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Shuffle theorems and sandpiles

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Abstract. We provide an explicit description of the recurrent configurations of the sandpile model on a family of graphs $\widehat{G}_{\mu,\nu}$, which we call *clique-independent* graphs, indexed by two compositions μ and ν. Moreover, we define a *delay* statistic on these configurations, and we show that, together with the usual *level* statistic, it can be used to provide a new combinatorial interpretation of the celebrated *shuffle theorem* of Carlsson and Mellit. More precisely, we will see how to interpret the polynomials $\langle \nabla e_n, e_\mu h_\nu \rangle$ in terms of these configurations.

Keywords: Shuffle theorem, sandpile model, recurrent configurations

1 Introduction

1.1 Shuffle theorem

The *shuffle theorem* of Carlsson and Mellit [4] is a recent breakthrough that provided a positive solution to a long-standing conjecture about a combinatorial formula for the Frobenius characteristic of the so-called diagonal harmonics. More precisely, this theorem provides the monomial expansion of the symmetric function ∇e_n, where e_n is the elementary symmetric function of degree n in the variables $x_1, x_2, \dots$, and ∇ is the famous *nabla* operator introduced by Bergeron and Garsia in the 90's. In this formula, to each *labelled Dyck path* of size n corresponds a monomial, where the variables $x_1, x_2, \dots$ keep track of the labels, while the variables q and t keep track of the bistatistic (dinv, area).

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In [14] Loehr and Remmel provided an alternative combinatorial interpretation of the same symmetric function in terms of the same objects, but using the bistatistic (area, pmaj). In particular, they showed bijectively that the two combinatorial formulas coincide. In the present article we show that this last combinatorial formula has a natural interpretation in terms of the sandpile model.

1.2 Sandpile model

The *(abelian) sandpile model* is a combinatorial dynamical system on graphs first introduced by Bak, Tang and Wiesenfeld [3] in the context of "self-organized criticality" in statistical mechanics. The sandpile model (and variants of it) have found applications in a wide variety of mathematical contexts including enumerative combinatorics, tropical geometry, and Brill–Noether theory, among others: see [13] for a nice introductory monograph. *In the present article we only consider the sandpile model with a sink.*

A well-known link between the combinatorics of this dynamical system and the one of the underlying graph is given by the so-called *recurrent configurations* (see Definition 9). For example, the recurrent configurations of the sandpile model are in bijection with the spanning trees of the graph (see e.g. [7]).

If the underlying graph presents some symmetries, then it is natural to look at the recurrent configurations "modulo" those symmetries. For example, for the complete graph we can identify recurrent configurations that are the same up to a permutation of the vertices (not moving the sink): perhaps not surprisingly, we still get an interesting combinatorics, as in this case we find Catalan many such "sorted" configurations.

More formally, consider the sandpile model on a graph G, and let $\mathsf{Aut}(G)$ be the automorphism group of G. Consider a subgroup Γ of the stabilizer of the sink. Now Γ acts naturally on the set $\mathsf{Rec}(G)$ of recurrent configurations: we are interested in the orbits of this action, that we will call *sorted recurrent configurations*.

1.3 Main result

We will consider an explicit family of graphs $\widehat{G}_{\mu,\nu}$ indexed by pairs of compositions μ and ν. For such a graph $\widehat{G}_{\mu,\nu}$ we will look at a subgroup Γ of its automorphism group that will be isomorphic to the Young subgroup $\mathfrak{S}_\mu \times \mathfrak{S}_\nu$ of the symmetric group $\mathfrak{S}_n$, where $n = |\mu| + |\nu|$. We denote by $\mathsf{SortRec}(\mu,\nu)$ the set of the corresponding sorted recurrent configurations of $\widehat{G}_{\mu,\nu}$.

For every recurrent configuration κ of $\widehat{G}_{\mu,\nu}$, we will define a new statistic, called the *delay* of κ (denoted $\mathsf{delay}(\kappa)$), which we will couple with the usual *level* statistic (denoted $\mathsf{level}(\kappa)$). To state our main result, we need a few more definitions.

Given a composition $\mu = (\mu_1, \mu_2, \dots)$, we denote by e_μ the product $e_{\mu_1} e_{\mu_2} \cdots$, and similarly $h_\mu = h_{\mu_1} h_{\mu_2} \cdots$, where h_n is the complete homogeneous symmetric function of degree n. Finally, we denote by $\langle -,- \rangle$ the Hall scalar product on symmetric functions.

