Séminaire Lotharingien de Combinatoire **91B** (2024)
Article #65, 12 pp.

Proceedings of the 36th Conference on Formal Power Series and Algebraic Combinatorics (Bochum)

A Galois structure on the orbit of large steps walks in the quadrant

Pierre Bonnet*[1] and Charlotte Hardouin†[2]

[1]*LaBRI, Université de Bordeaux, Bordeaux*
[2]*Institut de mathématiques, Université Paul Sabatier, Toulouse*

Abstract. The enumeration of weighted walks in the quarter plane reduces to studying a functional equation with two catalytic variables. When the steps of the walk are small, Bousquet-Mélou and Mishna defined a group called *the group of the walk* which turned out to be crucial in the classification of the small steps models. In particular, its action on the catalytic variables provides a convenient set of changes of variables in the functional equation. This particular set called the *orbit* has been generalized to models with arbitrary large steps by Bostan, Bousquet-Mélou and Melczer (BBMM). However, the orbit had till now no underlying group.

In this article, we endow the orbit with the action of a Galois group, which extends the notion of the group of the walk to models with large steps. As an application, we look into a general strategy to prove the algebraicity of models with small backwards steps, which uses the fundamental objects that are *invariants* and *decoupling*. The group action on the orbit allows us to develop a Galoisian approach to these two notions. Up to the knowledge of the finiteness of the orbit, this gives systematic procedures to test their existence and construct them. Our constructions lead to the first proofs of algebraicity of weighted models with large steps, proving in particular a conjecture of BBMM, and allowing to find new algebraic models with large steps.

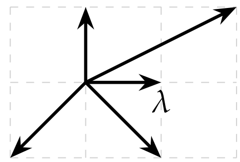
Keywords: functional equations, Galois theory, quadrant walks

1 Introduction and preliminaries

1.1 Walks in the quarter plane

A weighted walk in the quarter plane is defined as follows. Consider a finite subset $\mathcal{S}$ of $\mathbb{Z} \times \mathbb{Z}$. To each *step* s of $\mathcal{S}$ we attach its *weight* w_s which is a nonzero complex number. The tuple $\mathcal{W} = (\mathcal{S}, (w_s)_{s \in \mathcal{S}})$ is called a *weighted model of walks*. A *weighted walk in the quarter plane* of length n on the model $\mathcal{W}$ is then a sequence of points $P_0, \ldots, P_n$ in $\mathbb{N} \times \mathbb{N}$ such that for all i there exists $s_i \in \mathcal{S}$ satisfying $s_i + P_i = P_{i+1}$. The *weight* of the walk is the product $w_{s_0} w_{s_1} \ldots w_{s_{n-1}}$ of the weights of the steps taken by the walk.

*pierre.bonnet@u-bordeaux.fr
†hardouin@math.univ-toulouse.fr



The weighted model $\mathcal{G}_\lambda$ (for which we take any nonzero λ in $\mathbb{C}$) along with an example of a walk on $\mathcal{G}_\lambda$, of length 8, ending at $(3,0)$ and of weight λ^2.

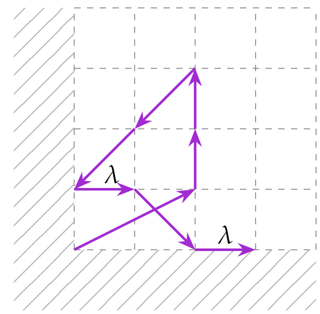


Figure 1: An example of weighted model and walk

The enumeration of weighted walks in the quarter plane has attracted a lot of attention over the past 20 years. Indeed, these objects are general enough to encode many objects in combinatorics (families of permutations, trees, maps), probability theory (stochastic processes, games of chance, sums of discrete random variables) or statistics (non-parametric tests). The attraction for this topic comes from the fact that the solution of this problem requires many different techniques and points of view, from combinatorics of course, but also from probability theory, computer algebra, differential Galois theory, complex analysis, geometry. . .

1.2 Generating function and classification

Given a model $\mathcal{W}$, denote by $q_{i,j,n}$ the sum of the weights of walks in the quadrant on $\mathcal{W}$ of length n starting at P_0 (taken as $(0,0)$ unless stated otherwise) and terminating at (i,j). The generating function for these walks is defined as

$$Q(X,Y,t) = \sum_{i,j,n\geq 0} q_{i,j,n} X^i Y^j t^n.$$

The weighted model is encoded as the *Laurent polynomial of the model* defined as $S(X,Y) = \sum_{s\in\mathcal{S}} w_s X^{s_x} Y^{s_y}$. From any weighted model $\mathcal{W}$, it is quite easy to form a functional equation for $Q(X,Y,t)$, as we demonstrate in the following example.

