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Excedance quotients, Quasisymmetric Varieties, and Temperley–Lieb algebras

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Abstract. Let $R_n = \mathbb{Q}[x_1, x_2, \ldots, x_n]$ be the ring of polynomials in n variables and consider the ideal $\langle \mathrm{QSym}_n^+ \rangle \subseteq R_n$ generated by quasisymmetric polynomials without constant term. It was shown by Aval–Bergeron–Bergeron that $\dim\left(R_n/\langle \mathrm{QSym}_n^+ \rangle\right) = C_n$ the nth Catalan number. We explain here this phenomenon by defining a set of permutations QSV_n with the following properties: first, QSV_n is a basis of the Temperley–Lieb algebra $\mathsf{TL}_n(2)$, and second, when considering QSV_n as a collection of points in $\mathbb{Q}^n$, the top-degree homogeneous component of the vanishing ideal $\mathbf{I}(\mathrm{QSV}_n)$ is $\langle \mathrm{QSym}_n^+ \rangle$. Our construction has a few byproducts which are independently noteworthy.

Résumé. Soit $R_n = \mathbb{Q}[x_1, x_2, \ldots, x_n]$ l'anneau des polynômes en n variables, et considérez l'idéal $\langle \mathrm{QSym}_n^+ \rangle \subseteq R_n$ engendré par les polynômes quasisymétriques sans terme constant. Il a été démontré par Ava–Bergeron–Bergeron que $\dim\left(R_n/\langle \mathrm{QSym}_n^+ \rangle\right) = C_n$ le n-ième nombre de Catalan. Nous expliquons ici ce phénomène en construisant un ensemble de permutations QSV_n ayant les propriétés suivantes: premièrement, QSV_n est une base de l'algèbre de Temperley–Lieb $\mathsf{TL}_n(2)$, et deuxièmement, en considérant QSV_n comme une collection de points dans $\mathbb{Q}^n$, la composante homogène de degré supérieur de l'idéal $\mathbf{I}(\mathrm{QSV}_n)$ est $\langle \mathrm{QSym}_n^+ \rangle$. Notre construction a quelques sous-produits qui sont indépendamment dignes d'intérêt.

Keywords: Quasisymmetric Polynomials, Bruhat order, Excedance, Temperley–Lieb

1 Introduction

Quasisymmetric functions originate in the work of Stanley [18], where they appear as enumeration series for P-partitions. Later, Gessel [8] gave a more algebraic treatment of the ring QSym spanned by all quasisymmetric functions, establishing a beautiful analogy with the classical ring of symmetric functions Sym. The importance of QSym has continued to increase: [1] established QSym as a universal setting for enumerative combinatorial invariants, and in recent years quasisymmetric functions have been at the center of a number of research programs (many examples can be found in [11, 15, 16] and references therein).

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In this abstract, based on the paper [4], we explore the striking similarity between quasisymmetric functions and the invariant theory of finite reflection groups. Chevalley's theorem states that each finite reflection group W acts naturally on a polynomial ring R, and the quotient of R by the ideal $\langle R_+^W \rangle$ generated by positive degree invariants is isomorphic to the regular module of W; see [13, Chapter 3]. Hivert [12] shows that the quasisymmetric polynomials QSym_n in $R_n = \mathbb{Q}[x_1, \ldots, x_n]$ are likewise the invariants of an action of the Temperley–Lieb algebra $\mathsf{TL}_n(2)$ on R_n. Writing $\langle \mathrm{QSym}_n^+ \rangle$ for the ideal generated by the positive degree quasisymmetric polynomials, [2, 3] show that the dimension of the coinvariant space $R_n/\langle \mathrm{QSym}_n^+ \rangle$ and $\mathsf{TL}_n(2)$ agree: both are the nth Catalan number C_n. Since $\mathsf{TL}_n(2)$ shares many nice properties with reflection groups, one might expect a Chevalley-type theorem from this coincidence, but there is no obvious $\mathsf{TL}_n(2)$-action on $R_n/\langle \mathrm{QSym}_n^+ \rangle$: Hivert's action is not multiplicative and $\langle \mathrm{QSym}_n^+ \rangle$ is not a $\mathsf{TL}_n(2)$-submodule.

Motivated by the discussion above, we revisit two modules which afford the left regular representation of the symmetric group S_n:

(1) the quotient $R_n/\langle \mathrm{Sym}_n^+ \rangle$ of the polynomial ring $R_n = \mathbb{Q}[x_1, \ldots, x_n]$ by the ideal generated by positive-degree symmetric polynomials Sym_n^+, and
(2) the coordinate ring $R_n/\mathbf{I}(S_n)$ for the vertices of the regular permutohedron S_n in $\mathbb{Q}^n$, which are the points $(\sigma_1, \ldots, \sigma_n)$ for each permutation σ on n letters.

