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Chain polynomials of generalized paving matroids

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Abstract. We prove that the chain polynomial of the lattice of flats of a paving matroid is real-rooted, and we define a class of matroids called *generalized paving matroids*. Generalized paving matroids associated to subspace lattices are shown to have real-rooted chain polynomials, by a study of a q-analog of the subdivision operator. We finish by studying single element extensions, and prove that the chain polynomials of the lattice of flats of single element extensions of U_n^n and U_n^{n-1} are real-rooted.

Keywords: matroid, chain polynomial, geometric lattice, real-rootedness

1 Introduction

The chain polynomial of a finite poset P is defined as

$$c_P(t) := \sum_{k \geq 0} c_k(P) t^k, \tag{1.1}$$

where $c_k(P)$ is the number of k-element chains in P. The chain polynomials of posets in several important classes have been proven to be real-rooted. For example face lattices of simplicial [8] and cubical polytopes [1], $(3+1)$-free posets [14, Corollary 2.9], and for some classes of distributive lattices [7, 18], but not all [16]. In [2] the authors asked for which posets the chain polynomial is real-rooted. In particular the following conjecture was formulated.

Conjecture 1.1. *[2, Conjecture 1.2] The chain polynomial $c_{\mathcal{L}}(t)$ is real-rooted for every geometric lattice $\mathcal{L}$.*

This conjecture can bee seen as a member of a family of recent conjectures about the real-rootedness of Kazhdan-Lusztig polynomials [9, 11] and Chow ring Poincaré polynomials [10, 17] associated matroids arising in the emerging Hodge theory of matroids [12].

In [2], Athanasiadis and Kalampogia-Evangelinou proved Conjecture 1.1 for the subspace lattices $\mathcal{L}_n(q)$, for the partition lattices Π_n and Π_n^B, and for the lattice of flats of near-pencils and uniform matroids.

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The purpose of this paper is to verify Conjecture 1.1 for further classes of geometric lattices. We prove Conjecture 1.1 for the lattice of flats of paving matroids, a class of matroids which is conjectured to correspond to almost all finite matroids. We define a new class of matroids, called *generalized paving matroids*, that includes the class of paving matroids and we prove Conjecture 1.1 for generalized paving matroids associated to subspace lattices. In the process we extend the well studied *subdivision operator* $\mathcal{E}$ (see [6, 8]) to subspace lattices, and prove several real-rootedness results concerning these generalized subdivision operators. We finish the paper by studying single-element extensions and proving that the chain polynomials of the lattice of flats of single-element extensions of some uniform matroids are real-rooted.

2 Interlacing polynomials

In this section we collect a few results and terminology that will be needed in subsequent sections, for proofs we refer to [6].

Let $f, g \in \mathbb{R}[t]$ be real-rooted polynomials with nonnegative coefficients and of degrees r and s, respectively. Let $x_r \leq \cdots \leq x_2 \leq x_1$ be the zeros of f, and let $y_s \leq \cdots \leq y_2 \leq y_1$ be the zeros of g. We say that g *interlaces* f (written $g \preceq f$) if either $r = s$ and

$$y_s \leq x_r \leq \cdots \leq y_2 \leq x_2 \leq y_1 \leq x_1$$

or $r = s + 1$ and

$$x_r \leq y_s \leq x_{r-1} \leq \cdots \leq y_2 \leq x_2 \leq y_1 \leq x_1.$$

We say that a sequence of polynomials $f_1, f_2, \dots, f_m \in \mathbb{R}[t]$ is *interlacing* if $f_i \preceq f_j$ whenever $i < j$.

Proposition 2.1. *Let $f, g, h \in \mathbb{R}[t]$.*

1. *If $f \preceq g$ and $f \preceq h$, then $f \preceq ag + bh$ for all $a, b \geq 0$.*
2. *If $g \preceq f$ and $h \preceq f$, then $ag + bh \preceq f$ for all $a, b \geq 0$.*

Proposition 2.2. *Let $f_0, f_1, \dots, f_m \in \mathbb{R}[t]$ be an interlacing sequence of real-rooted polynomials with positive leading coefficients.*

1. *Every nonnegative linear combination f of $f_0, f_1, \dots, f_m$ is real-rooted, and $f_0 \preceq f \preceq f_m$;*
2. *If we define*

$$g_k := t\sum_{i=0}^{k-1} f_i + \sum_{i=k}^{m} f_i,$$

for $k = 0, 1, \dots, m$, then $\{g_i\}_{i=0}^{m}$ is interlacing.

