Séminaire Lotharingien de Combinatoire **91B** (2024)
Article #69, 12 pp.

Proceedings of the 36th Conference on Formal Power Series and Algebraic Combinatorics (Bochum)

Asymptotics of Bounded Lecture-Hall Tableaux

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Abstract. We study the asymptotics of bounded lecture hall tableaux. Limit shapes form when the bounds of the lecture hall tableaux go to infinity linearly in the lengths of the partitions describing the large-scale shapes of these tableaux. We prove Conjecture 6.1 in [8], stating that the slopes of the rescaled height functions in the scaling limit satisfy a complex Burgers equation. We also show that the fluctuations of the unrescaled height functions converge to the Gaussian free field. The proof is based on new construction and analysis of Schur generating functions for the lecture hall tableaux, whose corresponding particle configurations do not form a Gelfand-Tsetlin scheme; and the corresponding dimer models are not doubly periodic.

Résumé. Nous étudions l'asymptotique des tableaux de la salle de cours bornés. Les formes limites se forment lorsque les bornes des tableaux de la salle de cours tendent vers l'infini linéairement par rapport aux longueurs des partitions décrivant les formes à grande échelle de ces tableaux. Nous démontrons la Conjecture 6.1 dans [8], affirmant que les pentes des fonctions de hauteur mises à l'échelle dans la limite d'échelle satisfont une équation de Burgers complexe. Nous montrons également que les fluctuations des fonctions de hauteur non mises à l'échelle convergent vers le champ libre gaussien. La preuve repose sur une nouvelle construction et une analyse des fonctions génératrices de Schur pour les tableaux de la salle de cours, dont les configurations de particules correspondantes ne forment pas un schéma de Gelfand-Tsetlin; et les modèles de dimères correspondants ne sont pas doublement périodiques.

Keywords: lecture hall tableaux, limit shape, Gaussian free field

1 Introduction

Lecture hall tableaux were introduced in [10] as fillings of Young tableaux satisfying certain conditions, which generalize both lecture hall partitions ([2, 3]) and anti-lecture

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hall compositions ([11]), and also contain reverse semistandard Young tableaux as a limit case. Lecture hall partitions and anti-lecture hall compositions have attracted considerable interest among combinatorists in the last two decades; see the recent survey [21] and references therein.

We now define the lecture hall tableaux. Recall that a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ is a sequence of nonnegative integers $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k \geq 0$. Each integer λ_i is called a part of λ. The length $l(\lambda)$ of λ is the number of parts. A partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ can be identified with its Young diagram, which consists of unit squares (cells) with integer coordinates (i, j) satisfying $1 \leq i \leq k$ and $1 \leq j \leq \lambda_i$. For two partitions λ and μ we write $\mu \subset \lambda$ to mean that the Young diagram of μ is contained in that of λ as a set. In this case, a skew shape λ/μ is defined to be the set-theoretic difference λ/μ of their Young diagrams. We denote by $|\lambda/\mu|$ the number of cells in λ/μ. A partition λ is also considered as a skew shape by $\lambda/\emptyset$; where $\emptyset$ represents the empty partition.

A tableau of shape λ/μ is a filling of the cells in λ/μ with nonnegative integers. In other words, a tableau is a map $T : \lambda/\mu \to \mathbb{N}$, where $\mathbb{N}$ is the set of nonnegative integers.

An n-lecture hall tableau of shape λ/μ is a tableau L of shape λ/μ satisfying the following conditions

$$\frac{L(i,j)}{n + c(i,j)} \geq \frac{L(i,j+1)}{n + c(i,j+1)}, \qquad \frac{L(i,j)}{n + c(i,j)} > \frac{L(i+1,j)}{n + c(i+1,j)}.$$

where $c(i,j) = j - i$ is the content of the cell (i,j). The set of n-lecture hall tableaux is denoted by $LHT_n(\lambda/\mu)$. For $L \in LHT_n(\lambda/\mu)$, let $\lfloor L \rfloor$ be the tableaux of shape λ/μ whose (i,j)th entry is $\lfloor \frac{L(i,j)}{(n-i+j)} \rfloor$.

See the left graph of Figure 1 for an example of a lecture hall tableaux.

We shall study lecture hall tableaux with an extra condition as follows:

$$L(i,j) < t(n + j - i)$$

We say these tableaux are bounded by $t > 0$. These tableaux are called bounded lecture hall tableaux and are enumerated in [9].