Example 1. Let $\mathcal{G}_\lambda = \{(-1,-1),(0,1),((1,0),\lambda),(2,1),(1,-1)\}$ as in Figure 1 (Example 2.1 in [2], see also Remark 2.2 for alternate weightings). Its Laurent polynomial is $S(X,Y) = \frac{1}{XY} + Y + \lambda X + \frac{X}{Y} + X^2 Y$. We now construct a recurrence on the walks: a walk terminating at coordinates (i,j) can be completed by a step s of $\mathcal{S}$ as long as $(i,j)+s$ is in $\mathbb{N}\times\mathbb{N}$. This translates into the following functional equation:

$$\begin{aligned}Q(X,Y,t) &= 1 + tX^2YQ(X,Y,t) + t\lambda XQ(X,Y,t) + tYQ(X,Y,t)\\ &+ t\frac{X}{Y}\left(Q(X,Y,t) - Q(X,0,t)\right) + t\frac{1}{XY}\left(Q(X,Y,t) - Q(X,0,t) - Q(0,Y,t) + Q(0,0,t)\right).\end{aligned}$$

Such an equation is then usually put in the following normal form:

$$\widetilde{K}(X,Y,t)Q(X,Y,t) = XY - t(X^2+1)Q(X,0,t) - tQ(0,Y,t) + tQ(0,0,t), \tag{1.1}$$

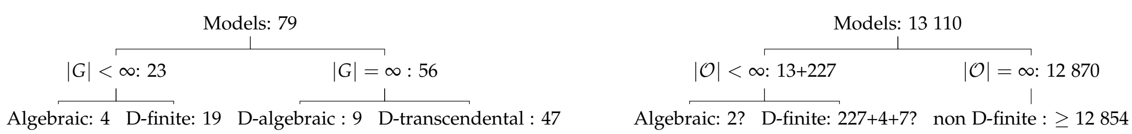
with $\widetilde{K}(X,Y,t)$ the *kernel polynomial of the walk* being equal here to $XY\,(1 - tS(X,Y))$.

Given a class of combinatorial objects, a natural question is to determine where its generating function fits in the classical hierarchy of power series of $\mathbb{C}(X,Y)[[t]]$

$$\textit{rational} \subset \textit{algebraic} \subset \textit{D-finite} \subset \textit{D-algebraic},$$

where algebraic series satisfy polynomial equations; D-finite series satisfy one linear differential equation in each variable X, Y, t; and D-algebraic series satisfy polynomial differential equations, all the coefficients being taken in the polynomial ring $\mathbb{C}[X,Y,t]$.

For walks, this hierarchy measures the complexity of a model: the lower its generating function in this hierarchy, the simpler the walks. The *catalytic variables equations* like (1.1) do not immediately allow to conclude to the position of their solutions in this hierarchy, hence the question of classifying the complexity of a model of walks in the quarter plane is highly nontrivial.



(a) $\mathcal{S} \subset \{-1,0,1\}^2$ (b) $\mathcal{S} \subset \{-2,-1,0,1\}^2$

Figure 2: The partial classifications for two families of unweighted models

For instance, consider the restriction of the problem to unweighted models with steps contained in $\{-1,0,1\}^2$ (these models are commonly called *small steps models*). The classification completed in 2018 and summarized in Figure 2a shows the numerous different behaviours that arise. Following the success of this first classification, the set of considered models has been extended to models with steps in $\{-2,1,0,-1\}^2$ in [3]. It is summarized in Figure 2b, and is currently incomplete. One of the reasons is that not all the tools used in the classification of small steps models extend to large steps.

For small steps models, [9, Chapter 4] proposes a complete strategy with Galois theoretic tools to classify solutions of functional equation of the form (1.1) when the kernel polynomial is biquadratic and the orbit is finite. Unfortunately, theses tools rely heavily on an elliptic uniformization of the algebraic curve associated with the kernel polynomial. We propose here a an extension of these tools with in sight a particular strategy to prove algebraicity, which goes beyond the elliptic framework since it deals with kernel polynomials of arbitrary degree. It has the advantage to stay within the realm of Laurent power series and is almost algorithmic until an algebraic characterization of certain invariants. Moreover, we hope that the geometric framework hidden behind our constructions will allow to adapt entirely the strategy of [1] to large steps models.

1.3 The group and the orbit

A fundamental object which arises in the study of models with small steps is the *group of the walk*, introduced by Bousquet-Mélou and Mishna in [5], following [9]. It is defined as follows. For a small steps model, we can write its Laurent polynomial in two ways:

$$\begin{aligned} S(X,Y) &= A_{-1}(X)/Y + A_0(X) + A_1(X)Y \\ &= B_{-1}(Y)/X + B_0(Y) + B_1(Y)X. \end{aligned}$$

Assume $A_{-1}(X)$, $A_1(X)$, $B_{-1}(Y)$ and $B_1(Y)$ to be nonzero. The polynomial $S(x,y)$ is left unchanged by the two *birational transformations* (that are involutions) of $\mathbb{C}\times\mathbb{C}$ defined as

$$\Phi\colon (u,v) \mapsto \left(\frac{B_{-1}(v)}{uB_1(v)}, v\right) \qquad \Psi\colon (u,v) \mapsto \left(u, \frac{A_{-1}(u)}{vA_1(u)}\right).$$

The *group of the walk* is then defined as $\langle \Phi, \Psi \rangle$, the subgroup of birational transformations of $\mathbb{C}\times\mathbb{C}$ generated by Φ and Ψ. This group turned out to be a crucial algebraic invariant of a model with small steps. For instance, an unweighted model with small steps has a D-finite generating function if and only if the group is finite (see the introduction of [1]).