Proposition 2.3. *Let $f_1, \ldots, f_m \in \mathbb{R}[t]$. If $f_1 \preceq f_2 \preceq \cdots \preceq f_m$ and $f_1 \preceq f_m$, then $f_i \preceq f_j$ for all $i \leq j$.*

Lemma 2.4. *Suppose $f_0, f_1, \ldots, f_n$ is an interlacing sequence of polynomials of degree d, such that for each $0 \leq j \leq n$, the polynomial f_j has nonnegative leading coefficient and all zeros in the interval $[-1, 0]$. Then the sequence $g_0, g_1, \ldots, g_{n+1}$ defined by*

$$g_k = t \sum_{j=0}^{k-1} f_j + (1+t) \sum_{j=k}^{n} f_j,$$

is interlacing.

Proof. Let $h_j(t) = (1-t)^d f_j(t/(1-t))$ and $r_j(t) = (1-t)^{d+1} g_j(t/(1-t))$. Then $\{h_j\}_{j=0}^n$ is an interlacing sequence of polynomials with nonnegative coefficients. Moreover,

$$r_j = t \sum_{j=0}^{k-1} h_j + \sum_{j=k}^{n} h_j,$$

and hence $\{r_j\}_{j=0}^{n+1}$ is interlacing by Proposition 2.2. Since $g_j = (1+t)^{d+1} r_j(t/(1+t))$, the lemma follows. □

3 Generalized subdivision operators

Let P be a locally finite and graded poset with a least element $\hat{0}$, such that $[\hat{0}, x]$ is isomorphic to $[\hat{0}, y]$ whenever x and y have the same rank. Define a linear operator $\mathcal{E}_P : \mathbb{R}[t] \to \mathbb{R}[t]$ by $\mathcal{E}_P(1) = 1$, and

$$\mathcal{E}_P(t^n) = \sum_{j=1}^{n} |\{\hat{0} < x_1 < \cdots < x_j = x\}| \cdot t^j = \frac{t}{(1+t)^2} \cdot c_{[\hat{0},x]}(t),$$

where x is any element in P of rank n. Let further $\mathcal{R}_P : \mathbb{R}[t] \to \mathbb{R}[t]$ be the linear operator defined by

$$\mathcal{R}_P(t^n) = \sum_{k=0}^{n} r_{n,k} t^k,$$

where $r_{n,k}$ is the number of elements in $[\hat{0}, x]$ of rank k, where x is any element in P of rank n. Hence, for $n \geq 1$,

$$\mathcal{E}_P(t^n) = t\mathcal{E}_P\big(\mathcal{R}_P(t^n) - t^n\big) \quad \text{and} \quad (1+t)\mathcal{E}_P(t^n) = t\mathcal{E}_P(\mathcal{R}_P(t^n)). \tag{3.1}$$

If P is a Boolean lattice, then $\mathcal{E}_P$ is the *subdivision operator* $\mathcal{E}$, which has the property that

$$\mathcal{E}(f_\Delta(t)) = f_{\mathrm{sd}(\Delta)}(t),$$

for any simplicial complex Δ, where f_Δ is the f-polynomial of Δ and $\mathrm{sd}(\Delta)$ is the barycentric subdivision of Δ, see [6, 8]. The subdivision operator is important in proving real-rootedness for polynomials associated to simplicial complexes, posets or Ehrhart theory [6]. In this section we will generalize and refine the following result to subspace lattices $\mathcal{L}_n(q)$.

Proposition 3.1. *[5, Section 4] The sequence* $\{\mathcal{E}\left(t^i(t+1)^{d-i}\right)\}_{i=0}^d$ *is interlacing.*

Let $\mathcal{L}(q)$ be the inverse limit, as $n \to \infty$, of the subspace lattices $\mathcal{L}_n(q)$ of all subspaces of $\mathbb{F}_q^n$, where $\mathbb{F}_q$ is a finite field with q elements. Denote by $\mathcal{E}_q$, the linear operator $\mathcal{E}_{\mathcal{L}(q)}$. Hence

$$\mathcal{E}_q(t^n) = t\mathcal{E}_q\big(G_n(t) - t^n\big), \tag{3.2}$$

where $G_n(t) = \sum_{k=0}^n \binom{n}{k}_q t^k$, and $\binom{n}{k}_q$, $0 \le k \le n$, are the *Gaussian polynomials* which may be defined recursively by $\binom{n}{0}_q = 1$, and

$$\binom{n}{k}_q = q^k \binom{n-1}{k}_q + \binom{n-1}{k-1}_q, \tag{3.3}$$

see [15, Section 1.7]. Henceforth we let q be any real number greater or equal to 1.