The main aim to study the asymptotics of bounded n-lecture hall tableaux as $n \to \infty$. We shall first recall a bijection between lecture hall tableaux and non-intersecting path configurations in [9], and then investigate the asymptotics (limit shape and height fluctuations) of the corresponding non-intersecting path configurations. Now we define the graph on which the non-intersecting path configurations correspond to the lecture hall tableaux.

1. Given a positive integer t, the lecture hall graph is a graph $\mathcal{G}_t = (V_t, E_t)$. This graph can be described through an embedding in the plane with vertex set V_t given by

- $\left(i, \frac{j}{i+1}\right)$ for $i \geq 0$ and $0 \leq j < t(i+1)$.

and the directed edges given by

- from $\left(i, k + \frac{r}{i+1}\right)$ to $\left(i+1, k + \frac{r}{i+2}\right)$ for $i \geq 0$, $0 \leq r \leq i$ and $0 \leq k < t$
- from $\left(i, k + \frac{r+1}{i+1}\right)$ to $\left(i, k + \frac{r}{i+1}\right)$ for $i \geq 0$ and $0 \leq r \leq i$ and $0 \leq k < t-1$ or for $i \geq 0$ and $0 \leq r < i$ and $k = t-1$.

2. Given a positive integer t and a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0$, a non-intersecting path configuration is a system of n paths on the graph $\mathcal{G}_t$. For each integer i satisfying $1 \leq i \leq n$, the ith path starts at $\left(n-i, t - \frac{1}{n-i+1}\right)$, ends at $(n - i + \lambda_i, 0)$ and moves only downwards and rightwards. The paths are said to be not intersecting if they do not share a vertex.

See the middle graph of 1 for an example of $\mathcal{G}_3$ and a configuration of non-intersecting paths on $\mathcal{G}_3$.

Given a positive integer t and a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ with $\lambda_1 \geq \ldots \geq \lambda_n \geq 0$, the non-intersecting path system is a system of n paths on the graph $\mathcal{G}_t$. The ith path starts at $\left(n-i, t - \frac{1}{n-i+1}\right)$ and ends at $(\lambda_i + n - i, 0)$. The paths are called non-intersection if they do not share a vertex.

Theorem 1. *([9])There is a bijection between the bounded lecture hall tableaux of shape λ and bounded by t and non-intersecting paths on $\mathcal{G}_t$ starting at $\left(n-i, t - \frac{1}{n-i+1}\right)$ and ending at $(n - i + \lambda_i, 0)$ for $i = 1, 2, \ldots, n$.*

More precisely, there are exactly $|\lambda|$ non-vertical edges present in the non-intersecting path configuration in $\mathcal{G}_t$ corresponding to a lecture-hall tableaux of shape λ. These edges have left endpoints located at $\left(n+j-i-1, \frac{L(i,j)}{n+j-i}\right)$. The non-intersecting path configuration corresponding to the lecture hall tableaux is the unique non-intersecting path configuration joining $\left(n-i, t - \frac{1}{n-i+1}\right)$ and $(n - i + \lambda_i, 0)$ for $i = 1, 2, \ldots, n$ obtained by adding only vertical edges to these present non-vertical edges.

One can see that for an n-lecture hall tableaux bounded by t, t is also the height of the corresponding lecture hall graph $\mathcal{G}_t$, and n is also the total number of paths in the corresponding non-intersecting path configuration on $\mathcal{G}_t$. See Figure 1 for an example of such a correspondence.

We shall investigate the asymptotics of bounded lecture hall tableaux as $n, t \to \infty$ by studying the asymptotics of the corresponding non-intersecting paths. These asymptotics were studied in [8] using the (not fully rigorous) tangent method; here we attack this problem by analyzing Schur polynomials. The tangent method gives the frozen