Theorem 1. *For every pair of compositions μ, ν such that $n = |\mu| + |\nu|$ we have*

$$\langle \nabla e_n, e_\mu h_\nu \rangle = \sum_{\kappa \in \mathsf{SortRec}(\mu,\nu)} q^{\mathsf{level}(\kappa)} t^{\mathsf{delay}(\kappa)}.$$

Notice that for $\mu = \varnothing$, the coefficient $\langle \nabla e_n, h_\nu \rangle$ is simply the coefficient of $x^\nu = x_1^{\nu_1} x_2^{\nu_2} \cdots$ in ∇e_n, hence this formula gives in particular a new combinatorial interpretation of the monomial expansion of the symmetric function ∇e_n in terms of the sandpile model.

The idea of the proof is to show that the sorted recurrent configurations with the bistatistic (level, delay) correspond bijectively to the labelled Dyck paths predicted by the shuffle theorem with the bistatistic (area, pmaj).

1.4 Comments

Theorem 1 extends several previous results in the literature: the case $\widehat{G}_{\varnothing,(n)}$ was already worked out in [8], (a slight modification of) the case $\widehat{G}_{(m,n-m),\varnothing}$ already appears in [1, 11], while the case $\widehat{G}_{(m),(n-m)}$ is dealt with in [10].

Other articles in which sorted recurrent configurations make their appearance are for example [2] and [9]. It should be noticed that the works [8] and [9] inspired the results in [6] and [5] respectively, which belong to tropical geometry and Brill-Noether theory.

We hope that the findings in the present article motivate further investigation of sorted recurrent configurations, and their relation to other parts of mathematics.

2 Combinatorics of the shuffle theorem

For every $n \in \mathbb{N}$, we set $[n] := \{1, 2, \ldots, n\}$.

The pmaj statistic was first introduced in [14]. The area statistic is classical.

Definition 1. A *Dyck path* of size n is a lattice path going from $(0,0)$ to (n,n), using only north and east steps and staying weakly above the line $x = y$ (also called the *main diagonal*). A *labelled Dyck path* is a Dyck path whose vertical steps are labelled with (not necessarily distinct) positive integers such that, when placing the labels in the square to the right of its step, the labels appearing in each column are strictly increasing from bottom to top. For us, a *parking function*[1] of size n is a labelled Dyck path of size n whose labels are precisely the elements of $[n]$. See Figure 1 for an example.

The set of all parking functions of size n is denoted by $\mathsf{PF}(n)$.

[1]These are in bijection with the functions $f\colon [n] \to [n]$ such that $\#\{1 \leq j \leq n \mid f(j) \geq i\} \leq n + 1 - i$, by defining $f(i)$ to be the column of the label i.

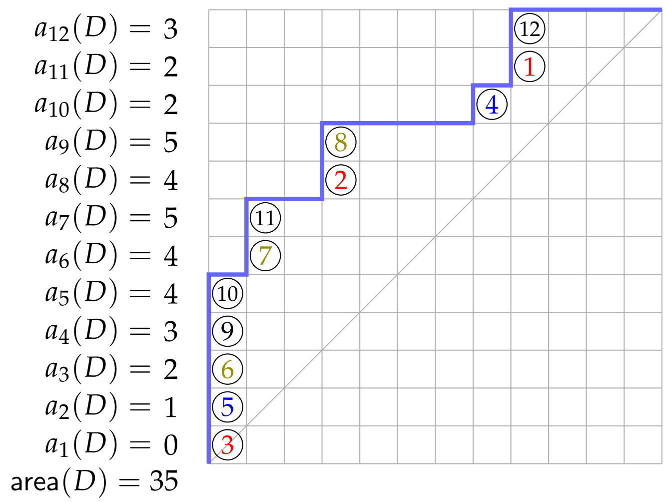

Figure 1: An element D of $\mathsf{PF}((4,3);(3,2))$.

Definition 2. Given $D \in \mathsf{PF}(n)$, we define its *area word* to be the string of integers $a(D) = a_1(D) \cdots a_n(D)$ where $a_i(D)$ is the number of whole squares in the i-th row (from the bottom) between the path and the main diagonal. We define the *area* of D as

$$\mathsf{area}(D) := \sum_{i=1}^{n} a_i(D).$$

Example 1. The area word of the path in Figure 1 is 012344545223 and its area is 35.