Lemma 3.2. *Let n be a nonnegative integer. Then*

$$\mathcal{E}_q(t^k G_{n+1-k}(t)) = t\sum_{j=0}^{k-1} \mathcal{E}_q(t^j G_{n-j}(qt)) + (1+t)\sum_{j=k}^{n} \mathcal{E}_q(t^j G_{n-j}(qt)), \quad 0 \le k \le n+1, \tag{3.4}$$

and

$$\mathcal{E}_q(t^k G_{n-k}(qt)) = \mathcal{E}_q(t^k G_{n-k}(t)) + (q^{n-k}-1)\mathcal{E}_q(t^{k+1} G_{n-(k+1)}(t)), \quad 0 \le k \le n. \tag{3.5}$$

Proof. The identity (3.3) implies

$$t^k G_{n+1-k}(t) - t^{n+1} = \sum_{j=k}^{n} t^j G_{n-j}(qt). \tag{3.6}$$

By (3.2) and (3.6),

$$\mathcal{E}_q(t^{n+1}) = t\mathcal{E}_q(G_{n+1}(t) - t^{n+1}) = t\sum_{j=0}^{n} \mathcal{E}_q(t^j G_{n-j}(qt)),$$

which combined with (3.6) gives (3.4).
Similarly, the identity

$$q^k \binom{n}{k}_q = \binom{n}{k}_q + (q^n - 1)\binom{n-1}{k-1}_q \tag{3.7}$$

implies (3.5). □

The following theorem generalizes Theorem 3.1 to any $q \geq 1$.

Theorem 3.3. *Let n be a nonnegative integer. The sequence of polynomials $\{\mathcal{E}_q(t^k G_{n-k}(t))\}_{k=0}^n$ is interlacing. Moreover all zeros of $\mathcal{E}_q(t^k G_{n-k}(t))$ lie in the interval $[-1,0]$.*

Proof. The proof is by induction over n, the case $n = 0$ being trivial.

Suppose true for $n-1 \geq 0$. Since $\{\mathcal{E}_q(t^k G_{n-k}(t))\}_{k=0}^n$ is interlacing, we have by (3.5) and [6, Corollary 8.6] that $\{\mathcal{E}_q(t^k G_{n-k}(qt))\}_{k=0}^n$ is interlacing. Moreover (3.5) implies

$$\mathcal{E}_q(t^k G_{n-k}(t)) \prec \mathcal{E}_q(t^k G_{n-k}(qt)) \prec \mathcal{E}_q(t^{k+1} G_{n-k-1}(t)),$$

so that all zeros of $\mathcal{E}_q(t^k G_{n-k}(qt))$ are in the interval $[-1,0]$. The lemma now follows by induction from (3.4) and Lemma 2.4. □

Corollary 3.4. *Suppose $f = \sum_{k=0}^d h_k t^k G_{d-k}$, where $h_k \geq 0$ for all $0 \leq k \leq d$. Then $\mathcal{E}_q(f)$ is real-rooted and $\mathcal{E}_q(G_d) \prec \mathcal{E}_q(f) \prec \mathcal{E}_q(t^d)$.*

Proof. Follows immediately from Proposition 2.2 and Theorem 3.3. □

For $d \leq n$, let $G_{n,k}^d$ be the polynomial obtained from $t^k G_{n-k}(t)$ by removing all terms t^j, where $j > d$.

Lemma 3.5. *Let d be a nonnegative integer.*

(a) If $n \geq d$, then $\{\mathcal{E}_q(G_{n,k}^d)\}_{k=0}^d$ is interlacing.

(b) If $0 \leq k \leq d$, then $\{\mathcal{E}_q(G_{n,k}^d)\}_{n=d}^\infty$ is interlacing.

