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Extremal weight crystals over affine Lie algebras of infinite rank

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Abstract. We explain extremal weight crystals over affine Lie algebras of infinite rank using combinatorial models: a spinor model due to Kwon, and an infinite rank analogue of Kashiwara–Nakashima tableaux due to Lecouvey. In particular, we show that the Lecouvey's tableau model combinatorially explains an extremal weight crystal structure of level zero. Using these combinatorial models, we explain an algebra structure of the Grothendieck ring for a category consisting of some extremal weight crystals.

Keywords: extremal weight crystals, affine Lie algebras of infinite rank, Jacobi–Trudi formula, Grothendieck ring

1 Introduction

Let $U_q(\mathfrak{g})$ be a quantum group associated with a Kac–Moody algebra $\mathfrak{g}$. For an integral weight λ, let $V(\lambda)$ be an extremal weight $U_q(\mathfrak{g})$-module of weight λ and $B(\lambda)$ be its associated crystal base (cf. [6]). It is significant to study extremal weight crystals because it is closely related to level-zero representations of quantum affine Lie algebras (of finite rank). For details, see [1, 2, 8, 17] and references therein. However, properties of extremal weight crystals over affine Lie algebras of infinite rank differ considerably from those over affine Lie algebras of finite rank. In this extended abstract, we discuss several properties of extremal weight crystals over affine Lie algebras of infinite rank.

An important observation by Naito and Sagaki [16] (see Proposition 3.2 also) is that for an integral weight λ of a nonnegative level, there exist $\lambda^0 \in E$ and $\lambda^+ \in P^+$ (and unique in some sense) such that

$$B(\lambda) \cong B(\lambda^0) \otimes B(\lambda^+). \tag{1.1}$$

This isomorphism suggests that a combinatorial model of $B(\lambda)$ ($\lambda \in P$) by combining that for $B(\lambda^0)$ ($\lambda^0 \in E$) and that for $B(\lambda^+)$ ($\lambda^+ \in P^+$).

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We associate $B(\lambda)$ ($\lambda \in E$) to a set $\mathbf{KN}^{\mathfrak{g}}(\lambda^{\dagger})$ ($\lambda^{\dagger} \in \mathcal{P}$) of $\mathfrak{g}_{\infty}$-type Kashiwara–Nakashima (simply KN) tableaux introduced by Lecouvey [15], which are an infinite rank analogue of KN tableaux. We define a $\mathfrak{g}_{\infty}$-crystal structure on $\mathbf{KN}^{\mathfrak{g}}(\lambda)$, and we construct an isomorphism between $\mathbf{KN}^{\mathfrak{g}}(\lambda)$ and $B(\varpi_{\lambda})$ (Theorem 3.10). On the other hand, we associate $B(\lambda)$ for $\lambda \in P^{+}$ to a spinor model introduced by Kwon [12, 13]. Indeed, the crystal structure of a spinor model is already known and a spinor model is isomorphic to extremal weight crystals of dominant weights (see Theorem 3.6).

As an application, we characterize the Grothendieck ring $\mathcal{K}$ for a category $\mathcal{C}$ consisting of some extremal weight crystals. In particular, as similarly as (1.1), the set $\mathcal{K}$ has the tensor decomposition

$$\mathcal{K} = \mathcal{K}^{0} \otimes \mathcal{K}^{+}$$

where $\mathcal{K}^{0}$ and $\mathcal{K}^{+}$ are the subalgebra of $\mathcal{K}$ generated by $[B(\lambda)]$ for $\lambda \in E$ and $\lambda \in P^{+}_{\mathrm{int}}$, respectively. It is known that $\mathcal{K}^{0}$ is isomorphic to the ring of symmetric functions (Proposition 5.2) and $\mathcal{K}^{+}$ is isomorphic to the ring of formal power series (Theorem 5.4). One can find a full version of this extended abstract including proofs and details in [3].