To introduce the other statistic, we need a couple of definitions.

Definition 3. Let $a_1 a_2 \cdots a_k$ be a string of integers. We define its *descent set*

$$\mathsf{Des}(a_1 a_2 \cdots a_k) := \{1 \leq i \leq k-1 \mid a_i > a_{i+1}\}$$

and its *major index* $\mathsf{maj}(a_1 a_2 \cdots a_k)$ as the sum of the elements of the descent set.

Definition 4. Let $D \in \mathsf{PF}(n)$. We define its *parking word* $p(D)$ as follows.

Let C_1 be the multiset containing the labels appearing in the first column of D, and let $p_1(D) := \max C_1$. At step i, let C_i be the multiset obtained from C_{i-1} by removing $p_{i-1}(D)$ and adding all the labels in the i-th column of the D; let

$$p_i(D) := \max \{x \in C_i \mid x \leq p_{i-1}(D)\}$$

if this last set is non-empty, and

$$p_i(D) := \max C_i$$

otherwise. We finally define the parking word of D as $p(D) := p_1(D) \cdots p_n(D)$.

Definition 5. We define the statistic *pmaj* on $D \in \mathsf{PF}(n)$ as

$$\mathsf{pmaj}(D) := \mathsf{maj}(p_n(D) \cdots p_1(D)).$$

Example 2. For example, the parking word of the parking function D in Figure 1 is[2] $\overline{10}976532\overline{11}841\overline{12}$. In fact, we have $C_1 = \{3,4,5,6,9,10\}$, $C_2 = \{3,4,5,6,9,7,11\}$, $C_3 = \{3,4,5,6,7,11\}$, and so on. The descent set of the reverse is $\{1,5\}$, so $\mathsf{pmaj}(D) = 6$.

Definition 6. For $D \in \mathsf{PF}(n)$ we set $l_i(D)$ to be the label of the i-th vertical step. Then the *pmaj reading word* of D is the sequence $l_1(D) \cdots l_n(D)$, i.e. the sequence of the labels read bottom to top.

For example, the labelled Dyck path in Figure 1 has pmaj reading word $3569\overline{10}7\overline{11}2841\overline{12}$.

Given two compositions $\mu = (\mu_1, \mu_2, \dots)$ and $\nu = (\nu_1, \nu_2, \dots)$ with $|\mu| + |\nu| = n$, let $K_{\mu_1} = \{n, n-1, \dots, n-\mu_1+1\}$, $K_{\mu_2} = \{n-\mu_1, n-\mu_1-1, \dots, n-\mu_1-\mu_2+1\}$, and so on, and let $I_{\nu_1} = \{1, 2, \dots, \nu_1\}$, $I_{\nu_2} = \{\nu_1+1, \nu_1+2, \dots, \nu_1+\nu_2\}$, and so on. Notice that the sets $K_{\mu_1}, K_{\mu_2}, \dots, I_{\nu_1}, I_{\nu_2}, \dots$ form a partition of $[n]$.

Let now $\uparrow K_{\mu_i}$ be the word consisting of the elements of K_{μ_i} in increasing order: for example $\uparrow K_{\mu_1} = (n-\mu_1+1)(n-\mu_1+2)\cdots(n-1)n$. Similarly, let $\downarrow I_{\nu_j}$ be the word consisting of the elements of I_{ν_j} in decreasing order: for example $\downarrow I_{\nu_1} = \nu_1(\nu_1-1)\cdots 21$.

Consider the shuffle

$$\mathsf{W}(\mu;\nu) := \uparrow K_{\mu_1} \sqcup \uparrow K_{\mu_2} \sqcup \cdots \sqcup \uparrow K_{\mu_{\ell(\mu)}} \sqcup \downarrow I_{\nu_1} \sqcup \downarrow I_{\nu_1} \sqcup \cdots \sqcup \downarrow I_{\nu_{\ell(\nu)}},$$

which we can think of as a set of permutations in $\mathfrak{S}_n$ in one-line notation. Let $\mathsf{PF}(\mu;\nu)$ be the set of parking functions whose pmaj reading word is in $\mathsf{W}(\mu;\nu)$.

For example[2], $\mathsf{W}((4,3);(3,2)) = 9\overline{10}\,\overline{11}\,\overline{12} \sqcup 678 \sqcup 54 \sqcup 321$, and the pmaj reading word of the parking function D in Figure 1 belongs to it, so that $D \in \mathsf{W}((4,3);(3,2))$.

