}, \tag{3.1}$$

where

$$\mathsf{CoInv}_G(F) := \mathsf{CoInv}_G(f_G(F)).$$

4 Interval orders, chromatic functions and LLT

Given any simple graph $G = ([n], E)$, Shareshian and Wachs defined in [15] its *chromatic quasisymmetric function* as

$$\chi_G[X; q] := \sum_{\kappa \in \mathsf{PC}(G)} q^{\mathsf{coinv}_G(\kappa)} x_\kappa.$$

The following theorem is a direct consequence of Proposition 3.4 and Theorem 3.3.

Theorem 4.1. *Given an interval graph $G = ([n], E)$, we have*

$$\chi_G[X; q] = \sum_{F \in \mathsf{ISF}(G)} q^{\mathsf{wt}_G(F)} \mathcal{Q}_F^{(G)}. \tag{4.1}$$

Example 4.2. For $G = ([3], \{(1,2), (1,3)\})$, the increasing spanning forests of G are

$$F_1 = (([3], \{(1,2), (1,3)\})), \quad F_2 = ((\{1,3\}, \{(1,3)\}), (\{2\}, \varnothing)),$$
$$F_3 = ((\{1,2\}, \{(1,2)\}), (\{3\}, \varnothing)), \quad F_4 = ((\{1\}, \varnothing), (\{2\}, \varnothing), (\{3\}, \varnothing)),$$

and we compute

$$\mathsf{wt}_G(F_1) = 2, \quad \mathsf{wt}_G(F_2) = \mathsf{wt}_G(F_3) = 1, \quad \mathsf{wt}_G(F_4) = 0,$$

$$\mathcal{Q}_{F_1}^{(G)} = L_{(1,2)} + L_{(1^3)}, \quad \mathcal{Q}_{F_2}^{(G)} = \mathcal{Q}_{F_3}^{(G)} = L_{(1^3)}, \quad \mathcal{Q}_{F_4}^{(G)} = L_{(2,1)} + L_{(1^3)},$$

hence finally

$$\chi_G[X; q] = L_{(2,1)} + q^2 L_{(1,2)} + (1 + 2q + q^2) L_{(1^3)}.$$

The following corollary is a reformulation of [15, Theorem 3.1] in our cases, and it follows immediately from Theorems 4.1 and 3.6.

Corollary 4.3. *Given an interval graph G on n vertices, we have*

$$\chi_G[X; q] = \sum_{\sigma \in \mathfrak{S}_n} q^{\mathsf{coinv}_G(\sigma)} L_{n, \mathsf{Des}_G(\sigma^{-1})}. \tag{4.2}$$

Given any simple graph $G = ([n], E) \in \mathcal{IG}_n$, we define its *LLT quasisymmetric function* as

$$\mathrm{LLT}_G[X; q] := \sum_{\kappa \in \mathsf{C}(G)} q^{\mathsf{inv}_G(\kappa)} x_\kappa.$$

The following formula is an immediate consequence of Proposition 2.3.

Theorem 4.4. *Given any interval graph $G = ([n], E)$, we have*

$$\mathrm{LLT}_G[X; q] = \sum_{\sigma \in \mathfrak{S}_n} q^{\mathsf{inv}_G(\sigma)} L_{n, \mathsf{Des}(\sigma^{-1})}.$$

The name LLT of these quasisymmetric functions comes from the following well-known facts: when G is a Dyck graph, $\mathrm{LLT}_G[X; q]$ is a symmetric function, and in fact it is a so called *unicellular LLT symmetric functions* (see e.g. [3, Section 3]). In this case the formula in Theorem 4.4 is well known (e.g. it can be deduced from [13, Theorem 8.6]).

5 The main identity

Recall from Section 2.2 the involutions ψ and ρ. We use the plethysm of quasisymmetric functions: cf. [11].