2 Preliminaries

2.1 Notations

Let $\mathbb{Z}_{+}$ be the set of nonnegative integers. Let $\mathcal{P}$ be the set of partitions and, for $n \in \mathbb{Z}_{+}$, $\mathcal{P}_n = \{\, \lambda \in \mathcal{P} \mid \ell(\lambda) \leq n \}$, where $\ell(\lambda)$ is the length of λ. Denote by $\lambda' = (\lambda'_1, \lambda'_2, \dots)$ the conjugate of λ.

For even $\ell \geq 2$, let G_{ℓ} be one of the algebraic groups: $\mathrm{Sp}_{\ell}, \mathrm{Pin}_{\ell}$, and O_{ℓ}. Let

$$\begin{gathered}\mathscr{P}(\mathrm{Sp}_{\ell}) = \mathcal{P}_{\frac{\ell}{2}}, \qquad \mathscr{P}(\mathrm{Pin}_{\ell}) = \mathcal{P}_{\frac{\ell}{2}}, \\ \mathscr{P}(\mathrm{O}_{\ell}) = \{\, \lambda \in \mathcal{P}_{\ell} \mid \lambda'_1 + \lambda'_2 \leq \ell \,\},\end{gathered}$$

and

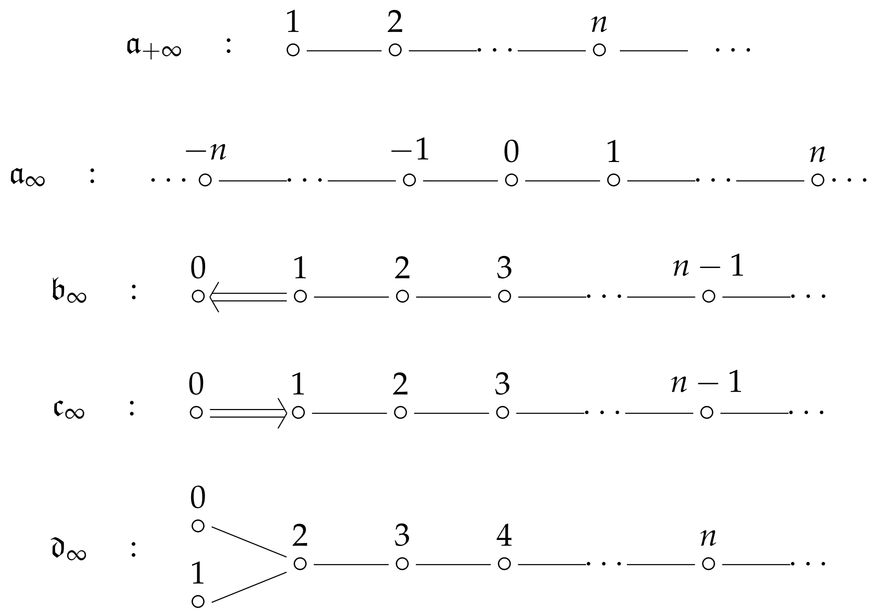
$$\mathscr{P}(G) = \{\, (\lambda, \ell) \mid \ell \in \mathbb{N}, \lambda \in \mathscr{P}(\mathrm{G}_{2\ell}) \,\}$$

for $G = \mathrm{Sp}, \mathrm{Pin}$, or O.

For an ordered set $\mathcal{A}$ and a skew shape λ/μ, denote by $SST_{\mathcal{A}}(\lambda/\mu)$ the set of semistandard (or $\mathcal{A}$-semistandard) tableaux of shape λ/μ, that is, tableaux with letters in $\mathcal{A}$ such that entries in each row (resp. column) are weakly (resp. strictly) increasing. We omit a subscript $\mathcal{A}$ from $SST_{\mathcal{A}}(\lambda/\mu)$ if there is no confusion or it does not depend on the choice of $\mathcal{A}$.