Proof. We first prove (a) by induction over $n \geq d$, the case $n = d$ being Theorem 3.3. Assume true for n. Equations (3.6) and (3.7) imply

$$t^k G_{n+1-k} - t^{n+1} = t^k G_{n-k} + \sum_{j=k+1}^{n} q^{n+1-j} t^j G_{n-j},$$

and thus

$$\mathcal{E}_q(G_{n+1,k}^d) = \mathcal{E}_q(G_{n,k}^d) + \sum_{j=k+1}^{d} q^{n+1-j} \mathcal{E}_q(G_{n,j}^d),$$

which by [6, Corollary 8.6] proves that $\{\mathcal{E}_q(G_{n+1,k}^d)\}_{k=0}^d$ is interlacing, and that $\mathcal{E}_q(G_{n,k}^d) \prec \mathcal{E}_q(G_{n+1,k}^d)$. This proves (a) by induction.
Notice that

$$\lim_{n\to\infty} \frac{\mathcal{E}_q(G_{n,k}^d)}{\binom{n-k}{d}_q} = \mathcal{E}_q(t^d),$$

and that $\mathcal{E}_q(G_{d,k}^d) \prec \mathcal{E}_q(t^d)$ by Theorem 3.3. Hence (b) now follows from Proposition 2.3. □

The next theorem generalizes a recent result [3] of Athanasiadis and Kalampogia-Evangelinou from Boolean lattices to subspace lattices. Suppose P is a graded poset, and $S = \{0 = s_0 < s_1 < s_2 < \cdots\} \subseteq \mathbb{N}$. Consider the rank selected poset $P_S := \{x \in P : \rho(x) \in S\}$. Define a linear operator $T_S : \mathbb{R}[t] \to \mathbb{R}[t]$ by

$$T_S(t^k) = \begin{cases} 0 & \text{if } k \notin S, \\ t^i & \text{if } k = s_i. \end{cases}$$

Theorem 3.6. *Let n be a positive integer, and let S be a subset of $\mathbb{N}$ containing 0. The sequence $\{\mathcal{E}_P(T_S(t^k G_{n-k}))\}_{k=0}^n$ is interlacing, where $P = \mathcal{L}(q)_S$. In particular the chain polynomial of $\mathcal{L}_n(q)_S$ is real-rooted.*

Proof. The proof is omitted in this extended abstract. □

4 Generalized paving matroids

Recall that a geometric lattice $\mathcal{L}$ of rank $d+1$ is the lattice of flats of a paving matroid on E if and only if

- the set $\mathcal{H}$ of hyperplanes of $\mathcal{L}$ form a *d-partition*, i.e., $|H| \geq d$ for each $H \in \mathcal{H}$, and for each set S of size d there exists a unique $H \in \mathcal{H}$ such that $S \subseteq H$;
- the flats of rank $k \leq d-1$ are the sets of size k of the Boolean lattice on E.

For example, if $\mathcal{P} = \binom{[n]}{d}$, then $\mathcal{P}$ is a d-partition of $[n]$ and, hence, there is a paving matroid whose set of hyperplanes is $\mathcal{H}_1 = \binom{[n]}{d}$. Another example, a 2-partition of [7], is $\mathcal{H}_2 = \{\{1,2,4\}, \{1,3,7\}, \{1,5,6\}, \{3,4,6\}, \{2,6,7\}, \{4,5,7\}, \{2,3\}, \{2,5\}, \{2,3\}, \{3,5\}\}$.

We will now generalize this construction to any geometric lattice, and prove that the chain polynomials of the lattice of flats of generalized paving matroids associated to subspace lattices and Boolean lattices are real-rooted.

Let $\mathcal{L}$ be a geometric lattice with rank function ρ on a ground set E. Suppose $d \geq 1$, and suppose $\mathcal{H} \subset \mathcal{L}$ satisfies

(a) $\rho(H) \geq d$ for each $H \in \mathcal{H}$,

(b) for each $F \in \mathcal{L}$ with $\rho(F) = d$, there exists a unique $H \in \mathcal{H}$ such that $F \leq H$.