Module (1) is a famous case of Chevalley's theorem: the S_n-invariants of R_n are the symmetric polynomials, and $R_n/\langle \mathrm{Sym}_n^+ \rangle$ is the S_n coinvariant ring. On the other hand, module (2) comes from the left multiplicative action of S_n on the permutohedron realized on the coordinate ring $R_n/\mathbf{I}(S_n)$ where $\mathbf{I}(S_n)$ is the vanishing ideal. However, as seen in the work of Garsia and Procesi [7] and reference therein, a careful inspection reveals that these modules determine one another! Consider the ideal

$$I_n = \langle f(x_1, \ldots, x_n) - f(1, \ldots, n) \mid f \in \mathrm{Sym}_n^+ \rangle \subseteq \mathbf{I}(S_n).$$

For each $f \in R_n$, let $\mathsf{h}(f)$ denote the top-degree homogeneous component of f, and for any ideal I in R_n write $\mathsf{gr}(I) = \langle \mathsf{h}(f) \mid f \in I \rangle$. Then $\mathsf{gr}(I_n) \supseteq \langle \mathrm{Sym}_n^+ \rangle$, and Gröbner basis theory gives a linear isomorphism $R_n/\mathsf{gr}(I_n) \cong R_n/I_n$. We therefore have

$$|S_n| = \dim\left(R_n/\langle \mathrm{Sym}_n^+ \rangle\right) \geqslant \dim\left(R_n/\mathsf{gr}(I_n)\right) = \dim\left(R_n/I_n\right) \geqslant \dim(R_n/\mathbf{I}(S_n)) = |S_n|,$$

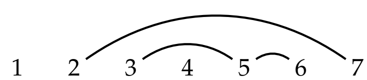
so that $I_n = \mathbf{I}(S_n)$ and $\mathsf{gr}(I_n) = \langle \mathrm{Sym}_n^+ \rangle$, and $R_n/\langle \mathrm{Sym}_n^+ \rangle \cong R_n/\mathbf{I}(S_n)$ as vector spaces. This isomorphism respects the S_n-action on each quotient: both $\mathbf{I}(S_n)$ and $\langle \mathrm{Sym}_n^+ \rangle$ are fixed spaces for the standard S_n-action on R_n, and this action coindices with the action on points for $R_n/\mathbf{I}(S_n)$. Thus, we have an S_n-module isomorphism $R_n/\langle \mathrm{Sym}_n^+ \rangle \cong R_n/\mathbf{I}(S_n)$, though the left hand side has a natural grading and the right hand side does not.

Our work in [4] applies this approach to quasisymmetric functions and Temperley–Lieb algebras. It is known that $\langle \mathrm{Sym}_n^+ \rangle \subseteq \langle \mathrm{QSym}_n^+ \rangle$, and that there is a surjective algebra homomorphism $\phi : \mathbb{C}S_n \to \mathsf{TL}_n(2)$. Guided by these relationships, we searched for a subset $\mathrm{QSV}_n \subseteq S_n \subseteq \mathbb{Q}^n$ which satisfies:

(i) $|\mathrm{QSV}_n| = C_n$,
(ii) the image $\phi(\mathrm{QSV}_n)$ is a basis of $\mathsf{TL}_n(2)$, and
(iii) considering the vanishing ideal $\mathbf{I}(\mathrm{QSV}_n)$, we have $\mathsf{gr}\big(\mathbf{I}(\mathrm{QSV}_n)\big) = \langle \mathrm{QSym}_n^+ \rangle$.

Assuming such a set exists, one can define an action of $\mathsf{TL}_n(2)$ on the space $R_n/\langle \mathrm{QSym}_n^+ \rangle$ using Gröbner basis theory and the multiplication constants for the basis obtained from QSV_n. However, QSV_n is not readily found: it took several years of computer exploration to find a list of candidates for small values of n. We have now found it, along with a number of remarkable properties that should be of interest to the wider community.

The set $\mathrm{QSV}_n \subseteq S_n$ is defined in Section 3. After discovering it, we noticed that the cycle structure of permutations in QSV_n determine a noncrossing partition, tying them to a more general story of Coxeter–Catalan combinatorics for the symmetric groups [5] (see also [17]). For example, writing Q_λ to denote the element of QSV_n indexed by the partition λ,

$\lambda =$ 

corresponds to $Q_\lambda = (1)(72)(653)(4)$.