Theorem 5.1. *Let $G = ([n], E)$ be an interval graph. Then*

$$(1-q)^n \rho\left(\psi \chi_G\left[X \frac{1}{1-q}\right]\right) = \mathrm{LLT}_G[X; q]. \tag{5.1}$$

Remark 5.2. This is really an extension of [6, Proposition 3.5]. Indeed, when G is a Dyck graph, $\chi_G[X; q]$ is symmetric (by [15, Theorem 4.5]), the plethysm reduces to the usual plethysm of symmetric functions (cf. [11]), ρ fixes the symmetric functions while ψ gives the usual ω involution of symmetric functions, and $\mathrm{LLT}_G[X; q]$ is precisely the unicellular LLT symmetric function corresponding to the Dyck graph G, so, our (5.1) is just a rewriting of [6, Proposition 3.5].

6 Expansions in the Ψ_α

In [5] the authors study a family of quasisymmetric functions that they call *type 1 quasisymmetric power sums*, and they denote Ψ_α. Actually $\{\Psi_\alpha \mid \alpha \text{ composition}\}$ is a basis of QSym, and these quasisymmetric functions refine the power symmetric functions, i.e. for any partition $\lambda \vdash n$

$$\sum_{\substack{\alpha \vDash n \\ \lambda(\alpha) = \lambda}} \Psi_\alpha = p_\lambda \ , \tag{6.1}$$

where $\lambda(\alpha)$ is the unique partition obtained by rearranging in weakly decreasing order the parts of α, and the $p_\lambda = p_{\lambda_1}p_{\lambda_2}\cdots$ are the usual *power symmetric functions*.

Given $G = ([n], E)$ a graph and $\sigma \in \mathfrak{S}_n$ a permutation, we say that $r \in [n]$ is a *left-to-right G-maximum* if for every $s \in [r-1]$ we have $\sigma(s) < \sigma(r)$ and $\{\sigma(s), \sigma(r)\} \notin E$. Notice that 1 is always a left-to-right G-maximum, that we call *trivial*. We set

$$\widetilde{\mathsf{inv}}_G(\sigma) := \{\{\sigma(i), \sigma(j)\} \in E \mid i < j \text{ and } \sigma(i) > \sigma(i+1)\},$$

and

$$\widetilde{\mathsf{Des}}_G(\sigma) := \{i \in [n-1] \mid \sigma(i) > \sigma(i+1) \text{ and } \{\sigma(i), \sigma(i+1)\} \notin E\}.$$

We say that $i \in [n-1]$ is a *G-descent* if $i \in \widetilde{\mathsf{Des}}_G(\sigma)$.

Given a composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \vDash n$, let $\mathcal{N}_{G,\alpha}$ be the set of $\sigma \in \mathfrak{S}_n$ such that if we break $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ into contiguous segments of lengths $\alpha_1, \alpha_2, \ldots, \alpha_k$, each contiguous segment has neither a G-descent nor a nontrivial left-to-right G-maximum.

Given a composition α, define $z_\alpha := z_{\lambda(\alpha)}$, where, as usual, for every partition $\lambda \vdash n$, if m_i denotes the number of parts of λ equal to i, then $z_\lambda := \prod_{i=1}^n m_i! \cdot i^{m_i}$.

Finally, recall the involution $\omega : \mathrm{QSym} \to \mathrm{QSym}$ from Section 2.2.

We state our conjecture.

Conjecture 6.1. *For any interval graph $G = ([n], E)$ we have*

$$\omega\chi_G[X; q] = \sum_{\alpha \vDash n} \frac{\Psi_\alpha}{z_\alpha} \sum_{\sigma \in \mathcal{N}_{G,\alpha}} q^{\widetilde{\mathsf{inv}}_G(\sigma)}.$$

This conjecture should generalize the following formula, proposed by Shareshian and Wachs [15, Conjecture 7.6] and later proved by Athanasiadis [4].

Theorem 6.2. *For any Dyck graph $G = ([n], E)$ we have*

$$\omega\chi_G[X; q] = \sum_{\lambda \vdash n} \frac{p_\lambda}{z_\lambda} \sum_{\sigma \in \mathcal{N}_{G,\lambda}} q^{\widetilde{\mathsf{inv}}_G(\sigma)}.$$

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