We can now state the shuffle theorem in the form that is suitable for our purposes: this is a combination of the main results in [4] and [14] combined with *superization*: see [12, Chapter 6].

Theorem 2. *For every pair of compositions μ and ν with $|\mu| + |\nu| = n$ we have*

$$\langle \nabla e_n, e_\mu h_\nu \rangle = \sum_{D \in \mathsf{PF}(\mu;\nu)} q^{\mathsf{area}(D)} t^{\mathsf{pmaj}(D)}.$$

3 The clique-independent graphs $\widehat{G}_{\mu,\nu}$

Definition 7. Let μ, ν be two compositions (i.e. tuples of positive integers). Set $n = |\mu| + |\nu|$. We define a graph $G_{\mu,\nu}$ with set of vertices $[n] := \{1, 2, \dots, n\}$ consisting of the following *components*[3]:

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[2]We put a bar on the two-digit numbers not to confuse them.

[3]Notice that the notation is coherent with the one used in Section 2.

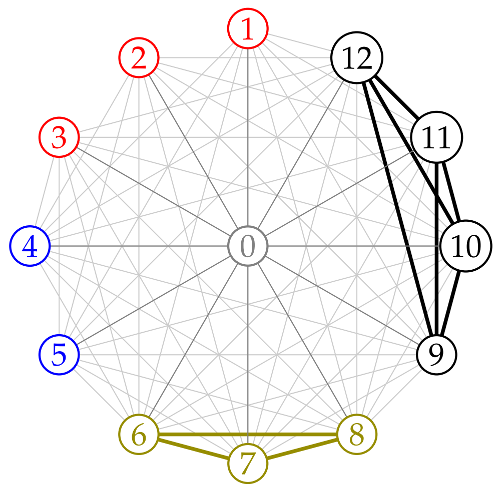

Figure 2: The graph $\widehat{G}_{(4,3),(3,2)}$.

- $\ell(\mu)$ *clique components*, i.e. complete graphs, $K_{\mu_1}, K_{\mu_2}, \dots$, on $\mu_1, \mu_k, \dots$ vertices respectively. The vertices of K_{μ_1} are $n, n-1, \dots, n-\mu_1+1$; the vertices of K_{μ_2} are $n-\mu_1, n-\mu_1-1, \dots, n-\mu_1-\mu_2+1$; and so on.
- $\ell(\nu)$ *independent components*, i.e. graphs without edges, $I_{\nu_1}, I_{\nu_2}, \dots$, on $\nu_1, \nu_2, \dots$ vertices respectively; the vertices of I_{ν_1} are $1, 2, \dots, \nu_1$; the vertices of I_{ν_2} are $\nu_1+1, \nu_1+2, \dots, \nu_1+\nu_2$; and so on.

Finally, two vertices in distinct components are always connected by an edge.

Example 3. If $\mu = \emptyset$, then $G_{\emptyset,\nu}$ is the complete multipartite graph $K_{\nu_1,\nu_2,\dots}$. If $\nu = \emptyset$, then $G_{\mu,\emptyset}$ is isomorphic to the complete graph $K_{|\mu|}$; however, for our purposes we will distinguish between $G_{(|\mu|),\emptyset}$ and $G_{(\mu_1,\mu_2,\dots),\emptyset}$, as we will consider the action of different groups of automorphisms, which will lead to different sorted configurations.

Given one of our labelled graphs $G_{\mu,\nu}$, we define the graph $\widehat{G}_{\mu,\nu}$ simply as $G_{\mu,\nu}$ to which we add a vertex 0, and we connect it with every other vertex. We will consider the sandpile on $\widehat{G}_{\mu,\nu}$, where 0 is the sink. Figure 2 is an illustration of the graph $\widehat{G}_{(4,3),(3,2)}$.

4 Basics of the sandpile model

In the present work with a *graph* we will always mean a simple graph, i.e. a graph with no loops and no multiple edges.

Definition 8. Let G be a finite, undirected graph on the vertex set $\{0, 1, \dots, n\}$.

A *configuration* of the *sandpile (model) on* G is a map $\kappa : [n] \cup \{0\} \to \mathbb{Z}$ that assigns a (integer) number of "grains of sand" to each vertex of G.