The group also acts on pairs of catalytic variables. The orbit of its action on the pair (x,y) has a graph structure: the vertices are the pairs (u,v) of the orbit of (x,y), and two pairs are adjacent if one can be obtained from the other by applying Φ or Ψ to it.

If the model contains a large step, the equation $S(x,y) = S(x,y')$ may have non-rational solutions in x and y because of the higher degree of the polynomials, therefore in that case this group cannot be defined as a group of birational transformations of $\mathbb{C}\times\mathbb{C}$. Nonetheless, Bostan, Bousquet-Mélou and Melczer noted in [3] that the graph could be defined independently from the group, and called it the *orbit of the walk*. It is defined as follows. Denote by $\mathbb{K} = \overline{\mathbb{C}(x,y)}$ the algebraic closure of $\mathbb{C}(x,y)$ for two indeterminates x and y.

Definition 2 (Definition 3.1 in [3]). Let (u,v) and (u',v') be in $\mathbb{K}\times\mathbb{K}$. Then (u,v) and (u',v') are called *x-adjacent* if $S(u,v) = S(u',v')$ and $u = u'$. Similarly, they are called *y-adjacent* if $S(u,v) = S(u',v')$ and $v = v'$. We denote these two equivalence relations by $\sim^x$ and $\sim^y$. Two pairs are then adjacent if they are either x-adjacent or y-adjacent, and this relation is denoted by $\sim$. We denote by $\sim^*$ the transitive closure of $\sim$. The *orbit of the walk* is the set of pairs (u,v) such that $(u,v) \sim^* (x,y)$, and is denoted $\mathcal{O}$.

Example 3. For the small steps model $S(X,Y) = 1/X + 1/Y + XY$, the orbit is the cycle

$$(x,y) \overset{\Phi}{\longleftrightarrow} \left(\frac{1}{xy}, y\right) \overset{\Psi}{\longleftrightarrow} \left(\frac{1}{xy}, x\right) \overset{\Phi}{\longleftrightarrow} (y,x) \overset{\Psi}{\longleftrightarrow} \left(y, \frac{1}{xy}\right) \overset{\Phi}{\longleftrightarrow} \left(x, \frac{1}{xy}\right) \overset{\Psi}{\longleftrightarrow} (x,y)$$

Example 4. For $\mathcal{G}_\lambda$, the equation $S(x,y) = S(x',y)$ has three solutions x, x_1 and x_2 with x_1 algebraic of degree two over $\mathbb{C}(x,y)$. Continuing the construction, it turns out that the orbit $\mathcal{O}$ is finite of size 12 with coordinates in $\mathbb{C}(x,y,x_1)$ (see Figure 3).

In Section 2, we introduce a Galois framework to study the orbit. It allows us to generalize the group of the walk to weighted models with arbitrarily large steps.

1.4 A strategy based on invariants and decoupling

In their article [1], Bernardi, Bousquet-Mélou and Raschel introduced a general strategy for proving algebraicity of a small steps model in the quadrant, later adapted to models of the three-quadrant cone in [6]. This method relies critically on objects called *invariants*, of which we use one flavor in the realm of formal power series, the *t-invariants*.

Define the subring $\mathbb{C}_{\text{mul}}(X,Y)[[t]]$ of power series of $\mathbb{C}(X,Y)[[t]]$ whose coefficients in the t-expansion are of the form $\frac{A_n(X,Y)}{B_n(X)C_n(Y)}$ for A_n, B_n and C_n polynomials over $\mathbb{C}$. A power series in $\mathbb{C}_{\text{mul}}(X,Y)[[t]]$ is said to have *poles of bounded order at 0* if there is a bound on the order of the poles at $X=0$ and $Y=0$ of its coefficients (Definition 2.1 in [6]).

Definition 5. Let $F(X,Y,t)$ and $G(X,Y,t)$ be two power series of $\mathbb{C}_{\text{mul}}(X,Y)[[t]]$. They are *t-equivalent* (with respect to $\widetilde{K}$) if the power series $\frac{F(X,Y,t)-G(X,Y,t)}{\widetilde{K}(X,Y,t)} \in \mathbb{C}_{\text{mul}}(X,Y)[[t]]$ has poles of bounded order at 0. This equivalence is denoted by $F(X,Y,t) \equiv G(X,Y,t)$.

The t-equivalence relation is compatible with ring operations, and is used to define the notions of *t-invariants* and *t-decoupling*.

Definition 6 (Invariants, Def. 2.3 in [6])**.** Let $F(X,t), G(Y,t) \in \mathbb{C}_{\text{mul}}(X,Y)[[t]]$. The pair $(F(X,t), G(Y,t))$ is called a *pair of t-invariants* if $F(X,t) \equiv G(Y,t)$.

Definition 7 (Decoupling)**.** Let $H(X,Y,t)$ be a series of $\mathbb{C}_{\text{mul}}(X,Y)[[t]]$. Then $H(X,Y,t)$ admits a *t-decoupling* if there exist $F(X,t)$ in $\mathbb{C}_{\text{mul}}(X)[[t]]$ and $G(Y,t)$ in $\mathbb{C}_{\text{mul}}(Y)[[t]]$ such that $H(X,Y,t) \equiv F(X,t) + G(Y,t)$.