2.2 Affine Lie algebras of infinite rank

A Lie algebra $\mathfrak{g}$ is of infinite rank if it is the Kac–Moody algebra associated with a generalized Cartan matrix of infinite rank. A Lie algebra of infinite rank is of affine type if every principal minor (of finite rank) of associated generalized Cartan matrix is positive. There are five (non-isomorphic) affine Lie algebras of infinite rank whose Dynkin diagram is connected and these are referred to Lie algebras $\mathfrak{a}_{+\infty}, \mathfrak{a}_\infty, \mathfrak{b}_\infty, \mathfrak{c}_\infty$, and $\mathfrak{d}_\infty$ (cf. [4]). The followings are Dynkin diagrams corresponding to stated affine Lie algebras of infinite rank.



In this extended abstract, we focus on providing results for $\mathfrak{g} = \mathfrak{b}_\infty, \mathfrak{c}_\infty$, or $\mathfrak{d}_\infty$. The corresponding results to ours can be found in [10] when $\mathfrak{g} = \mathfrak{a}_{+\infty}$ and in [11] when $\mathfrak{g} = \mathfrak{a}_\infty$. We use the following notations for affine Lie algebras $\mathfrak{g}_\infty$ of infinite rank.

- $I = \mathbb{Z}_+$: the index set
- $\{\alpha_i \,|\, i \in I\}$: the set of simple roots
- $\{\Lambda_i^{\mathfrak{g}} \,|\, i \in I\}$: the set of fundamental weights
- $P = \mathbb{Z}\Lambda_0^{\mathfrak{g}} \oplus \bigoplus_{i=1}^{\infty} \mathbb{Z}\epsilon_i$: the weight lattice
- P^+ : the set of dominant weights, $\qquad E = \bigoplus_{i=1}^{\infty} \mathbb{Z}\epsilon_i \subseteq P$
- W : the Weyl group

In this paper, we take the simple roots α_i as below. Then we can derive the following equations on $\Lambda_i^{\mathfrak{g}}$.

$$\begin{array}{ll}
\mathfrak{c}_\infty & \alpha_0 = -2\epsilon_1, \quad \alpha_i = \epsilon_i - \epsilon_{i+1} \quad (i \geq 1) \\
& \Lambda_i^{\mathfrak{c}} = \Lambda_0^{\mathfrak{c}} + (\epsilon_1 + \cdots + \epsilon_i) \quad (i \geq 1) \\
\mathfrak{b}_\infty & \alpha_0 = -\epsilon_1, \quad \alpha_i = \epsilon_i - \epsilon_{i+1} \quad (i \geq 1) \\
& \Lambda_i^{\mathfrak{b}} = 2\Lambda_0^{\mathfrak{b}} + (\epsilon_1 + \cdots + \epsilon_i) \quad (i \geq 1) \\
\mathfrak{d}_\infty & \alpha_0 = -\epsilon_1 - \epsilon_2, \quad \alpha_i = \epsilon_i - \epsilon_{i+1} \quad (i \geq 1) \\
& \Lambda_1^{\mathfrak{d}} = \Lambda_0^{\mathfrak{d}} + \epsilon_1, \quad \Lambda_i^{\mathfrak{d}} = 2\Lambda_0^{\mathfrak{d}} + (\epsilon_1 + \cdots + \epsilon_i) \quad (i \geq 2)
\end{array}$$

For an integer $n \geq 2$, let $\mathfrak{g}_n$ be the Lie subalgebra of $\mathfrak{g}_\infty$ generated by e_i, f_i ($i = 0, 1, \dots, n-1$). We write the expression $\mathfrak{g} = \mathfrak{b}, \mathfrak{c}, \mathfrak{d}$ when we don't have to specify its rank.

For $\lambda = (\lambda_1, \lambda_2, \dots) \in \mathcal{P}$, define

$$\varpi_\lambda = \sum_{i \geq 1} \lambda_i \epsilon_i \in E.$$

For simplicity, we write $\varpi_i = \varpi_{(1^i)}$ for $i \geq 1$. On the other hand, we suppose that a Lie algebra $\mathfrak{g}_\infty$ corresponds to an algebraic group G (and vice versa) as follows:

$$(\mathfrak{g}, G) \quad : \quad (\mathfrak{b}_\infty, \mathrm{Pin}),\ (\mathfrak{c}_\infty, \mathrm{Sp}),\ (\mathfrak{d}_\infty, \mathrm{O}) \tag{2.1}$$