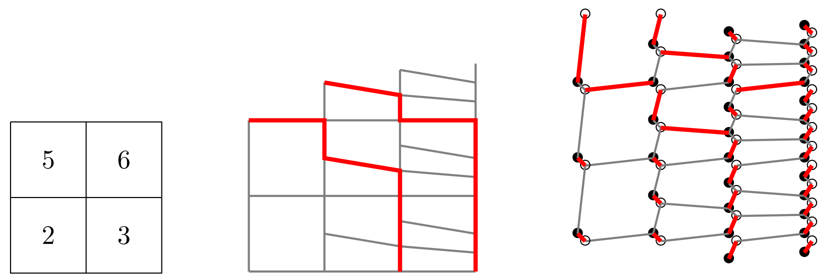

Figure 1: Tableau, non-intersecting paths, and dimers (Figure 1 in [8]). The left graph represents a lecture hall tableaux L of shape $\lambda = (2,2)$ with $L(1,1) = 5$, $L(1,2) = 6$, $L(2,1) = 2$, $L(2,2) = 3$ and $n = 2$. Then $\frac{L(1,1)}{n+1-1} = \frac{5}{2}$; $\frac{L(2,1)}{n+1-2}$=2;$\frac{L(1,2)}{n+2-1} = 2$;$\frac{L(2,2)}{n+2-2} = \frac{3}{2}$. The lecture hall tableaux is bounded by $t = 3$. The middle graph represents the corresponding non-intersecting path configuration. The right graph represents a dimer configuration on a graph which is not doubly-periodic.

boundary without the full limit shape; instead Conjecture 6.1 were made in [8], indicating that the slopes of the rescaled height functions in the scaling limit satisfy the complex Burgers equation. The complex Burgers equation was proved to be the governing equation of height functions in the scaling limit for uniform lozenge tilings and for other doubly periodic dimer models [13]. This equation naturally arises through a variational problem, we refer to [1] for a detailed study of the variational problem. Here we note that for lecture hall tableaux no variational principle has been established and although lecture hall tableaux naturally corresponds to non-interacting paths configurations and dimer configurations on a hexagon-octagon lattice ([8]), the corresponding hexagon-octagon lattice in this case is not doubly periodic as in the setting in [13]; see the right graph of Figure 1.

The Schur generating function approach was applied to study uniform dimer model on a hexagonal lattice in a trapezoid domain in [5, 6], and for uniform dimer model on a rectangular square grid in [7]. A generalized version of the Schur generating function was defined to study the non-uniform dimer model on rail-yard graphs in [4, 16, 15, 17, 19]. Schur processes are specializations of the Macdonald processes when $q = t$, hence the asymptotics of Schur processes can also be obtained by investigating the more general Macdonald processes; see [20, 18]. All the existing Schur-generating functions seem to be defined in the setting of the Gelfand-Tsetlin scheme; however the lecture hall tableaux are novel in the sense that on a skew shape they cannot be computed by skew Schur functions; and the corresponding particle configurations induced by the non-intersecting path configurations of the lecture hall tableaux do not satisfy the interlacing conditions required by the Gelfand-Tsetlin scheme; see Figure 2 for an example.

By constructing a novel Schur generating function specifically for the lecture hall tableaux and analyzing its asymptotics, in this paper we obtain a full description of the limit shape, including the moment formulas for the counting measures and the complex Burgers equation; resolving Conjecture 6.1 in [8].

The Gaussian free field, as a high dimensional time analog of the Brownian motion, was proved to be the rule of height fluctuations for dimer models on a large class of graphs ([12, 14]). In this paper we show that the unrescaled height fluctuations of the lecture hall tableaux converge to the Gaussian free field when t goes to infinity linearly as n goes to infinity.

The main results (with exact statements given in later sections after a number of precise definitions) are as follows.

- In Section 2, we discuss the moment formula for the limit counting when $n \to \infty$, $t \to \infty$ and $\frac{t}{n} \to \alpha \in (0, \infty)$ (Theorem 2); the equation of the boundary curve separating different phases (Theorem 3) and that the slopes of the (rescaled) height function in the scaling limit satisfy the complex Burgers equation; confirming Conjecture 6.1 in [8] (Theorem 4).

- In Section 3, we prove the convergence of the (unrescaled) height fluctuation to the Gaussian free field (GFF) $n \to \infty$, $t \to \infty$ and $\frac{t}{n} \to \alpha \in (0, \infty)$ (Theorem 5)

2 Limit Shape when $t \to \infty$ and Complex Burger's Equation

Let $\mathcal{M}$ be a random non-intersecting path configuration on $\mathcal{G} = \mathcal{G}_t$. Let n be the total number of non-intersecting paths. Let $\kappa \geq 0$ be an integer. Let $\epsilon > 0$ be sufficiently small such that the region $y \in (\kappa, \kappa + \epsilon]$ does not intersect any non-vertical edge of $\mathcal{G}$. We associate a partition $\lambda^{(\kappa)}$ as follows:

- $\lambda_1^{(\kappa)}$ is the number of absent vertical edges of $\mathcal{M}$ intersecting $y = \kappa + \epsilon$ to the left of the rightmost vertical edges present in $\mathcal{M}$.