Through this connection, [9, 10] and [20] have studied bases of general Temperley–Lieb algebras which specialize to $\phi(\mathrm{QSV}_n)$ for $\mathsf{TL}_n(2)$, so only condition (iii) remains.

Our initial attempts to prove condition (ii) also led us to an exciting discovery about how QSV_n sits in S_n. In Section 4 we define an equivalence relation $\sim$ on S_n using the weak excedance set of a permutation and its inverse. We call the equivalence classes of $S_n/\!\sim$ *excedance classes*, and show that each noncrossing partition λ bijectively determines an excedance class $\mathcal{C}_\lambda$. Surprisingly, the Bruhat order induces a well-defined quotient order on excedance classes. In the following, $\preceq$ denotes the order on noncrossing partitions which is dual to Young's lattice, described further in Section 3.

Theorem 4.2. *Writing $\leqslant$ for the relation on excedance classes $S_n/\!\sim$ induced by the Bruhat order, $\mathcal{C}_\lambda \leqslant \mathcal{C}_\mu$ if and only if $\lambda \preceq \mu$.*

This exhibits a duality between sub- and quotient orders of the Bruhat poset: a parallel result is given by [10] for the set QSV_n as a sub-poset of the Bruhat order (see Section 3). The result of [10] also simplifies the proof of Theorem 4.2 we give in [4].

Corollary 4.3. *Each excedance class $\mathcal{C}_\lambda$ is an interval in the Bruhat order, with upper bound $Q_\lambda \in \mathrm{QSV}_n$ and lower bound given by a 321-avoiding permutation.*

The combinatorics of excedance classes are very rich, and there is much left to explore. In Section 5, we use excedance classes of S_n to produce bases of $\mathsf{TL}_n(2)$. Using results of [10] and [20], our Theorem 5.1 restates the fact that QSV_n satisfies condition (ii) above. However, our technique is more general, and produces many (often novel) bases of $\mathsf{TL}_n(2)$ coming from the surjection $\phi : \mathbb{C}S_n \to \mathsf{TL}_n(2)$.

Theorem 5.2. *Let $n \geqslant 0$ and for each noncrossing partition λ of size n, fix an element $w_\lambda \in \mathcal{C}_\lambda$. Then the set $\{\phi(w_\lambda) \mid \text{noncrossing partitions } \lambda\}$ is a basis of $\mathsf{TL}_n(2)$.*

Finally, in Section 6 we outline our approach to proving that the set QSV_n satisfies condition (iii) above. The space of positive-degree quasisymmetric polynomials QSym_n has a homogeneous basis of monomial quasisymmetric functions M_α indexed by the compositions $\alpha \vDash d$ of positive integers $d > 0$ with length $\ell(\alpha) \leqslant n$. For each such composition α, we construct a nonhomogeneous polynomial $P_\alpha \in R_n$ for which $\mathsf{h}(P_\alpha) = M_\alpha$ and show the following.

Theorem 6.3. *The ideal $\langle P_\alpha \mid \alpha \vDash d \text{ with } d > 0 \text{ and } \ell(\alpha) \leqslant n \rangle \subseteq R_n$ is the vanishing ideal $\mathbf{I}(\mathrm{QSV}_n)$ and $\langle \mathrm{QSym}_n^+ \rangle = \mathsf{gr}\big(\mathbf{I}(\mathrm{QSV}_n)\big)$.*

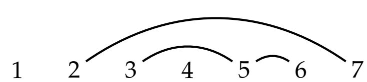
From this, we obtain a linear isomorphism $R_n/\mathbf{I}(\mathrm{QSV}_n) \cong R_n/\langle \mathrm{QSym}_n^+ \rangle$.

2 Noncrossing partitions and Bruhat order

Noncrossing partitions: Let n be a nonnegative integer. A *noncrossing partition* of size n is a diagram λ consisting of:

1. the positive integers $1, \ldots, n$, placed from left to right along a horizontal axis; and
2. a set of left-to-right arcs $i \frown j = (i, j)$, $i < j$ drawn above the axis with no intersections or coterminal points: λ contains no pair $i \frown k$, $j \frown l$ with $i \leqslant j < k \leqslant l$.

For example,

$\lambda =$  (2.1)

is a noncrossing partition of size 7 containing three arcs: $2 \frown 7$, $3 \frown 5$, and $5 \frown 6$.