Let $\mathcal{L}(\mathcal{H})$ be the graded meet semi-lattice of rank $d+1$,

$$\mathcal{L}(\mathcal{H}) = \{F \in \mathcal{L}(\mathcal{H}) : \rho(F) \leq d-1\} \cup \mathcal{H} \cup \{E\}.$$

Lemma 4.1. *$\mathcal{L}(\mathcal{H})$ is a geometric lattice with set of hyperplanes $\mathcal{H}$.*

Proof. Let ρ' be the rank function of $\mathcal{L}(\mathcal{H})$, and let $\vee$ and $\vee'$ (respectively, $\wedge$ and $\wedge'$) be the joins (respectively, the meets) in $\mathcal{L}$ and $\mathcal{L}(\mathcal{H})$, respectively. We will prove that $\mathcal{L}(\mathcal{H})$ is (1) a lattice, (2) atomistic and (3) semimodular:

1. $\mathcal{L}(\mathcal{H})$ **is a lattice -** Since $\mathcal{L}(\mathcal{H})$ is a finite meet semi-lattice with a largest element, then, by [15, Proposition 3.3.1], it is a lattice;

2. $\mathcal{L}(\mathcal{H})$ **is atomistic -** Let $F \in \mathcal{L}(\mathcal{H})$. If $\rho'(F) \leq d-1$, then, by definition of $\mathcal{L}(\mathcal{H})$, if $F = \bigvee_i F_i$, then $F = \bigvee'_i F_i$. If $\rho'(F) = d$, then there exist a unique $G \in \mathcal{L}$ such that $\rho(G) = d$ and $G \leq F$. Since $G = \bigvee_i G_i$ for atoms $G_i \in \mathcal{L}$, then $F = \bigvee'_i G_i$. Finally, if $F = E$, then F is the join of two elements in $\mathcal{H}$ and hence a join of atoms, by the above;

3. $\mathcal{L}(\mathcal{H})$ **is semimodular -** Let $F, G \in \mathcal{L}(\mathcal{H})$. We want to prove that
$$\rho'(F) + \rho'(G) \geq \rho'(F \vee' G) + \rho'(F \wedge' G). \tag{4.1}$$
There are three different cases to deal with:
   - F **and** G **are in** $\mathcal{H}$**.** Then $\rho'(F) = \rho'(G) = d$, $\rho'(F \vee' G) = d+1$ and $\rho'(F \wedge' G) \leq d-1$, which implies (4.1);
   - F **is not in** $\mathcal{H}$ **and** G **is in** $\mathcal{H}$**.** If $F \leq' G$, then there is nothing to prove. Otherwise, $\rho'(F \wedge' G) \leq \rho'(F) - 1$ and $F \vee' G = E$, and (4.1) holds;
   - F **and** G **are not in** $\mathcal{H}$**.** We may assume F and G are smaller than E. Then
$$\begin{aligned}\rho'(F) + \rho'(G) &= \rho(F) + \rho(G) \geq \rho(F \vee G) + \rho(F \wedge G)\\ &= \rho(F \vee G) + \rho'(F \wedge' G).\end{aligned}$$
Hence it remains to prove
$$\rho'(F \vee' G) \leq \rho(F \vee G). \tag{4.2}$$
If $\rho(F \vee G) \leq d-1$, then $F \vee G = F \vee' G$ and so (4.2) holds. If $\rho(F \vee G) = d$, then there exists $H \in \mathcal{H}$ such that $F \vee G \leq H$, and hence $\rho'(F \vee' G) \leq d$. If $\rho(F \vee G) \geq d+1$, then (4.2) holds since $\rho'(E) = d+1$.

□

If $\mathcal{L}$ is the Boolean algebra, then the lattices $\mathcal{L}(\mathcal{H})$ are precisely the lattices of flats of paving matroids.

Notice that
$$c_{\mathcal{L}(\mathcal{H})}(t) = c_{\mathcal{L}^d}(t) + t \sum_{H \in \mathcal{H}} c_{[\hat{0},H]^d}(t), \tag{4.3}$$
where $\mathcal{L}^d = \{F \in \mathcal{L} : \rho(F) \leq d-1\} \cup \{E\}$ is the truncation of $\mathcal{L}$ to rank d.

The next theorem verifies Conjecture 1.1 for the lattice of flats of paving matroids and generalized paving matroids associated to subspace lattices.