- for $j \geq 2$, $\lambda_j^{(\kappa)}$ is the number of absent vertical edges of $\mathcal{M}$ intersecting $y = \kappa + \epsilon$ to the left of the jth rightmost vertical edges present in $\mathcal{M}$.

See Figure 2 for an example.

For $\mathbf{x} = (x_0, x_1, \ldots)$ Let $s_{\lambda/\mu}(\mathbf{x})$ be the skew Schur function. For any tableaux T of shape λ/μ, let

$$\mathbf{x}^T = \prod_{(i,j) \in \lambda/\mu} x_{T(i,j)};$$

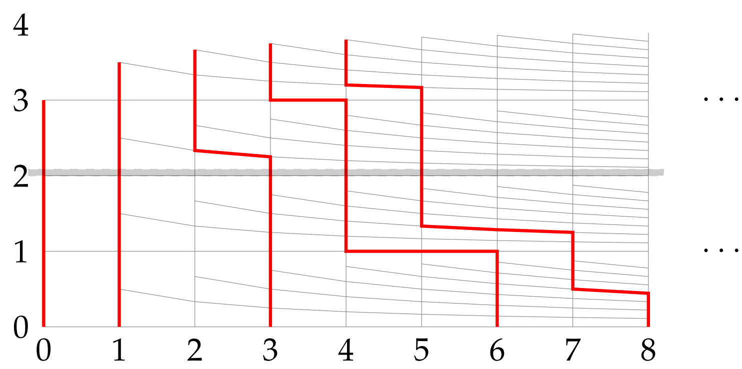

Figure 2: Non-intersecting lattice paths on $\mathcal{G}_4$ for $n = 5$. We have $\lambda^{(3)} = (1,0,0,0,0)$, $\lambda^{(2)} = (1,1,1,0,0)$, $\lambda^{(1)} = (3,3,1,0,0)$ and $\lambda^{(0)} = (4,3,1,0,0)$. The sequence of partitions $(\lambda^{(0)}, \lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)})$ do not form a Gelfand-Tsetlin scheme.

we define

$$L^n_{\lambda/\mu}(\mathbf{x}) = \sum_{T \in LHT_n(\lambda/\mu)} \mathbf{x}^{\lfloor T \rfloor}$$

Let ρ_κ be the probability distribution of $\lambda^{(\kappa)}$. Define the Schur generating function for ρ_κ as follows:

$$\mathcal{S}_{\rho_\kappa}(|\mathbf{x}|, \mathbf{u}) = \sum_{\lambda \in \mathbb{Y}} \rho_\kappa(\lambda) \frac{s_\lambda(|\mathbf{x}| + \mathbf{u})}{s_\lambda(|\mathbf{x}|)} \tag{2.1}$$

where

$$\mathbf{u} = (u_1, u_2, \ldots, u_n); \qquad \mathbf{x} = (x_1, x_2, \ldots, x_t); \qquad |\mathbf{x}| = x_1 + x_2 + \ldots + x_t$$

and

$$s_\lambda(|\mathbf{x}| + \mathbf{u}) := s_\lambda(|\mathbf{x}| + u_1, |\mathbf{x}| + u_2, \ldots, |\mathbf{x}| + u_n) \tag{2.2}$$

$$s_\lambda(|\mathbf{x}|) := s_\lambda(|\mathbf{x}|, \ldots, |\mathbf{x}|) \tag{2.3}$$

Let λ be a length-N partition. We define the counting measure $m(\lambda)$ as a probability measure on $\mathbb{R}$ as follows:

$$m(\lambda) = \frac{1}{N} \sum_{i=1}^{N} \delta\left(\frac{\lambda_i + N - i}{N}\right).$$

If λ is random, then we can define the corresponding random counting measure.

let $S_{\mathbf{m}}(z) = z + \sum_{k=1}^{\infty} M_k(\mathbf{m}) z^{k+1}$ be the moment generating function of the measure $\mathbf{m}$, where $M_k(\mathbf{m}) = \int x^k d\mathbf{m}(x)$, and $S_{\mathbf{m}}^{(-1)}$ be its inverse for the composition. Let $R_{\mathbf{m}}(z)$ be the *Voiculescu R-transform* of $\mathbf{m}$ defined as

$$R_{\mathbf{m}}(z) = \frac{1}{S_{\mathbf{m}}^{(-1)}(z)} - \frac{1}{z}.$$

Then

$$H_{\mathbf{m}}(u) = \int_0^{\ln u} R_{\mathbf{m}}(t) dt + \ln\left(\frac{\ln u}{u-1}\right). \tag{2.4}$$