Considering a noncrossing partition λ as an (undirected) graph, the connected components of λ give a partition of the set $[n] = \{1, \ldots, n\}$, which is the origin of the term. For example, the noncrossing partition shown in Equation (2.1) corresponds to the set partition $\big\{\{1\}, \{2, 7\}, \{3, 5, 6\}, \{4\}\big\}$. Let

$$\mathrm{NCP}_n = \{\text{noncrossing partitions of size } n\}.$$

The size of NCP_n is the nth Catalan number, $C_n = \frac{1}{n+1}\binom{2n}{n}$ [19].

Given an arc $i \frown j \in \lambda$, say that i is the *left endpoint* and j is the *right endpoint*, and let

$$\lambda^+ = \{\text{left endpoints in } \lambda\} \qquad \text{and} \qquad \lambda^- = \{\text{right endpoints in } \lambda\}.$$

For example, with the noncrossing partition λ in (2.1), $\lambda^+ = \{2,3,5\}$ and $\lambda^- = \{5,6,7\}$. The arcs in λ give a bijection between the sets λ^+ and λ^-, so that $|\lambda^+| = |\lambda^-|$.

Permutations and the Bruhat order: Let S_n denote the group of permutations of $[n]$. We represent elements of S_n either by using the standard one- and two-line notations or as a product of cycles. We also write ℓ for the length function, so that for $w \in S_n$, $\ell(w)$ is the number of inversions of w: $\ell(w) = |\{(i,j) \mid 1 \leqslant i < j \leqslant n \text{ and } w_i > w_j\}|$.

The *Bruhat order* on S_n is the partial order generated by the relation

$$v < w \qquad \text{if and only if} \qquad wv^{-1} \text{ is a transposition } (i\,j) \text{ and } \ell(v) < \ell(w).$$

This order is ubiquitous in the study of S_n and related objects (for examples, see [6]).

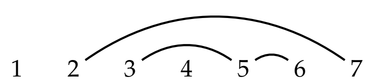
3 The set QSV_n

Let λ be a noncrossing partition of size n. Define a permutation $Q_\lambda \in S_n$ by

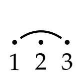
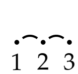
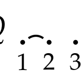
$$Q_\lambda(j) = \begin{cases} i & \text{if } j \in \lambda^- \text{ and } i \frown j \in \lambda \\ k & \text{if } j \notin \lambda^- \text{ and } k \text{ is the largest element connected to } j \text{ in } \lambda \end{cases}$$

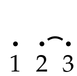
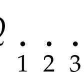
Thus, Q_λ sends each $j \in [n]$ to its leftward neighbor in λ, if such a neighbor exists, and otherwise sends j to the rightmost element of its connected component.

The cycles of Q_λ correspond to the connected components of λ, for example, with

$\lambda =$  we have $Q_\lambda = (1)(72)(653)(4) = 1764352.$

Let $\mathrm{QSV}_n = \{Q_\lambda \mid \lambda \in \mathrm{NCP}_n\}$. For example, the elements of QSV_3 are:

Q  $= 321$, Q  $= 312$, Q  $= 213$,

Q  $= 132$, and Q  $= 123$.

Remark 3.1. Given any n-cycle $c \in S_n$, [5] gives a bijection between NCP_n and the interval between the identity and c in the absolute order on S_n. Our construction of the permutations Q_λ realize this bijection for the n-cycle $c = (n \cdots 21)$.

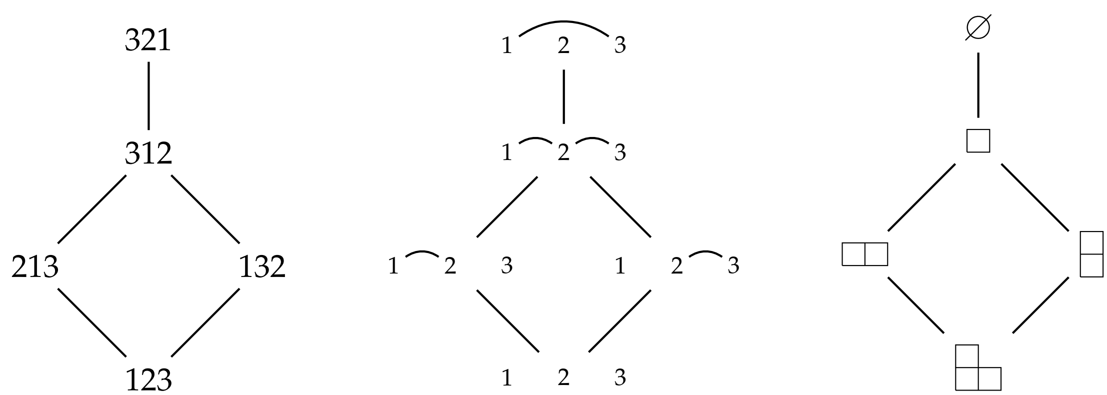


Figure 1: From left to right, the Hasse diagrams of: QSV_3 with the Bruhat order; NCP_3 with $\preceq$; and the dual interval in the Young's lattice.