Put

- $\Pi_i^{\mathfrak{c}} = \Lambda_i^{\mathfrak{c}} \quad (i \geq 0)$
- $\Pi_0^{\mathfrak{b}} = 2\Lambda_0^{\mathfrak{b}},\ \Pi_i^{\mathfrak{b}} = \Lambda_i^{\mathfrak{b}} \quad (i \geq 1)$
- $\Pi_0^{\mathfrak{d}} = 2\Lambda_0^{\mathfrak{d}},\ \overline{\Pi}_0^{\mathfrak{d}} = 2\Lambda_1^{\mathfrak{d}},\ \Pi_1^{\mathfrak{d}} = \Lambda_0^{\mathfrak{d}} + \Lambda_1^{\mathfrak{d}}$, and $\Pi_i^{\mathfrak{d}} = \Lambda_i^{\mathfrak{d}} \quad (i \geq 2)$

and let

$$\Pi^{\mathfrak{g}}(\lambda, \ell) = \ell\, \Pi_0^{\mathfrak{g}} + \varpi_{\lambda'} \in P^+$$

for $(\lambda, \ell) \in \mathscr{P}(G)$. For $(\lambda, \ell) \in \mathscr{P}(G)$ with $\ell(\lambda) = t$, we have

$$\Pi^{\mathfrak{g}}(\lambda, \ell) = \begin{cases} \Pi_{\lambda_1}^{\mathfrak{g}} + \cdots + \Pi_{\lambda_\ell}^{\mathfrak{g}} & \text{if } t \leq \ell, \\ \Pi_{\lambda_1}^{\mathfrak{d}} + \cdots + \Pi_{\lambda_{2\ell - t}}^{\mathfrak{d}} + (t - \ell)\overline{\Pi}_0^{\mathfrak{d}} & \text{if } t > \ell. \end{cases}$$

The condition $(\lambda, \ell) \in \mathscr{P}(G)$ with $\ell(\lambda) > \ell$ holds only when $(\mathfrak{g}, G) = (\mathfrak{d}_\infty, \mathrm{O})$.

Let

$$P_{\mathrm{int}}^+ = \{ \Pi^{\mathfrak{g}}(\lambda, \ell) \,|\, (\lambda, \ell) \in \mathscr{P}(G) \} \ \subseteq\ P^+.$$

Note that when $\mathfrak{g} = \mathfrak{c}_\infty$, we have $P_{\mathrm{int}}^+ = P^+$, and when $\mathfrak{g} = \mathfrak{b}_\infty$ or $\mathfrak{d}_\infty$, we have $P_{\mathrm{int}}^+ \subsetneq P^+$ and P_{int}^+ is the set of dominant weights with a positive even level. Recall that the level of $\lambda \in P$ is the value $\langle \lambda, K \rangle$, where K is the canonical central element of $\mathfrak{g}_\infty$ (cf. [4, Section 7.12]).

Remark 2.1. *The correspondence* (2.1) *between Lie algebras and algebraic groups originates from dual pairs due to Howe. For details, see [3, Remark 2.1].*

3 A combinatorial realization of extremal weight crystals

3.1 Extremal weight crystals

We recall the notion of an extremal weight crystal, which is introduced by Kashiwara (cf. [6, 8]). For $\lambda \in P$, let $V(\lambda)$ be an extremal weight module generated by an extremal weight vector. In particular, $V(\lambda)$ is an irreducible highest weight module when $\lambda \in P^+$ (cf. [5]). It is proved in [6] that $V(\lambda)$ has a crystal base $(L(\lambda), B(\lambda))$, which provides a tool to interpret a given module in a combinatorial way. We simply say that $B(\lambda)$ is an extremal weight crystal.

When $\mathfrak{g}$ is a general Kac–Moody algebra, for $\lambda \in P$ and $w \in W$, there exists an isomorphism $B(\lambda) \cong B(w\lambda)$ of $\mathfrak{g}$-crystals [6]. Moreover, the converse of the above statement holds when $\mathfrak{g}$ is an affine Lie algebra of infinite rank.