In particular, $H_{\mathbf{m}}(1) = 0$, and

$$H'_{\mathbf{m}}(u) = \frac{1}{u S_{\mathbf{m}}^{(-1)}(\ln u)} - \frac{1}{u-1}. \tag{2.5}$$

Assume as $n \to \infty$, the rescaled graph $\frac{1}{n}\mathcal{G}$ approximate a bounded simply-connected region $\mathcal{R} \subset \mathbb{R}^2$. Let $\mathcal{L}$ be the set of (χ, y) inside $\mathcal{R}$ such that the density $d\mathbf{m}_y(\frac{\chi}{1-y})$ is not equal to 0 or 1. Then $\mathcal{L}$ is called the liquid region. Its boundary $\partial\mathcal{L}$ is called the frozen boundary. Let

$$\widetilde{\mathcal{L}} := \{(\chi, s) : (\chi, y) \in \mathcal{L}\}$$

where s, y are given as Theorem 2.

Theorem 2. *Let n be the the total number of non-interacting paths in $\mathcal{G}$, and let t be the height of $\mathcal{G}$. Let $\rho_\kappa(n)$ be the probability distribution of $\lambda^{(\kappa)}$. Assume*

$$y := \lim_{n\to\infty} \frac{\kappa}{n}; \qquad s := \lim_{n\to\infty} \frac{|\mathbf{x}_\kappa|}{|\mathbf{x}|}; \qquad \alpha := \lim_{n\to\infty} \frac{t}{n}; \tag{2.6}$$

such that

$$s \in (0,1); \qquad y \in (0, \alpha).$$

Then random measures $\mathbf{m}_{\rho_\kappa(n)}$ converge as $n \to \infty$ in probability, in the sense of moments to a deterministic measure $\mathbf{m}_y$ on $\mathbb{R}$, whose moments are given by

$$\int_{\mathbb{R}} x^j \mathbf{m}_y(dx) = \frac{1}{2(j+1)\pi \mathbf{i}} \oint_1 \frac{dz}{z-1+s} \left((z-1+s) H'_{\mathbf{m}_0}(z) + \frac{z-1+s}{z-1} \right)^{j+1}$$

Here $\mathbf{m}_0$ is the limit counting measure for the boundary partition $\lambda^{(0)} \in \mathbb{Y}_n$ as $n \to \infty$, and $H_{\mathbf{m}_0}$ is defined as in (2.4).

The main idea to prove Theorem 2 is to use a differential operator acting on the Schur generating function defined by (2.1), which gives the moments of $\int_{\mathbb{R}} x^j \mathbf{m}_{\rho_\kappa(n)}$; by proving that

$$\lim_{n\to\infty} \mathbb{E}\left[\int_{\mathbb{R}} x^j \mathbf{m}_{\rho_\kappa(n)}\right]^2 = \lim_{n\to\infty}\left[\mathbb{E}\int_{\mathbb{R}} x^j \mathbf{m}_{\rho_\kappa(n)}\right]^2;$$

it follows that the limit counting measure is deterministic. The explicit integral formula for $\int_{\mathbb{R}} x^j \mathbf{m}_{\rho_\kappa(n)}$ follows from the Residue theorem.

Theorem 3. *Let*

$$U_y(z) := (z-1+s)H'_{\mathbf{m}_0}(z) + \frac{z-1+s}{z-1} \tag{2.7}$$

Assume the liquid region is nonempty, and assume that for any $x \in \mathbb{R}$, the equation $U_y(z) = x$ has at most one pair of complex conjugate roots. Then for any point (χ, y) lying on the frozen boundary, the equation $U_y(z) = \chi$ has double roots.

The main idea to prove Theorem 3 is to compute the density of the measure $d\mathbf{m}_y(x)$ by the Stieljes transform

$$\frac{d\mathbf{m}_y(x)}{dx} = -\lim_{\epsilon\to 0+}\frac{1}{\pi}\Im(\mathrm{St}_{\mathbf{m}_y}(x+\mathbf{i}\epsilon)) \tag{2.8}$$

where $\Im(\cdot)$ represents the imaginary part of a complex number and $\mathrm{St}_{\mathbf{m}_y}$ is the Stieljes transform of the measure $\mathbf{m}_y$; and then find the boundary of the region where the density is 0 or 1 (frozen region).