The Bruhat order on QSV_n: The Bruhat order on S_n described in Section 2 restricts to a partial order on the set QSV_n. This order turns out to be very natural, as is described in the paper [10], and we recall the description for use in later sections.

Define a partial order $\preceq$ on the set NCP_n of noncrossing partitions as the extension of the covering relation: λ is covered by μ if and only if λ is obtained from μ in one of the following ways:

1. removing an arc of the form $i \frown i{+}1$ from μ, or
2. replacing any arc $i \frown k$ in μ with two arcs $i \frown j$ and $j \frown k$ for some $i < j < k$ which do not intersect or share a left or right endpoint with any other arc in μ.

Proposition 3.2 ([10, Theorem 1.1 and Corollary 7.5])**.** *Let λ and μ be noncrossing partitions of size n. The following are equivalent:*

1. $\lambda \preceq \mu$,

2. $Q_\lambda \leqslant Q_\mu$ in the Bruhat order.

Moreover, the partial orders on NCP_n and QSV_n are each dual to the interval between the empty diagram and the staircase in Young's lattice; see Figure 1.

Remark 3.3. In fact, [10] describes the Bruhat order on the set $\{\omega_0 w \omega_0^{-1} \mid w \in \mathrm{QSV}_n\}$, where ω_0 is the longest element of S_n. Vis-a-vis Remark 3.1, these are the non-crossing partitions associated with the cycle $(12\ldots n)$ instead of $(n\ldots 21)$. Since conjugation by ω_0 is an automorphism of the Bruhat order, this result is equivalent to Proposition 3.2.

4 The excedance quotient of the Bruhat order

In this section we describe a novel equivalence relation $\sim$ on S_n and show that it induces a quotient of the Bruhat order. This equivalence relation is defined in a simple way using the weak excedances of a permutation. We have discovered a number of nice properties of the equivalence classes in $S_n/{\sim}$, which we summarize after our initial definition.

Given a permutation $w \in S_n$, a *weak excedance* of w is a pair (i, w_i) for which $i \leqslant w_i$. We define the *excedance values* $E_{val}(w)$ and *excedance positions* $E_{pos}(w)$ to be the sets

$$E_{val}(w) = \{w_i \mid (i, w_i) \text{ is a weak excedance of } w\}, \text{ and}$$

$$E_{pos}(w) = \{i \mid (i, w_i) \text{ is a weak excedance of } w\}.$$

The sets $E_{val}(w)$ and $E_{pos}(w)$ are most easily seen using two-line notation for permutations. For example, marking the non-excedances of a permutation in red,

$$w = \overset{12345678}{35142658}, \qquad E_{pos}(w) = \{1, 2, 4, 6, 8\}, \qquad \text{and} \qquad E_{val}(w) = \{3, 4, 5, 6, 8\}.$$

We define the *excedance relation* $\sim$ on S_n by:

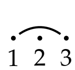
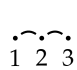
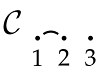
$$v \sim w \qquad \text{if and only if} \qquad E_{val}(v) = E_{val}(w) \text{ and } E_{pos}(v) = E_{pos}(w), \tag{4.1}$$

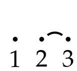
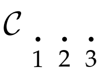
and say that each equivalence class of $S_n/{\sim}$ is an *excedance class*.

We now summarize our main results on excedance classes. Each noncrossing partition λ of size n determines an excedance class:

$$\mathcal{C}_\lambda = \{w \in S_n \mid E_{val}(w) = [n] - \lambda^+ \text{ and } E_{pos}(w) = [n] - \lambda^-\}.$$

This construction is bijective, so that the excedance classes are counted by the Catalan numbers. For example, the five excedance classes of S_3 are:

$\mathcal{C}$  $= \{\overset{123}{321}, \overset{123}{231}\}$, $\mathcal{C}$  $= \{\overset{123}{312}\}$, $\mathcal{C}$  $= \{\overset{123}{213}\}$,

$\mathcal{C}$  $= \{\overset{123}{132}\}$, and $\mathcal{C}$  $= \{\overset{123}{123}\}$.

The Bruhat order induces a relation on $S_n/{\sim}$. Recall the order $\preceq$ from Section 3.