Example 8. Consider the model defined by $S(X,Y) = XY + \frac{1}{X} + \frac{1}{Y}$ (the same as Example 3). The fraction XY admits the obvious decoupling $XY \equiv \frac{1}{t} - \frac{1}{X} - \frac{1}{Y}$. Moreover, the following identity induces a pair of rational invariants: $X + \frac{1}{tX} - \frac{1}{X^2} \equiv Y + \frac{1}{tY} - \frac{1}{Y^2}$.

When all the components of a pair of t-invariants or a t-decoupling are rational fractions, we speak of *rational* t-invariants or t-decoupling. This notion of invariants intervenes in the following result, on which the strategy of [1, 6] relies crucially.

Lemma 9 (Lemma 2.6 in [6])**.** *Let $(F(X,t), G(Y,t))$ be a pair of t-invariants. If the coefficients of the power series $\frac{F(X,t)-G(Y,t)}{\widetilde{K}(X,Y,t)} \in \mathbb{C}_{\text{mul}}(X,Y)[[t]]$ have no pole at $X=0$ nor $Y=0$, then there exists a series $A(t)$ in $\mathbb{C}[[t]]$ such that $F(X,t) = G(Y,t) = A(t)$.*

The strategy of [1, 6] applies verbatim to large steps models with small backward steps, and goes as follows. Using a *rational t-decoupling* of XY (or more generally

$X^{k+1}Y^{l+1}$ for other starting points) and the special shape of the equation (e.g. Equation (1.1)), we construct a first pair of t-invariants. Next, we combine it with a pair of *non-constant rational t-invariants* using ring operations, to eventually obtain a third pair of invariants that satisfy the conditions of Lemma 9. As this pair involves $Q(X,0,t)$ and $Q(0,Y,t)$, the lemma gives two equations with one catalytic variable on these series. By a result of Bousquet-Mélou and Jehanne in [4], they must be algebraic, so is $Q(X,Y,t)$.

The existence of non-constant rational t-invariants and decoupling is crucial to conduct this strategy. In sections 3.2 and 3.3, we give a Galois approach to these two notions, which exploits the notion of the group of the walk introduced in Section 2, providing a *systematic* construction of these objects up to their existence and the finiteness of the orbit. This is an alternative approach to the one developed in [8] where the authors search for a polynomial decoupling (see the discussion of Example 3.19 in [2]).

Using our systematic approach, we were able to conduct the strategy on the model $\mathcal{G}_\lambda$, proving a conjecture of Bostan, Bousquet-Mélou and Melczer in [3]. We detail the proof in Section 4 as an illustration of the strategy. Moreover, for a family of models with large steps $(\mathcal{H}_n)_n$ whose orbits are conjectured to be finite, we were able to conjecture that $X^{i+1}Y^{j+1}$ (which appears in place of XY in equations of the form (1.1) when considering a starting point (i,j) other than $(0,0)$) admits a decoupling for several (i,j) (namely, $(n-1,0)$ and $((n+1)k-1,k-1)$ for every k). We successfully proved the algebraicity for several of these starting points for $n \leq 4$, hinting a possibly infinite family of algebraic models with arbitrarily large steps (see Appendix E of [2]).

2 A Galois structure on the orbit

The proofs and constructions in Sections 2, 3 and 4 are detailed in our upcoming paper [2]. We consider a weighted model $\mathcal{W}$ with a non-univariate step polynomial. We denote by k the field $\mathbb{C}(S(x,y))$. Recall also that $\mathbb{K} = \overline{\mathbb{C}(x,y)}$, and that if $M|L$ is a subextension of $\mathbb{K}|L$, a L-algebra automorphism $\sigma\colon M \to M$ is a ring homomorphism such that $\sigma_{|L} = \mathrm{id}_L$. We denote by $\mathrm{Aut}(M|L)$ the group of L-algebra automorphisms of M.

We first endow the orbit with a group action as follows. If $\sigma : \mathbb{K} \to \mathbb{K}$ is a $\mathbb{C}$-algebra automorphism, define its action on a pair $(u,v) \in \mathbb{K} \times \mathbb{K}$ by $\sigma \cdot (u,v) = (\sigma \cdot u, \sigma \cdot v)$.

Lemma 10 (Lemmas 3.7 and 3.8 in [2])**.** *The orbit is stable under the action of $k(x)$ and $k(y)$-algebra automorphisms of $\mathbb{K}$, which all preserve the relations $\sim^x$ and $\sim^y$.*

This lemma has a field theoretic counterpart: define $k(\mathcal{O})$ to be the subextension of $\mathbb{K}|k$ generated by the coordinates of the pairs of $\mathcal{O}$.