If $0 \leq \kappa(v) \leq \deg(v)$, we say that v is *stable*, and otherwise it is *unstable*. Any vertex can *topple* (or *fire*), and "donate a single grain" to each of its neighbors: the result is a new configuration κ' in which $\kappa'(v) = \kappa(v) - \deg(v)$ and for any $w \neq v$

$$\kappa'(w) = \begin{cases} \kappa(w) + 1, & \text{if } (v, w) \text{ is an edge} \\ \kappa(w), & \text{otherwise.} \end{cases}$$

For any $v \in \{0, \dots, n\}$ we write ϕ_v for the *toppling operator* at vertex v. That is $\phi_v(\kappa)$ is a new configuration obtained from κ by toppling the vertex v.

The vertex 0 is special in this model, and we call it the *sink*, while we call all the others *nonsink* vertices. We say that a configuration κ is *non-negative* if all of its nonsink vertices are non-negative, *stable* if all of its nonsink vertices are stable, and *unstable* if at least one of its nonsink vertices is unstable.

Remark 1. Notice that the notion of stable configuration has no dependency on the value on the sink. Therefore, as it is customary, we will ignore the value of a configuration on the sink, and consider the configurations as restricted on the nonsink vertices. Moreover, we will identify every configuration κ with the word $\kappa(n)\kappa(n-1)\cdots\kappa(2)\kappa(1)$.

Example 4. Consider the graph $\widehat{G}_{(4,3),(3,2)}$ (see Figure 2), whose vertices are $\{0\} \cup [12]$, and let 0 be the sink. The configuration κ given by[2] $3\overline{10}\,\overline{11}\,\overline{11}8\overline{10}\,\overline{11}\,\overline{10}4973$ is a stable configuration. We compute a few topplings:

$$\begin{aligned}
\phi_0(\kappa) &= 4\overline{11}\,\overline{12}\,\overline{12}9\overline{11}\,\overline{12}\,\overline{11}5\overline{10}84, \\
(\phi_{\overline{10}} \circ \phi_0)(\kappa) &= 5\overline{12}0\overline{13}\,\overline{10}\,\overline{12}\,\overline{13}\,\overline{12}6\overline{11}95, \\
(\phi_9 \circ \phi_{\overline{10}} \circ \phi_0)(\kappa) &= 6\overline{13}11\overline{11}\,\overline{13}\,\overline{14}\,\overline{13}7\overline{12}\,\overline{10}6, \\
(\phi_7 \circ \phi_9 \circ \phi_{\overline{10}} \circ \phi_0)(\kappa) &= 7\overline{14}22\overline{12}1\overline{15}\,\overline{14}8\overline{13}\,\overline{11}7, \\
(\phi_6 \circ \phi_7 \circ \phi_9 \circ \phi_{\overline{10}} \circ \phi_0)(\kappa) &= 8\overline{15}33\overline{13}23\overline{15}9\overline{14}\,\overline{12}8, \\
(\phi_5 \circ \phi_6 \circ \phi_7 \circ \phi_9 \circ \phi_{\overline{10}} \circ \phi_0)(\kappa) &= 9\overline{16}44\overline{14}3449\overline{15}\,\overline{13}9, \\
(\phi_3 \circ \phi_5 \circ \phi_6 \circ \phi_7 \circ \phi_9 \circ \phi_{\overline{10}} \circ \phi_0)(\kappa) &= \overline{10}\,\overline{17}55\overline{15}455\overline{10}5\overline{13}9, \\
(\phi_2 \circ \phi_3 \circ \phi_5 \circ \phi_6 \circ \phi_7 \circ \phi_9 \circ \phi_{\overline{10}} \circ \phi_0)(\kappa) &= \overline{11}\,\overline{18}66\overline{16}566\overline{11}539.
\end{aligned}$$

Definition 9. Let κ be a stable configuration, and consider the configuration $\phi_0(\kappa)$. We say that κ is *recurrent*[4] if there is an order of all the nonsink vertices such that toppling the vertices in that order we always stay non-negative. Of course at the end of this sequence of topplings we will be back to κ. More precisely, a configuration κ is recurrent if there is a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in \mathfrak{S}_n$ such that

$$\phi_0(\kappa), (\phi_{\sigma(1)} \circ \phi_0)(\kappa), (\phi_{\sigma(2)} \circ \phi_{\sigma(1)} \circ \phi_0)(\kappa), \dots, (\phi_{\sigma(n)} \circ \cdots \circ \phi_{\sigma(1)} \circ \phi_0)(\kappa) = \kappa$$

[4] In the literature "recurrent" is sometimes used in a broader sense than in this paper. Configurations that are recurrent in our sense are called *critical* in these settings.

are all non-negative configurations. In this case, σ is the *toppling word* of this sequence of topplings, and we say that this sequence *verifies the recurrence* of κ.