Theorem 4.2. *Writing $\leqslant$ for the relation on excedance classes $S_n/{\sim}$ induced by the Bruhat order, $\mathcal{C}_\lambda \leqslant \mathcal{C}_\mu$ if and only if $\lambda \preceq \mu$.*

Our proof Theorem 4.2 in [4] includes the intermediate result that each excedance class $\mathcal{C}_\lambda$ contains unique Bruhat-minimal and Bruhat-maximal elements, and moreover these are respectively a 321-avoiding permutation and the element $Q_\lambda \in \mathrm{QSV}_n$. Combined with Theorem 4.2, this implies the following corollary.

Corollary 4.3. *Each excedance class $\mathcal{C}_\lambda$ is an interval in the Bruhat order, with maximum $Q_\lambda \in \mathrm{QSV}_n$ and minimum given by a* 321*-avoiding permutation.*

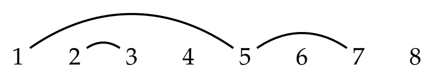
We now identify the minimal element of each excedance class. For a noncrossing partition λ of size n, enumerate the sets λ^+, λ^-, $[n] - \lambda^+$, and $[n] - \lambda^-$ in increasing order as

$$\lambda^+ = \{a_1 < a_2 < \cdots < a_s\}, \qquad \lambda^- = \{b_1 < b_2 < \cdots < b_s\},$$
$$[n] - \lambda^+ = \{x_1 < x_2 < \cdots < x_{n-s}\}, \qquad \text{and} \qquad [n] - \lambda^- = \{y_1 < y_2 < \cdots < y_{n-s}\}.$$

Let $T_\lambda \in S_n$ be the permutation with

$$T_\lambda(i) = \begin{cases} a_r & \text{if } i \in \lambda^- \text{ and } i = b_r \\ x_r & \text{if } i \notin \lambda^- \text{ and } i = y_r. \end{cases}$$

Thus, the two-line notation for T_λ can be obtained by placing the elements of λ^+ in increasing left-to-right order below the elements of λ^-, and placing the elements of $[n] - \lambda^+$ below the elements of $[n] - \lambda^-$ in the same manner. For example, with $n = 8$ and

$\lambda =$ 

we have $\lambda^+ = \{1, 2, 5\}$ and $\lambda^- = \{3, 5, 7\}$, $[8] - \lambda^+ = \{3, 4, 6, 7, 8\}$, and $[8] - \lambda^- = \{1, 2, 4, 6, 8\}$, and consequently

$$T_\lambda = \begin{matrix} 12345678 \\ 34162758 \end{matrix},$$

where non-excedances are marked in red, as at the beginning of Section 4.

Proposition 4.4. *For all noncrossing partitions λ, $T_\lambda \in \mathcal{C}_\lambda$, is the Bruhat-minimum element of $\mathcal{C}_\lambda$, and is* 321*-avoiding.*

Remark 4.5. Proposition 4.4 implicitly defines a bijection between 321-avoiding permutations and noncrossing partitions. This bijection is equivalent to one used by Zinno in [20] and Gobet in [9].

5 Bases for the Temperley–Lieb Algebra $\mathsf{TL}_n(2)$

The Temperley–Lieb algebra $\mathsf{TL}_n(2)$ is the $\mathbb{C}$-algebra generated by elements $e_1, \ldots, e_{n-1}$ subject to the following relations for each $1 \leqslant i, j \leqslant n$

$$e_i^2 = 2e_i; \qquad e_ie_j = e_je_i \text{ if } |i - j| > 1; \qquad e_ie_je_i = e_i \text{ if } |i - j| = 1.$$

There is a surjective algebra morphism from the symmetric group algebra $\mathbb{C}S_n$ to $\mathsf{TL}_n(2)$ given by $\phi : \mathbb{C}S_n \to \mathsf{TL}_n(2)$ where $\phi(s_i) = 1 - e_i$. In particular $\mathsf{TL}_n(2) \cong S_n/\ker(\phi)$.

It is well-known that the images of all 321-avoiding permutations under ϕ forms a basis for $\mathsf{TL}_n(2)$. Gobet [9] shows that the set QSV_n has a similar property.

Theorem 5.1 ([9, Theorem 7.21])**.** *For all $n \geqslant 0$, the set $\phi(\mathrm{QSV}_n)$ is a basis for $\mathsf{TL}_n(2)$.*

In our investigation of excedance classes we found an application of their structure the problem of computing sets of permutations which give bases of $\mathsf{TL}_n(2)$ under ϕ. We include it here as it is a nice result of our current investigation.