Theorem 11 (Theorem 3.9 in [2])**.** *The field extensions $k(\mathcal{O})|k(x)$ and $k(\mathcal{O})|k(y)$ are Galois.*

We denote by $G_x = \mathrm{Aut}(k(\mathcal{O})|k(x))$ and $G_y = \mathrm{Aut}(k(\mathcal{O})|k(y))$ their respective Galois groups, and by G_{xy} their intersection $G_x \cap G_y$ (which is the Galois group of the extension

$k(\mathcal{O})|k(x,y))$. We recall that the algebraic extension $k(\mathcal{O})|k(x)$ is *Galois* if $k(x)$ coincides with the subfield of $k(\mathcal{O})$ formed by the elements fixed by every automorphism in G_x.

Definition 12 (Group of the walk)**.** We define the *group of the walk* $G = \langle G_x, G_y \rangle$ to be the subgroup of k-algebra automorphisms of $k(\mathcal{O})$ generated by G_x and G_y.

It is easy to see from its definition that G acts by graph automorphisms on the graph of $\mathcal{O}$, and that its action is faithful. Moreover, while the group G is *a priori* not finitely generated, the left cosets $G_x/G_{x,y}$ and $G_y/G_{x,y}$ are of finite cardinal, respectively $d_x = \deg_X \widetilde{K}$ and $d_y = \deg_Y \widetilde{K}$ (Lemma 3.14 in [2]). We then fix $I_x = \{\mathrm{id}, \iota_1^x, \ldots, \iota_{d_x-1}^x\}$ and $I_y = \{\mathrm{id}, \iota_1^y, \ldots, \iota_{d_y-1}^y\}$ two respective sets of representatives for these two cosets.

Theorem 13 (Theorem 3.16 in [2])**.** *The subgroup $\langle I_x, I_y \rangle$ of G acts transitively on $\mathcal{O}$.*

Thus, the orbit $\mathcal{O}$ is realized as the action of a finite set of automorphisms on the pair (x,y), completing the analogy with the small steps setting.

Example 14 (Examples 3.10 and 3.18 in [2])**.** For $\mathcal{W}$ a model with small steps that has both positive and negative steps in each direction, $k(\mathcal{O}) = \mathbb{C}(x,y)$. Therefore, $G_{xy} = 1$, of index two in G_x and G_y. Hence, $G_x = \langle \psi \rangle$ and $G_y = \langle \phi \rangle$ with $\psi^2 = \phi^2 = 1$, and we find $G = \langle \phi, \psi \rangle$. The identities $\psi(h(x,y)) = h(\Psi(x,y))$ and $\phi(h(x,y)) = h(\Phi(x,y))$ yield an isomorphism between G and the group of small steps (§ 1.3).

Example 15 (continuing Example 4)**.** For $\mathcal{G}_\lambda$, we saw that $k(\mathcal{O}) = \mathbb{C}(x,y,x_1)$, with x_1 algebraic of degree 2 over $\mathbb{C}(x,y)$. Hence, $G_{xy} = \langle \tau \rangle$ with $\tau^2 = 1$, and after some computation we find $G_x = \langle \tau, \tau' \rangle \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $G_y = \langle \tau, \sigma \rangle \simeq S_3$ for $\tau'^2 = 1$ and $\sigma^3 = 1$. In the end, $G = \langle \tau, \tau', \sigma \rangle \simeq S_4$, which in this particular case coincides with the full group of graph automorphisms of $\mathcal{O}$.

3 Construction of invariants and decoupling

3.1 Fractions as elements of $k(\mathcal{O})$

To apply the Galois framework of Section 2 to the construction of rational invariants and decoupling, we define an evaluation of some fractions of $\mathbb{C}(X,Y,t)$ into $k(\mathcal{O})$. Its definition relies crucially on the fact that the kernel polynomial $\widetilde{K}(X,Y.t)$ is irreducible in $\mathbb{C}[X,Y,t]$ (Lemma 3.10 in [2]).

Definition 16. We call a fraction $H(X,Y,t)$ of $\mathbb{C}(X,Y,t)$ *regular* if the denominator of H is not divisible by $\widetilde{K}(X,Y,t)$.

Note that fractions of $\mathbb{C}(X,Y)$, $\mathbb{C}(X,t)$ and $\mathbb{C}(Y,t)$ are automatically regular because $\widetilde{K}(X,Y,t)$ is irreducible and trivariate by assumption on $\mathcal{W}$.

Definition 17. If (u,v) is a pair of the orbit and $H(X,Y,t)$ is a regular fraction of $\mathbb{C}(X,Y,t)$, define its evaluation on (u,v) to be $H_{(u,v)} = H(u,v,1/S(x,y)) \in k(\mathcal{O})$.

The evaluation on a pair of the orbit naturally extends to $\mathbb{C}$-linear combinations of pairs of the orbit (called 0-*chains*):

$$\text{for } c = \sum_{(u,v)\in\mathcal{O}} c_{u,v}(u,v), \quad \text{define} \quad H_c = \sum_{(u,v)\in\mathcal{O}} c_{u,v} H_{(u,v)}$$

Proposition 18 (Proposition 3.23 in [2])**.** *The evaluation homomorphism sending a regular fraction H to its evaluation $H_{(x,y)}$ maps bijectively $\mathbb{C}(X,t)$ to $k(x)$, $\mathbb{C}(Y,t)$ to $k(y)$ and $\mathbb{C}(X,Y)$ to $k(x,y)$. A regular fraction evaluates to 0 if and only if its numerator is divisible by $\widetilde{K}(X,Y,t)$.*