Theorem 4.2. *Suppose $\mathcal{L}$ is a subspace lattice $\mathcal{L}_n(q)$ or a Boolean lattice, and that $\mathcal{H}$ satisfies (a) and (b). Then $c_{\mathcal{L}(\mathcal{H})}(t)$ is real-rooted and $c_{\mathcal{L}^d}(t) \prec c_{\mathcal{L}(\mathcal{H})}(t)$.*

Proof. Clearly,

$$c_{\mathcal{L}^d}(t) = (1+t)\cdot \mathcal{E}_q(G^{d-1}_{n,d-1}(t)) \quad \text{and} \quad c_{[\hat{0},H]^d}(t) = (1+t)\cdot \mathcal{E}_q(G^{d-1}_{m,d-1}(t)),$$

where $m \le n$, and $q = 1$ for the Boolean case. The theorem now follows from Proposition 2.1, Lemma 3.5 and (4.3).

□

5 Single-element extensions

Let N be a matroid with ground set E. Recall that, if $T \subset E$, then the *deletion of T in N* is $N\backslash T$, the matroid on $E\backslash T$ with independent sets given by

$$\{I : I \text{ is independent in } N \text{ and } I \cap T = \varnothing\}.$$

In this case, N is called an *extension* of $N\backslash T$. For the particular case when $|T| = 1$, we call N a *single-element extension* of $N\backslash T$.

A *modular cut* $\mathcal{M}$ of a matroid M is a set of flats of M that satisfies the following:

(i) if $F \in \mathcal{M}$ and F' is a flat of M containing F, then $F' \in \mathcal{M}$;

(ii) if $F_1, F_2 \in \mathcal{M}$ and $r(F_1) + r(F_2) = r(F_1 \cup F_2) + r(F_1 \cap F_2)$, then $F_1 \cap F_2 \in \mathcal{M}$.

There is a one-to-one correspondence between the modular cuts of a matroid M and single-element extensions of M, see [13, Chapter 7.2]. Hence, for each modular cut $\mathcal{M}$ of M we can associate a single-element extension $M +_{\mathcal{M}} e$ of M, where e is an element not in the ground set of M, whose lattice of flats fall into the three following disjoint classes (see [13, Corollary 7.2.]):

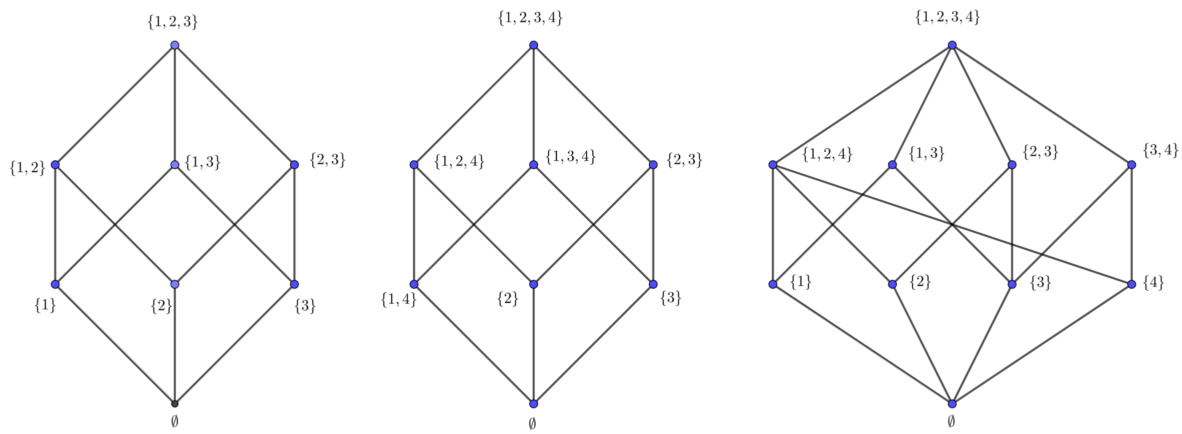
(i) flats F of M that are not in $\mathcal{M}$;

(ii) sets $F \cup e$, where F is a flat of M that is in $\mathcal{M}$;

(iii) sets $F \cup e$, where F is a flat of M that is not in $\mathcal{M}$ and F is not contained in a member F' of $\mathcal{M}$ of rank $r(F) + 1$.

Moreover, we can use this construction to determine all matroids: any matroid M is obtained from a uniform matroid U^n_n by a sequence of single-element extensions.