Theorem 5.2. *Let $n \geqslant 0$ and for each noncrossing partition λ of size n, fix an element $w_\lambda \in \mathcal{C}_\lambda$. Then the set $\{\phi(w_\lambda) \mid$ noncrossing partitions $\lambda\}$ is a basis of $\mathsf{TL}_n(2)$.*

Here, we discuss its implications: taking $w_\lambda = Q_\lambda$ in the theorem gives yet another proof of Theorem 5.1, confirming the results of [10] and [20]. In general, however, many bases obtained via Theorem 5.2 are novel. The smallest novel example can be found with $n = 4$: the set

$$\{\overset{1234}{4312}, \overset{1234}{4231}, \overset{1234}{4213}, \overset{1234}{3142}, \overset{1234}{1432}, \overset{1234}{4123}, \overset{1234}{3214}, \overset{1234}{3124}, \overset{1234}{2143}, \overset{1234}{1323}, \overset{1234}{2134}, \overset{1234}{1324}, \overset{1234}{1243}, \overset{1234}{1234}\}$$

meets the criteria of Theorem 5.2, and accordingly maps to a basis of $\mathsf{TL}_n(2)$ under ϕ. This set is neither QSV_4 nor the set of 321-avoiding permutations ($4312 \notin \mathrm{QSV}_4$ and is not 321-avoiding). Moreover, the set above is not described in [10, 20]: each subset of S_4 in these sources which is not QSV_4 contains more than one element from certain excedance classes and none from others.

6 The quasisymmetric variety

In this section, we summarize Theorem 6.3 and its proof, which is given in full in our paper [4]. As in the introduction, let QSym_n denote the quasisymmetric polynomials in $R_n = \mathbb{Q}[x_1, \ldots, x_n]$ and write M_α for the monomial quasisymmetric function indexed by the composition α. In Section 6.1, we define a family of non-homogeneous polynomials P_α which are also indexed by compositions and we show that

$$P_\alpha = M_\alpha + \text{lower degree terms.} \tag{6.1}$$

For a permutation $\sigma \in S_n$, we write $P_\alpha(\sigma)$ for the evaluation of P_α at $x_1 = \sigma_1$, $x_2 = \sigma_2$, and so on. Recall the set QSV_n defined in Section 3.

Theorem 6.2. *For each non-empty integer composition α with at most n parts and any $\sigma \in \mathrm{QSV}_n$ we have $P_\alpha(\sigma) = 0$.*

Our proof of Theorem 6.2 in [4] uses the noncrossing cycle structure of each element $\sigma \in \mathrm{QSV}_n$, as well as a sign-reversing involution to establish desired vanishing property.

Now recall that for any $f \in R_n$, $\mathsf{h}(f)$ denotes the homogeneous top-degree component of f, and that for any ideal $I \subseteq R_n$, we write $\mathsf{gr}(I) = \langle \mathsf{h}(f) \mid f \in I \rangle$. Standard results in Gröbner basis theory give a linear isomorphism $R_n/I \cong R_n/\mathsf{gr}(I)$. With Theorem 6.2 and the dimension considerations set out in the introduction, this proves of our main result.

Theorem 6.3. *The ideal* $\langle P_\alpha \mid$ *non-empty compositions* α *of length* $\ell(\alpha) \leqslant n\rangle \subseteq R_n$ *is the vanishing ideal* $\mathbf{I}(\mathrm{QSV}_n)$ *and*

$$\langle \mathrm{QSym}_n^+ \rangle = \mathsf{gr}\big(\mathbf{I}(\mathrm{QSV}_n)\big),$$

where QSym_n^+ *denotes the set of positive-degree quasisymmetric functions.*

Using Gröbner basis theory again, we obtain the following corollary.

Corollary 6.4. *We have* $R_n/\langle \mathrm{QSym}_n^+\rangle \cong R_n/\mathbf{I}(\mathrm{QSV}_n)$ *as vector spaces.*

Remark 6.5. Remarks 3.1 and 3.3 describe the combinatorics of the sets $\{w\sigma w \mid \sigma \in \mathrm{QSV}_n\}$, each of which corresponds to a unique n-cycle $c \in S_n$. It is natural to consider how Theorems 6.2 and 6.3 generalize to these sets as well, and we explain this below.