Example 5. The configuration $\kappa = 3\overline{10}\,\overline{11}\,\overline{11}8\overline{10}\,\overline{11}\,\overline{10}4973$ is a recurrent configuration for $\widehat{G}_{(4,3),(3,2)}$: indeed it is easy to check that $\sigma = \overline{10}976532\overline{11}841\overline{12}$ verifies the recurrence of κ (cf. Example 2).

Remark 2. It is well known (see e.g. [2, Theorem 2.4]) that the condition for κ to be recurrent is equivalent to say that starting from $\phi_0(\kappa)$ there is no proper (possibly empty) subset A of $[n]$ such that toppling all the vertices of A brings $\phi_0(\kappa)$ to a stable configuration.

Definition 10. Given a recurrent configuration κ of G, we define its *level* as

$$\mathsf{level}(\kappa) := -|E_s(G)| + \sum_{i=1}^{n} \kappa(i)$$

where $E_s(G)$ is the set of edges of G that are not incident to the sink.

It is well-known that $\mathsf{level}(\kappa) \geq 0$, and there exists a recurrent configuration of level 0 if G is connected [15].

Remark 3. For $\widehat{G}_{\mu,\nu}$ with $|\mu| + |\nu| = n$ we have

$$|E_s(\widehat{G}_{\mu,\nu})| = \binom{n}{2} - \sum_{i \geq 0} \binom{\nu_i}{2}.$$

Example 6. The configuration $\kappa = 3\overline{10}\,\overline{11}\,\overline{11}8\overline{10}\,\overline{11}\,\overline{10}4973$ for $\widehat{G}_{(4,3),(3,2)}$ has level

$$\mathsf{level}(\kappa) = -\binom{12}{2} + \binom{3}{2} + \binom{2}{2} + 97 = 35.$$

5 Toppling algorithm and delay

Consider the sandpile on a graph G with vertices $\{0\} \cup [n]$, where 0 is the sink. Let κ be a recurrent configuration of G. Consider Algorithm 1.
Before discussing the algorithm, let us look at an example.

Example 7. Consider again the configuration κ from Example 4: in that example we actually computed the sequence of toppling given by the first iteration of the **for** loop of Algorithm 1 applied to κ. We compute the second iteration of the **for** loop:

$$(\phi_{\overline{11}} \circ \phi_2 \circ \phi_3 \circ \phi_5 \circ \phi_6 \circ \phi_7 \circ \phi_9 \circ \phi_{\overline{10}} \circ \phi_0)(\kappa) = \overline{12}677\overline{17}677\overline{12}64\overline{10},$$
$$(\phi_8 \circ \phi_{\overline{11}} \circ \phi_2 \circ \phi_3 \circ \phi_5 \circ \phi_6 \circ \phi_7 \circ \phi_9 \circ \phi_{\overline{10}} \circ \phi_0)(\kappa) = \overline{13}7885788\overline{13}75\overline{11},$$
$$(\phi_4 \circ \phi_8 \circ \phi_{\overline{11}} \circ \phi_2 \circ \phi_3 \circ \phi_5 \circ \phi_6 \circ \phi_7 \circ \phi_9 \circ \phi_{\overline{10}} \circ \phi_0)(\kappa) = \overline{14}89968982 86\overline{12},$$
$$(\phi_1 \circ \phi_4 \circ \phi_8 \circ \phi_{\overline{11}} \circ \phi_2 \circ \phi_3 \circ \phi_5 \circ \phi_6 \circ \phi_7 \circ \phi_9 \circ \phi_{\overline{10}} \circ \phi_0)(\kappa) = \overline{15}9\overline{10}\,\overline{10}79\overline{10}93862,$$

and finally the third and last iteration of the **for** loop:

$$(\phi_{\overline{12}}\circ\phi_1\circ\phi_4\circ\phi_8\circ\phi_{\overline{11}}\circ\phi_2\circ\phi_3\circ\phi_5\circ\phi_6\circ\phi_7\circ\phi_9\circ\phi_{\overline{10}}\circ\phi_0)(\kappa) = 3\overline{10}\,\overline{11}\,\overline{11}8\overline{10}\,\overline{11}\,\overline{10}4973 = \kappa.$$

Hence, the output of Algorithm 1 applied to κ is the word $\overline{10}976532\overline{11}841\overline{12}$ (cf. Example 2 and Example 5).