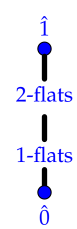
For example, consider the uniform matroid U^3_3. Then, $\mathcal{M}_1 = [[1],[3]]$ and $\mathcal{M}_2 = [[2],[3]]$ are modular cuts. The lattices of flats of U^3_3, $U^3_3 +_{\mathcal{M}_1} 4$ and $U^3_3 +_{\mathcal{M}_2} 4$ are given as follows:



U_3^3 $\qquad$ $U_3^3 +_{\mathcal{M}_1} 4$ $\qquad$ $U_3^3 +_{\mathcal{M}_2} 4$

Lemma 5.1. *Conjecture 1.1 is true for all matroids of ranks 1, 2 or 3.*

Proof. The result is trivial for matroids of ranks 1 and 2. For matroids of rank 3, its lattice of flats is given by



and, hence, its chain polynomial is $c_{\mathcal{L}}(t) = [1 + (m_1 + m_2)t + et^2](1 + t)^2$, where m_i is the number of i-flats and e is the number of edges between 1-flats and 2-flats. Since $e \leq m_1 m_2$, then $c_{\mathcal{L}}(t)$ is real-rooted. $\square$

Now, consider uniform matroids U_n^n, $n \geq 4$. The lattice of flats of $\mathcal{M}$ is B_n. So, if $\mathcal{M}$ is a modular cut of U_n^n (and, in general, a modular cut of U_n^r), then $\mathcal{M} = \emptyset$ or $\mathcal{M} = [X, [n]]$, where $X \subseteq [n]$. Hence, every flat of the lattice of flats of $U_n^n +_{\mathcal{M}} \{n + 1\}$ fall into one of the following disjoint classes:

(i) $F \subset \{1, \ldots, n, n + 1\}$ such that $\{1, \ldots, m, n + 1\}$ is not a subset of F. In this case, the rank of F is $|F|$;

(ii) $F \subset \{1, \ldots, n, n + 1\}$ such that $\{1, \ldots, m, n + 1\} \subseteq F$. In this case, the rank of F is $|F| - 1$.

It follows that $U_n^n +_{\mathcal{M}} \{n+1\}$ is isomorphic to the direct product $U_{m+1}^m \times U_{n-m}^{n-m}$.

Lemma 5.2. *[6, Theorem 7.6] If all zeros of $\mathcal{E}(f)$ and $\mathcal{E}(g)$ lie in the interval $[-1,0]$, then so does $\mathcal{E}(fg)$.*

Lemma 5.3. *Let P and Q be two posets with a least and a greatest element such that $|P|, |Q| \geq 2$, and define*

$$\hat{c}_P(t) = \frac{t}{(1+t)^2} \cdot c_P(t).$$

Then

$$\hat{c}_{P\times Q} = \hat{c}_P \diamond \hat{c}_Q := \mathcal{E}(\mathcal{E}^{-1}(\hat{c}_P)\mathcal{E}^{-1}(\hat{c}_Q)).$$

Proof. Omitted in the long abstract. □

Lemma 5.4 ([5]). *If*

$$f(x) = \sum_{k=0}^{d} h_k x_k (1+x)^{d-k}$$

has $h_k \geq 0$ for all $0 \leq k \leq d$, then all zeros of $\mathcal{E}(f)$ are real, simple and located in $[-1,0]$. In particular, the h-polynomial of a Cohen-Macaulay poset is real-rooted.

Now, we can prove the following:

Theorem 5.5. *If the h-polynomials of the order complexes of the posets P and Q have nonnegative coefficients, then the chain polynomial of $P \times Q$ is real-rooted.*

Proof. It follows directly from Lemmas 5.4, 5.2 and 5.3. □

Corollary 5.6. *The chain polynomial of $U_n^n +_{\mathcal{M}} \{n+1\}$ is real-rooted for any modular cut of U_n^n.*

Proof. As mentioned before, $U_n^n +_{\mathcal{M}} \{n+1\}$ is isomorphic to $U_{m+1}^m \times U_{n-m}^{n-m}$. So, by Lemma 5.3,

$$\hat{c}_{\mathcal{L}(U_n^n +_{\mathcal{M}} \{n+1\})} = \hat{c}_{\mathcal{L}(U_{m+1}^m)} \diamond \hat{c}_{\mathcal{L}(U_{n-m}^{n-m})}.$$