Example 1. *Assume the bottom boundary partition is given by*

$$\lambda^{(0)}(n) := ((p-1)n, (p-1)(n-1), \ldots, p-1) \in \mathbb{Y}_n$$

where p, n are positive integers. We have

$$U_y(z) = \frac{pz^{p-1}(z-1+s)}{z^p - 1}$$

Assume $p = 3$. then for each $\chi \in \mathbb{R}$ the equation $U_y(z) = \chi$ has at most one pair of nonreal conjugate roots. The condition that $U_y(z) = \chi$ has double roots gives

$$\begin{cases} U_y(z) = \chi. \\ U'_y(z) = 0 \end{cases}$$

which gives the parametric equation for (x, s) as follows.

$$\begin{cases} \chi = \frac{3z^3}{z^3+2} \\ s = \frac{z^3-3z+2}{z^3+2} \end{cases}$$

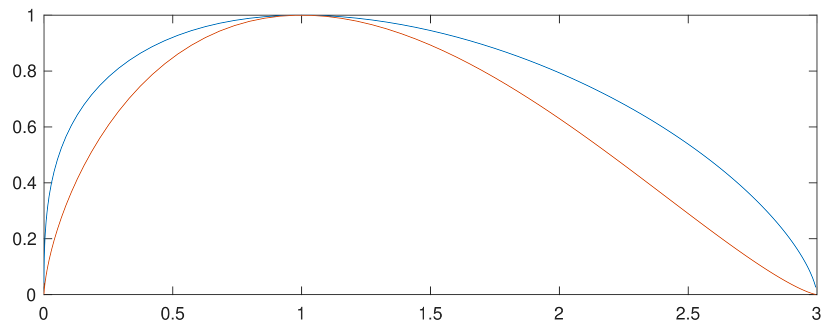

Figure 3: Frozen boundary for the scaling limit of weighted non-interaction paths. The blue curve is for the uniform weight; the red curve is when the limit weight function s satisfies $y = (1-s)^2$.

1. *When $x_1 = x_2 = \ldots = x_n$, and $\alpha = 1$, we have $s = 1 - y$. The frozen boundary is given by the blue curve of Figure 3.*
2. *When $\alpha = 1$, and $y = (1-s)^2$. The frozen boundary is given by the red curve of Figure 3.*

On the lecture hall graph $\mathcal{G}$, define a random height function h associated to a random non-intersecting path configuration as follows. The height at the lower left corner is 0, and the height increases by 1 whenever crossing a path from the left to the right. Define the rescaled height function by

$$h_n(\chi, y) := \frac{1}{n} h(n\chi, ny)$$

Following similar computations before Lemma 8.1 of [4], we obtain that when (χ, y) is in the liquid region,

$$\lim_{n\to\infty} \frac{dh_n(\chi, y)}{d\chi} = \frac{1}{\pi}\mathrm{Arg}(\mathbf{z}_+(\chi, y) - 1 + s).$$

where $\mathbf{z}_+(\chi, y)$ is the unique root in the upper half plane of the equation $U_y(z) = \chi$.

Theorem 4. *Assume $\mathcal{G}$ is uniformly weighted such that $s = 1 - y$. Suppose that the assumptions of Theorem 3 holds. Let*

$$u = \frac{1}{\mathbf{z}_+(\chi, y) S_{\mathbf{m}_0}^{(-1)}(\ln \mathbf{z}_+(\chi, y))}$$

Then

$$\frac{\partial h}{\partial x} = \frac{1}{\pi}\left(2 - \mathrm{Arg}(u)\right); \qquad \frac{\partial h}{\partial y} = \frac{1}{\pi}\Im u \tag{2.9}$$

where $\operatorname{Arg}(\cdot)$ *is the branch of the argument function taking values in* $[0, 2\pi)$. *Moreover,* u *satisfies the complex Burgers equation*