Algorithm 1 Toppling algorithm

Input: A graph G and a recurrent configuration κ
Output: The word of nonsink vertices in the order they have been toppled

```
Topple the sink, i.e. compute φ_0(κ)
Initialize the output word as empty
while there are nonsink vertices that are untoppled do
    for i going from n to 1 (in decreasing order) do
        if vertex i is unstable then
            Topple vertex i
            Append i to the output word
        end if
    end for
end while
```

Observe that by construction the algorithm terminates: since κ is recurrent, $\phi_0(\kappa)$ is non-negative and at least one of the vertices adjacent to the sink is unstable; then every time we topple we stay non-negative, and since κ is recurrent the process must go through all the nonsink vertices (otherwise we found a subset A of nonsink vertices such that after we topple its vertices we are in a stable configuration, cf. Remark 2).

By construction the algorithm outputs a toppling sequence that verifies the recurrence of κ. We can now define our new statistic on recurrent configurations.

Definition 11. Let κ be a recurrent configuration of G. For every $i \in [n]$, let $r_i(\kappa)$ be the number of **for** loop iterations in Algorithm 1 that occurred before the one in which the vertex i is toppled (so if i is toppled in the first iteration, then $r_i(\kappa) = 0$). Then we define the *delay* of κ as

$$\mathsf{delay}(\kappa) := \sum_{i=1}^{n} r_i(\kappa).$$

Remark 4. If σ is the output of Algorithm 1 applied to κ, then clearly

$$\mathsf{delay}(\kappa) = \mathsf{maj}(\sigma_n\sigma_{n-1}\cdots\sigma_1).$$

Example 8. For the configuration κ of Example 4, we got in Example 7 that Algorithm 1 gives $\sigma = \overline{10}976532\overline{11}841\overline{12}$, so that $\mathsf{delay}(\kappa) = \mathsf{maj}(\overline{12}148\overline{11}235679\overline{10}) = 1+5 = 6$. Indeed, looking at the computation of the algorithm, we find that the word $r_1(\kappa)r_2(\kappa)\cdots$ in this case is indeed 100100010012, whose letters add up to 6 (cf. Example 2).

6 Sorted recurrent configurations of $\widehat{G}_{\mu,\nu}$

Consider the Young subgroup $\mathfrak{S}_\mu \times \mathfrak{S}_\nu$ of the symmetric group $\mathfrak{S}_n$ consisting of the permutations that preserve the components of $G_{\mu,\nu}$. We want to consider configurations "modulo" the natural action of $\mathfrak{S}_\mu \times \mathfrak{S}_\nu$ on the set of configurations. More precisely, a *sorted configuration*[5] of the sandpile on $\widehat{G}_{\mu,\nu}$ is a configuration κ that is weakly decreasing inside each clique component of $\widehat{G}_{\mu,\nu}$ and weakly increasing inside each independent component of $\widehat{G}_{\mu,\nu}$: if $i, j \in K_{\mu_r}$ and $i < j$, then $\kappa(i) \leq \kappa(j)$; if $i, j \in I_{\nu_s}$ and $i < j$, then $\kappa(i) \geq \kappa(j)$.

Example 9. The configuration $\kappa = 3\overline{10}\,\overline{11}\,\overline{11}8\overline{10}\,\overline{11}\,\overline{10}4973$ is a sorted recurrent configuration for $\widehat{G}_{(4,3),(3,2)}$ (recall that in our notation $\kappa = \kappa(n)\kappa(n-1)\cdots\kappa(1)$).

Let κ be a sorted recurrent configuration of $\widehat{G}_{\mu,\nu}$. Let $\sigma \in \mathfrak{S}_n$ be the toppling word produced by Algorithm 1 applied to κ.