Thus, we can consider regular fractions as elements of the field $k(\mathcal{O})$, so as to benefit from our Galois-theoretic formalism. The homomorphism induces a relation on regular fractions: two regular fractions of $\mathbb{C}(X,Y,t)$ are called *Galois-equivalent* if their evaluations induce the same element in $k(\mathcal{O})$. Like the t-equivalence (Definition 5), this equivalence relation induces notions of invariants and decoupling:

- A pair of regular fractions $(I(X,t),J(Y,t))$ that are Galois equivalent is called a pair of *Galois invariants*. By Proposition 18, the evaluation homomorphism gives a correspondence between pairs of Galois invariants and elements of the subfield $k(x)\cap k(y)$ of $k(\mathcal{O})$, which we denote by k_{inv}, whose elements are fixed by G.

- Likewise, we say that a regular fraction $H(X,Y,t)$ admits a *Galois decoupling* pair $(F(X.t),G(Y,t))$ if H is Galois-equivalent to $F+G$. As above, a fraction H admitting a Galois decoupling corresponds through the evaluation homomorphism to a fraction h in $k(x,y)$ that writes as $h = f + g$ for some $f \in k(x)$ (fixed by G_x) and $g \in k(y)$ (fixed by G_y).

Proposition 3.23 in [2] implies that t-equivalent regular fractions are Galois equivalent. Therefore, the existence of rational t-invariants or t-decoupling of a fraction H is conditioned to the existence of their Galois counterparts (of which we give a complete treatment in the next two subsections). Once we have obtained non-constant Galois invariants or decoupling, we simply check if the Galois-equivalences involved are also t-equivalences, so that we obtain non-constant t-invariants and t-decoupling. If one of these two steps fails, then we know that non-constant rational t-invariants or t-decoupling of H do not exist.

3.2 Galois invariants

A pair of constant invariants $(F(t),F(t))$ is mapped by the evaluation homomorphism at (x,y) to an element of k. Therefore, the existence of non-constant Galois invariants is reduced to the field-theoretic question of whether the inclusion $k \subset k_{\text{inv}}$ is proper or not. This question is answered through the following result, which is a special instance of a theorem proved in the more general context of finite algebraic correspondences by Fried in [10], which we translate in the context of walks. This extends Theorem 4.6 in [1] and Corollary 4.6.11 in [9] to the large steps case.

Theorem 19 (Theorem 4.3 in [2])**.** *The following statements are equivalent:*

1. *the orbit $\mathcal{O}$ is finite,*
2. *G is a finite group,*
3. *there exists a pair of non-constant Galois invariants.*

When finite, the orbit is described by two explicit polynomials that cancel the left and right coordinates. In this case, the extension $k_{\text{inv}}|k$ is purely transcendental of transcendence degree 1, and any nonconstant coefficient of these polynomials is a generator, making the construction of rational invariants systematic (e.g. Equation (4.3) for $\mathcal{G}_\lambda$). Finally, as $k(\mathcal{O})^G = k_{\text{inv}}$, then $k(\mathcal{O})|k_{\text{inv}}$ is a finite Galois extension with Galois group G.

3.3 Galois decoupling

We now assume that the orbit is finite. Given a regular fraction $H(X, Y, t)$, we want to find a criterion for whether it admits a Galois decoupling, and if it does, to compute it. To this end, we define a notion of decoupling in the orbit.

Definition 20 (Definition 5.7 in [2])**.** Let $(\gamma_x, \gamma_y, \alpha)$ be a tuple of 0-chains such that $(x, y) = \gamma_x + \gamma_y + \alpha$ (with (x, y) being considered as a vertex in the orbit). This is called a *decoupling of (x, y) in the orbit* if for every regular $H(X, Y, t)$ the following conditions hold: (1) $H_{\gamma_x} \in k(x)$, $H_{\gamma_y} \in k(y)$ and (2) $H_\alpha = 0$ when H admits a Galois decoupling.

In the proof of Theorem 4.11 in [1], the authors construct an explicit decoupling of (x, y) for the cyclic orbits of small steps models. We extend their result to an arbitrary finite orbit using our Galois-theoretic framework and graph homology.

Theorem 21 (Theorem 5.10 in [2])**.** *If the orbit is finite, the pair (x, y) always admits a decoupling $(\gamma_x, \gamma_y, \alpha)$ in the orbit (in the sense of Definition 20).*

Thanks to this result, the question of the existence of the Galois decoupling of a regular fraction may be decided through an evaluation:

Proposition 22. *If $(\gamma_x, \gamma_y, \alpha)$ is a such a tuple, then a regular fraction $H(X, Y, t)$ admits a Galois decoupling if and only if $H_\alpha = 0$, and the decoupling is given by $H_{(x,y)} = H_{\gamma_x} + H_{\gamma_y}$.*

Note however that for an arbitrary 0-chain c, it is not always convenient to compute the evaluation H_c, because the coordinates of elements of the orbit are random algebraic elements. A friendlier family for computer algebra is composed of 0-chains of the form

$$c = \sum_{(u,v)\in\mathcal{O}, P(u)=0} (u,v) \quad \text{or} \quad c = \sum_{(u,v)\in\mathcal{O}, P(v)=0} (u,v)$$

with P a polynomial over $\mathbb{C}(x,y)$, which we call *symmetric chains*. It is easy to compute the evaluation of regular fractions over symmetric chains via *Newton's identities*. Some symmetric chains are presented as *level lines* for a well chosen distance in the graph of the orbit. They are denoted by X_i and Y_i in Figure 3.