By Theorem 3.3, $c_{\mathcal{L}(U_{m+1}^m)}$ and $c_{\mathcal{L}(U_{n-m}^{n-m})}$ are real-rooted. So, by Lemma 5.2, $c_{\mathcal{L}(U_n^n +_{\mathcal{M}} \{n+1\})}$ is real-rooted. □

Corollary 5.7. *The chain polynomial of $U_n^{n-1} +_{\mathcal{M}} \{n+1\}$ is real-rooted for any modular cut of U_n^{n-1}.*

Proof. First, observe that U_n^{n-1} is a truncation of U_n^n and that modular cuts of U_n^{n-1} are also intervals. Let $\hat{\mathcal{M}} = [[m],[n]]$ be a modular cut of U_n^{n-1} and $\mathcal{M} = [[m],[n]]$ be a modular cut of U_n^n. So $U_n^{n-1} +_{\hat{\mathcal{M}}} \{n+1\}$ is a truncation of $U_n^n +_{\mathcal{M}} \{n+1\}$. Hence, if $\mathcal{H}$ is the set of hyperplanes of $U_n^n +_{\mathcal{M}} \{n+1\} = U_{m+1}^m \times U_{n-m}^{n-m}$, then $H \in \mathcal{H}$ if and only if

$H = A \times [n-m]$ or $H = [m+1] \times B$, where $A \in \mathcal{A}$ is a hyperplane of U^m_{m+1} and $B \in \mathcal{B}$ is a hyperplane of U^{n-m}_{n-m}. So,

$$\begin{aligned}\hat{c}_{\mathcal{L}(U_n^{n-1}+_{\hat{\mathcal{M}}}\{n+1\})}(t) &= \hat{c}_{\mathcal{L}(U_n^n+_{\mathcal{M}}\{n+1\})}(t) - t\sum_{H_i\in\mathcal{H}} \hat{c}_{[\emptyset,H_i]}(t)\\ &= \hat{c}_{\mathcal{L}(U^m_{m+1})} \diamond \hat{c}_{\mathcal{L}(U^{n-m}_{n-m})}(t) - t\sum_{A_i\in\mathcal{A}} \hat{c}_{[\emptyset,A_i\times[n-m]]}(t)\\ &\quad - t\sum_{B_j\in\mathcal{B}} \hat{c}_{[\emptyset,[m+1]\times B_j]}(t)\\ &= \hat{c}_{\mathcal{L}(U^m_{m+1})} \diamond \hat{c}_{\mathcal{L}(U^{n-m}_{n-m})}(t) - t\sum_{A_i\in\mathcal{A}} \hat{c}_{[\emptyset,A_i]}(t) \diamond \hat{c}_{\mathcal{L}(U^{n-m}_{n-m})}(t)\\ &\quad - t\sum_{B_j\in\mathcal{B}} \hat{c}_{\mathcal{L}(U^m_{m+1})}(t) \diamond \hat{c}_{[\emptyset,B_j]}(t).\end{aligned}$$

Since $\hat{c}_{[\emptyset,A_i]}(t) \preceq \hat{c}_{\mathcal{L}(U^m_{m+1})}(t)$ for all $A_i \in \mathcal{A}$ and $\hat{c}_{[\emptyset,B_j]}(t) \preceq \hat{c}_{\mathcal{L}(U^{n-m}_{n-m})}(t)$ for all $B_j \in \mathcal{B}$,

$$\hat{c}_{[\emptyset,A_i\times[n-m]]}(t) \preceq \hat{c}_{\mathcal{L}(U^m_{m+1})} \diamond \hat{c}_{\mathcal{L}(U^{n-m}_{n-m})}(t), \text{ for all } A_i \in \mathcal{A}$$

and

$$\hat{c}_{[\emptyset,[m+1]\times B_j]}(t) \preceq \hat{c}_{\mathcal{L}(U^m_{m+1})} \diamond \hat{c}_{\mathcal{L}(U^{n-m}_{n-m})}(t), \text{ for all } B_j \in \mathcal{B}.$$

by [4, Theorem 3]. Hence $\hat{c}_{\mathcal{L}(U_n^{n-1}+_{\hat{\mathcal{M}}}\{n+1\})}(t)$ is real-rooted. □

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