For every independent component I_{ν_s} of $G_{\mu,\nu}$, we order its vertices in decreasing order, and if $v_j^{(s)}$ is the j-th vertex of I_{ν_s}, we set

$$\widetilde{\kappa}(v_j^{(s)}) := \kappa(v_j^{(s)}) + \nu_s - j.$$

For every vertex v in a clique component K_{μ_r} we set

$$\widetilde{\kappa}(v) := \kappa(v).$$

For every $i \in [n]$, we set

$$u_{\sigma^{-1}(i)} := \sigma^{-1}(i) + \widetilde{\kappa}(i) - n.$$

Example 10. For the configuration $\kappa = 3\overline{10}\,\overline{11}\,\overline{11}8\overline{10}\,\overline{11}\,\overline{10}4973$ in Example 9, we found in Example 7 that Algorithm 1 gives $\sigma = \overline{10}976532\overline{11}841\overline{12}$. Hence $\widetilde{\kappa} = 3\overline{10}\,\overline{11}\,\overline{11}8\overline{11}\,\overline{11}\,\overline{11}4\overline{11}83$, and the word $u := u_1u_2\cdots$ is 011345365223.

Given a permutation $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$, we add $\sigma(0) := 0$ in front of it, and we define its *runs* as its maximal consecutive decreasing substrings. Now for every $i \in [n]$, we define $w_{\sigma^{-1}(i)} = w_{\sigma^{-1}(i)}(\sigma)$ as

$$\begin{aligned} w_{\sigma^{-1}(i)}(\sigma) &:= \#\{\text{numbers in the same run of } i \text{ and larger than } i\} \\ &+ \#\{\text{numbers smaller than } i \text{ in the run immediately to the left of the one containing } i\}. \end{aligned}$$

Example 11. The runs of $\sigma = \overline{10}976532\overline{11}841\overline{12}$ are separated by bars: $0|\overline{10}976532|\overline{11}841|\overline{12}$, so that the word $w = w_1(\sigma)w_2(\sigma)\cdots$ is 123456776434.

[5] The relation with the general definition of *sorted configuration* given in Section 1.2 is simply that we are picking a specific convenient element in each orbit.

The following propositions characterize the sorted recurrent configurations of $\widehat{G}_{\mu,\nu}$.

Proposition 1. *Let κ be a sorted recurrent configuration of $\widehat{G}_{\mu,\nu}$. Let $\sigma \in \mathfrak{S}_n$ be the toppling word produced by Algorithm 1 applied to κ. Then for every $i \in [n]$*

$$0 \leq u_{\sigma^{-1}(i)} < w_{\sigma^{-1}(i)}.$$

Proposition 2. *Let κ be a sorted stable configuration of $\widehat{G}_{\mu,\nu}$, and let $\sigma \in \mathfrak{S}_n$ be such that for every $i \in [n]$*

$$0 \leq u_{\sigma^{-1}(i)} < w_{\sigma^{-1}(i)}.$$

Then κ is recurrent and σ is the toppling word given by Algorithm 1 applied to κ.

We omit the proofs, but an instance can be checked by comparing Examples 10 and 11.

7 Bijection with parking functions

We now provide a bijection between recurrent sorted configurations of $\widehat{G}_{\mu,\nu}$ and the parking functions in $\mathsf{PF}(\mu;\nu)$.

Let $\mathsf{SortRec}(\mu,\nu)$ be the set of sorted recurrent configurations of $\widehat{G}_{\mu,\nu}$. Define the function $\Phi : \mathsf{SortRec}(\mu,\nu) \to \mathsf{PF}(\mu,\nu)$ in the following way: given $\kappa \in \mathsf{SortRec}(\mu,\nu)$, in the notation of Section 6, we set $\Phi(\kappa)$ to be the (unique) parking function of size $n = |\mu| + |\nu|$ such that the label i occurs in column $n - \widetilde{\kappa}(i)$ (we number the columns increasingly from left to right) for every $i \in [n]$.

Example 12. The parking function $D \in \mathsf{PF}((4,3);(3,2))$ in Figure 1 is the image $\Phi(\kappa)$ of the configuration κ in Example 4 ($\widetilde{\kappa}$ is computed in Example 10).

We can finally state the main result of our article.

Theorem 3. *The map Φ is a well-defined bijection such that $\mathsf{area}(\Phi(\kappa)) = \mathsf{level}(\kappa)$ and such that the σ obtained from the Algorithm 1 applied to κ equals the pmaj word of $\Phi(\kappa)$, so that $\mathsf{pmaj}(\Phi(\kappa)) = \mathsf{delay}(\kappa)$.*

Now Theorem 1 is an immediate consequence of this result combined with Theorem 2. It can be checked in the instance of Example 12 (cf. Examples 7, 8, 2, 1 and 6).

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