1. For the set $\{\omega_0\sigma\omega_0 \mid \sigma \in \mathrm{QSV}_n\}$ corresponding to the Coxeter element $c = (1\,2\ldots n)$, our results generalize completely. In particular, the modified polynomials

$$\omega_0 P_\alpha \omega_0 = P_\alpha(-x_n + n + 1, \ldots, -x_2 + n + 1, -x_1 + n + 1)$$

vanish on every permutation $\omega_0\sigma\omega_0$ for $\sigma \in \mathrm{QSV}_n$. Moreover,

$$\mathsf{h}\big(\omega_0 P_\alpha \omega_0\big) = M_\alpha(-x_n, \ldots, -x_2, -x_1) = (-1)^{|\alpha|} M_{\overleftarrow{\alpha}},$$

where for a composition $\alpha = (\alpha_1, \ldots, \alpha_k)$, $M_{\overleftarrow{\alpha}}$ denotes the monomial quasisymmetric function corresponding to the reverse $\overleftarrow{\alpha} = (\alpha_k, \ldots, \alpha_1)$. This is closely related to the automorphisms of the ring of quasisymmetric functions (see, for example [14]).

2. For the sets corresponding to n-cycles other than $(1\,2\ldots n)$ and $(n\ldots 2\,1)$, the vanishing ideal does not have top-degree homogeneous component $\langle \mathrm{QSym}_n^+\rangle$.

6.1 The vanishing polynomial P_α

In this section we define the polynomials P_α and prove Theorem 6.2. We begin with a short review of compositions and the refinement order as they relate to QSym.

A *composition* is a sequence of positive integers $\alpha = (\alpha_1, \ldots, \alpha_k)$. We refer to k as the *length* of α and to $d = \sum_{i=1}^k \alpha_i$ as the *size* of α. Compositions are partially ordered by refinement: the composition α refines another composition $\beta = (\beta_1, \ldots, \beta_\ell)$ if there exists a sequence $1 = f_1 < f_2 < \cdots < f_{\ell+1} = k + 1$ for which $\beta_i = \alpha_{f_i} + \alpha_{f_i+1} + \cdots + \alpha_{f_{i+1}-1}$, and in this case we write $\beta \succeq \alpha$. Whenever we have a refinement relation $\beta \succeq \alpha$, we will use the notation $f_1, f_2, \ldots, f_{\ell+1}$ to refer to the sequence of indices in the definition.

For each composition of length $k \geqslant 1$, the monomial quasisymmetric function $M_\alpha \in R_n$ is defined by

$$M_\alpha = \sum_{1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k},$$

where the sum is over subsets $\{i_1, \ldots, i_k\}$ of $[n]$, enumerated in increasing order. Using the same convention we define the vanishing polynomial $P_\alpha \in R_n$ to be

$$P_\alpha = \sum_{\beta \geq \alpha} \sum_{1 \leqslant i_1 < i_2 < \cdots < i_\ell \leqslant n} \prod_{j=1}^{\ell} \Big((x_{i_j}^{\alpha_{f_j}} - i_j^{\alpha_{f_j}}) \prod_{s=f_j+1}^{f_{j+1}-1} (-i_j)^{\alpha_s} \Big).$$

While this formula appears to be quite dense, expanding it reveals an intuitive combinatorial structure. We compute one example in its entirety for the sake of exposition:

$$\begin{aligned} P_{(1,2,1)}(x_1, \ldots, x_4) = &(x_1 - 1)(x_2^2 - 2^2)(x_3 - 3) + (x_1 - 1)(x_2^2 - 2^2)(x_4 - 4) \\ &+ (x_1 - 1)(x_3^2 - 3^2)(x_4 - 4) + (x_2 - 2)(x_3^2 - 3^2)(x_4 - 4) \\ &\quad - (x_1 - 1)(x_2^2 - 2^2)2 - (x_1 - 1)(x_3^2 - 3^2)3 - (x_1 - 1)(x_4^2 - 4^2)4 \\ &\quad - (x_2 - 2)(x_3^2 - 3^3)3 - (x_2 - 2)(x_4^2 - 4^2)4 - (x_3 - 3)(x_4^2 - 4^2)4 \\ &\qquad - (x_1 - 1)1^2(x_2 - 2) - (x_1 - 1)1^2(x_3 - 3) - (x_1 - 1)1^2(x_4 - 4) \\ &\qquad - (x_2 - 2)2^2(x_3 - 3) - (x_2 - 2)2^2(x_4 - 4) - (x_3 - 3)3^2(x_4 - 4) \\ &\qquad\quad + (x_1 - 1)1^3 + (x_2 - 2)2^3 + (x_3 - 3)3^3 + (x_4 - 4)4^3, \end{aligned}$$

where summands corresponding to the same index $\beta \succeq (1,2,1)$ are grouped horizontally and by alignment. These values of β are respectively $(1,2,1)$, $(1,3)$, $(3,1)$, and (4).

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