Assuming a *distance transitivity property* on the graph of the orbit, we refine Theorem 21 by showing that the pair (x,y) admits a decoupling $(\gamma_x, \gamma_y, \alpha)$ with γ_x and γ_y composed of level lines, whose expression is explicit (Theorem 5.34 in [2]). This assumption was verified for all finite orbits arising from weighted models with steps in $\{-1,0,1,2\}$ (Figure 10 in [3]) which includes the one of Figure 3. Other families of orbits have been checked such as the ones arising from *Hadamard* (Section 11 of [3]) and *Tandem* models (Section 3.2 of [7]).

Example 23. For the orbits of the same type as in Figure 3, the pair (x,y) admits a decoupling in terms of the symmetric chains X_i and Y_i. It reads

$$(x,y) = \left(\frac{X_0}{2} - \frac{X_1}{8} + \frac{X_2}{8}\right) + \left(\frac{Y_0}{4} - \frac{Y_1}{4}\right) + \alpha.$$

We evaluated the fraction XY on it to obtain its Galois decoupling (4.2).

4 An example: the model $\mathcal{G}_\lambda$ is algebraic

We illustrate here with the model $\mathcal{G}_\lambda$ the generic strategy for proving algebraicity of large steps models with small backward steps described in Section 1.4. First, recall the equation found for the generating function of walks on $\mathcal{G}_\lambda$ in Example 1:

$$\widetilde{K}(X,Y,t)Q(X,Y,t) = XY - t(X^2+1)Q(X,0,t) - tQ(0,Y,t) + tQ(0,0,t). \tag{1.1}$$

Note that the generating function $Q(X,Y,t)$ has polynomial coefficients, hence the left hand-side of the equation is t-equivalent to 0, so the functional equation translates into

$$XY \equiv \Big(t(X^2+1)Q(X,0,t) - tQ(0,0)\Big) + tQ(0,Y,t). \tag{4.1}$$

Moreover, using the decoupling of (x,y) in the orbit of $\mathcal{G}_\lambda$ (Example 23), Proposition 22 gives a Galois decoupling of the fraction XY, which is checked to be the t-decoupling

$$XY \equiv -\frac{3\lambda X^2 t - \lambda t - 4X}{4t(X^2+1)} + \frac{-\lambda Y - 4}{4Y}. \tag{4.2}$$

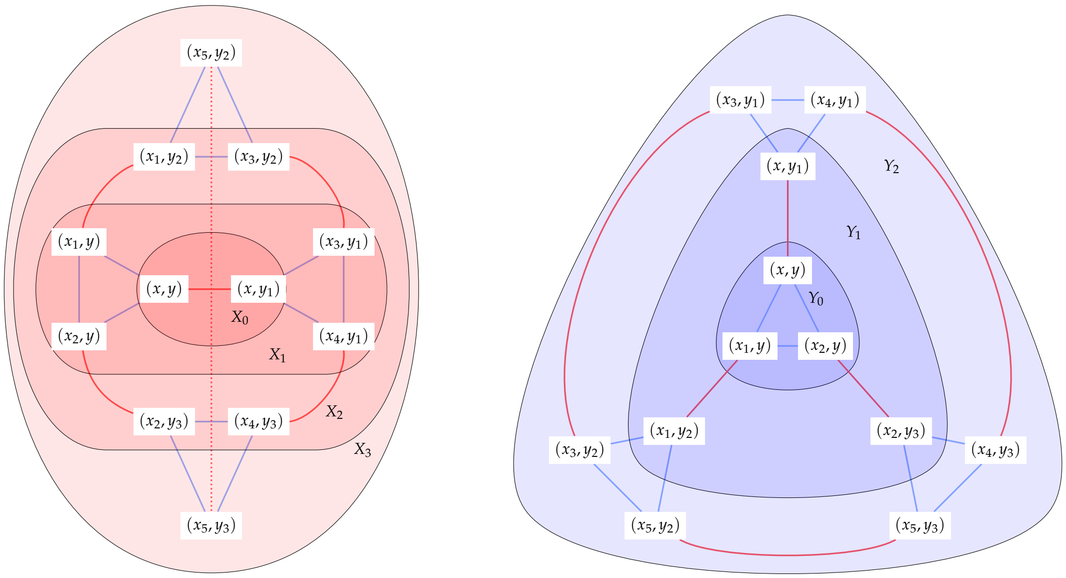


Figure 3: The orbit $\mathcal{O}_{12}$ of $\mathcal{G}_\lambda$ in two perspectives, illustrating a distance transitivity property (the 0-chains X_i and Y_i are the sums of vertices in their respective regions).