$$u_x - uu_y = 0. \tag{2.10}$$

3 Height Fluctuations and Gaussian Free Field

Let C_0^∞ be the space of smooth real-valued functions with compact support in the upper half plane $\mathbb{H}$. The **Gaussian free field** (GFF) Ξ on $\mathbb{H}$ with the zero boundary condition is a collection of Gaussian random variables $\{\xi_f\}_{f\in C_0^\infty}$ indexed by functions in C_0^∞, such that the covariance of two Gaussian random variables ξ_{f_1}, ξ_{f_2} is given by

$$\operatorname{Cov}(\xi_{f_1}, \xi_{f_2}) = \int_{\mathbb{H}}\int_{\mathbb{H}} f_1(z) f_2(w) G_{\mathbb{H}}(z, w) dz d\bar{z} dw d\overline{w},$$

where

$$G_{\mathbb{H}}(z, w) := -\frac{1}{2\pi} \ln \left| \frac{z - w}{z - \overline{w}} \right|, \qquad z, w \in \mathbb{H}$$

is the Green's function of the Dirichlet Laplacian operator on $\mathbb{H}$. The Gaussian free field Ξ can also be considered as a random distribution on C_0^∞ of $\mathbb{H}$, such that for any $f \in C_0^\infty$, we have

$$\Xi(f) = \int_{\mathbb{H}} f(z)\Xi(z)dz := \xi_f;$$

where $\Xi(z)$ is the generalized function corresponding to the linear functional Ξ. Note that GFF is conformally invariant; in the sense that for any simply-connected domain $\mathcal{D} \subsetneq \mathbb{C}$, and let $\phi : \mathcal{D} \to \mathbb{H}$ be a conformal map from $\mathcal{D}$ to $\mathbb{H}$. Then the GFF on $\mathcal{D}$ is

$$\Xi_{\mathcal{D}}(z) := \Xi(\phi(z))$$

See [22] for more about GFF.

Let f be a function of r variables. Define the symmetrization of f as follows

$$\operatorname{Sym}_{x_1,\ldots,x_r} f(x_1, \ldots, x_r) := \frac{1}{r!} \sum_{\sigma \in S_r} f(x_{\sigma(1)}, \ldots, x_{\sigma(r)}); \tag{3.1}$$

Assumption 1. *Let* l *be a fixed positive integer. Assume there exists*

$$0 = a_1 < b_1 < a_2 < b_2 < \ldots < a_l < b_l$$

such that $\mathbf{m}_0$*, the limit counting measure corresponding to the partition on the bottom boundary satisfies*

$$\frac{d\mathbf{m}_0}{dx} = \begin{cases} 1 & \text{if } a_i < x < b_i \\ 0 & \text{if } b_j < x < a_{j+1} \end{cases}$$

where $i \in [l]$ *and* $j \in [l-1]$*.*

Theorem 5. *Suppose that Assumption 1 holds. For each* $z \in \mathbb{H}$*, let*

$$\mathbf{\Delta}_n(z) := \Delta_n(n\chi_{\widetilde{\mathcal{L}}}(z), ns_{\widetilde{\mathcal{L}}}(z)) := \sqrt{\pi}\left|\{g \in [n] : \lambda_g^{(n-ny(s_{\widetilde{\mathcal{L}}}(z)))} - n + g \geq n\chi_{\widetilde{\mathcal{L}}(z)}\}\right|$$

Under the assumption of Theorem 2, $\mathbf{\Delta}_n(z) - \mathbb{E}\mathbf{\Delta}_n(z)$ *converge to GFF in the upper half plane in the sense that for each* $s \in (0,1)$

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\chi^j\left(\Delta_n(n\chi, ns) - \mathbb{E}\Delta_n(n\chi, ns)\right)d\chi = \int_{z\in\mathbb{H}:s_{\widetilde{\mathcal{L}}}(z)=s}\chi_{\widetilde{\mathcal{L}}}^j(z)\frac{d\chi_{\widetilde{\mathcal{L}}}(z)}{dz}\Xi(z)dz$$

The main idea to prove Theorem 5 is to first show that a collection of certain observables converge to a Gaussian vector in the scaling limit by applying the Wick's moment formula; then find an explicit diffeomorphism from the liquid region to the upper half plane, which gives the convergence of the observables to the pull-back of GFF in $\mathbb{H}$.

Acknowledgements

We thank Sylvie Corteel for asking the questions solved in the paper and for helpful discussions. DK and IP are grateful to the Workshop on 'Randomness, Integrability, and Universality', held on Spring 2022 at the Galileo Galilei Institute for Theoretical Physics, for hospitality and support at some stage of this work.

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