Combining Equations (4.1) and (4.2), we obtain the t-equivalence

$$\Big(t(X^2+1)Q(X,0,t) - tQ(0,0,t)\Big) + \frac{3\lambda X^2 t - \lambda t - 4X}{4t(X^2+1)} \equiv \frac{-\lambda Y - 4}{4Y} - tQ(0,Y,t),$$

which gives a first pair of invariants $P_1 = (I_1(X,t), J_1(Y,t))$. Note that $J_1(Y,t)$ has a pole at $Y = 0$, so this pair does not satisfy the conditions of Lemma 9.

The orbit of the model $\mathcal{G}_\lambda$ being finite, we also obtain automatically the following pair $P_2 = (I_2(X,t), J_2(Y,t))$ of Galois invariants, which we check to be t-invariants:

$$\left(\frac{\left(-\lambda^2 X^3 - 1\, X^4 - X^6 + X^2 + 1\right) t^2 - X^2 \lambda \left(X^2 - 1\right) t + X^3}{t^2 X \left(X^2+1\right)^2}, \frac{-t\, Y^4 + \lambda t Y + Y^3 + t}{Y^2 t} \right). \tag{4.3}$$

In order to find a pair of invariants satisfying the conditions of Lemma 9, the heuristic is to combine the pairs of invariants P_1 and P_2 using ring operations in order to remove their poles both in X and Y, by examining their Taylor expansions in their respective variables. Unlike the previous steps, the pole elimination is not systematic and requires a case by case treatment. This leads us to define $P_3 = (I_3(X,t), J_3(Y,t))$ to be

$$P_2\left(P_1 - \tfrac{\lambda}{4}\right) - P_1^3 + \left(2tQ(0,0) - \tfrac{\lambda}{4}\right)P_1^2 + \left(2t\tfrac{\partial Q}{\partial y}(0,0) - t^2 Q(0,0)^2 + \tfrac{5\lambda^2}{16}\right)P_1.$$

Using the functional equation, we are now able to check that this pair of t-invariants indeed satisfies the conditions of Lemma 9. Therefore, there exists a power series $A(t)$

in $\mathbb{C}[[t]]$ such that $I_3(X,t) = J_3(Y,t) = A(t)$. These are equations with one catalytic variable for $Q(X,0,t)$ and $Q(0,Y,t)$ that satisfy the assumptions of Theorem 3 in [4]. This allows us to conclude that these series are algebraic over $\mathbb{C}(X,Y,t)$, so that the same holds for the generating function $Q(X,Y,t)$ of the model $\mathcal{G}_\lambda$. Following the method of [4], we found an explicit minimal polynomial for the series $Q(0,0,t)$ of degree 32 with coefficients in $\mathbb{Q}(\lambda,t)$, proving in particular the algebraicity conjecture on the excursion series of the two models in [3] (lines 2 and 3 in Table 4, which are the reversed models of $\mathcal{G}_0$ and $\mathcal{G}_1$ but sharing the same excursion series).

Acknowledgements

We warmly thank Mireille Bousquet-Mélou for her invaluable advice and proofreading.

References

[1] O. Bernardi, M. Bousquet-Mélou, and K. Raschel. "Counting quadrant walks via Tutte's invariant method". *Comb. Theory* **1** (2021), Paper No. 3, 77. DOI.

[2] P. Bonnet and C. Hardouin. "A Galois structure on the orbit of large steps walks in the quadrant". Preprint. 2023.

[3] A. Bostan, M. Bousquet-Mélou, and S. Melczer. "Walks with large steps in an orthant". *J. Eur. Math. Soc. (JEMS)* **23**.7 (2021), pp. 2221–2297.

[4] M. Bousquet-Mélou and A. Jehanne. "Polynomial equations with one catalytic variable, algebraic series and map enumeration". *J. Combin. Theory Ser. B* **96** (2006), pp. 623–672.

[5] M. Bousquet-Mélou and M. Mishna. "Walks with small steps in the quarter plane". Contemp. Math. **520** (2010), pp. 1–39.

[6] M. Bousquet-Mélou. "Enumeration of three-quadrant walks via invariants: some diagonally symmetric models". *Canadian Journal of Mathematics* **75**.5 (2023), pp. 1566–1632. DOI.

[7] M. Bousquet-Mélou, E. Fusy, and K. Raschel. "Plane bipolar orientations and quadrant walks". *Sém. Lothar. Combin.* **81** (2020), Art. B81l, 64.

[8] M. Buchacher, M. Kauers, and G. Pogudin. "Separating variables in bivariate polynomial ideals". *ISSAC'20—Proceedings of the 45th International Symposium on Symbolic and Algebraic Computation*. ACM, New York, [2020] ©2020, pp. 54–61.

[9] G. Fayolle, R. Iasnogorodski, and V. Malyshev. *Random walks in the quarter-plane*. Vol. 40. Applications of Mathematics (New York). Algebraic methods, boundary value problems and applications. Springer-Verlag, Berlin, 1999, pp. xvi+156. DOI.

[10] M. D. Fried. "Poncelet correspondences: Finite correspondences; Ritt's theorem; and the Griffiths-Harris configuration for quadrics". *Journal of Algebra* **54**.2 (1978), pp. 467–493. DOI.