Proposition 3.1 ([16, Proposition 3.9])**.** *When $\mathfrak{g}$ is an affine Lie algebra of infinite rank and $\lambda, \mu \in P$, we have $B(\lambda) \cong B(\mu)$ if and only if $\lambda \in W\mu$.*

From now on, we assume that all Lie algebras in this article are affine Lie algebras of infinite rank without otherwise stated. In particular, we use the notation $\mathfrak{g}_\infty$ to emphasize the infinite rank.

The key observation of this paper is that an extremal weight crystal $B(\lambda)$ ($\lambda \in P$) is decomposed into the tensor product of two extremal weight crystals.

Proposition 3.2 ([16, Section 4.2])**.** *For a nonnegative level $\lambda \in P$, there exist $\lambda^0 \in E$ and $\lambda^+ \in P^+$ such that*

$$B(\lambda) \;\cong\; B(\lambda^0) \otimes B(\lambda^+). \tag{3.1}$$

Moreover, for $\lambda^0, \mu^0 \in E$ and $\lambda^+, \mu^+ \in P^+$, we have

$$B(\lambda^0) \otimes B(\lambda^+) \cong B(\mu^0) \otimes B(\mu^+) \iff \lambda^+ = \mu^+,\ \lambda^0 \in W\mu^0.$$

By Proposition 3.2, we shift our focus to understand extremal weight crystals $B(\lambda)$ for $\lambda \in E$ or $\lambda \in P^+$. In particular, for given $\lambda \in E$, there exists unique $\mu \in W\lambda$ such that μ is of the form ϖ_α for some $\alpha \in \mathcal{P}$, and we denote by $\lambda^\dagger \in \mathcal{P}$ such a (unique) partition α. Since $B(\lambda) \cong B(\varpi_{\lambda^\dagger})$ by Proposition 3.1, we may assume that $\lambda \in E$ is of the form ϖ_α for some $\alpha \in \mathcal{P}$, indeed $\alpha = \lambda^\dagger$.

Remark 3.3. *For a nonpositive level $\lambda \in P$, we have an isomorphism*

$$B(\lambda) \;\cong\; B(\lambda^-) \otimes B(\lambda^0)$$

for some $\lambda^0 \in E$ and $\lambda^- \in -P^+$, which is obtained from (3.1) *by applying dual crystals (cf. [7, Section 7.4]).*

Example 3.4 ([3, Example 3.11]). *When $\mathfrak{g}_\infty = \mathfrak{c}_\infty$, consider $\lambda = 4\Pi^{\mathfrak{c}}_0 + 2\epsilon_1 + 5\epsilon_3 - 3\epsilon_4 - \epsilon_5 + 4\epsilon_6 \in P$. Then we have $\nu = 4\Pi^{\mathfrak{c}}_0 + 4\epsilon_1 + 3\epsilon_2 + 2\epsilon_3 - \epsilon_4 - 3\epsilon_5 \in W\lambda$ with*

$$\begin{aligned} \nu^+ &= 4\Pi^{\mathfrak{c}}_0 + 4\epsilon_1 + 3\epsilon_2 + 2\epsilon_3 = \Pi^{\mathfrak{c}}((3,3,2,1),4), \\ \nu^0 &= -\epsilon_4 - 3\epsilon_5. \end{aligned}$$

In this case, $(\nu^0)^\dagger (= (\lambda^0)^\dagger) = (3,1)$. Thus, we have $\lambda^0 = \varpi_{(3,1)}$ and $\lambda^+ = \Pi^{\mathfrak{c}}((3,3,2,1),4)$.

3.2 Spinor model

For $a,b,c \in \mathbb{Z}_+$, let $\lambda(a,b,c) = (2^{b+c}, 1^a)/(1^b)$ be a skew shape with two columns. Suppose that $T \in SST(\lambda(a,b,c))$ for some $a,b,c \in \mathbb{Z}_+$ and T' is the tableau obtained from T by sliding the right column of T by k positions down for $0 \le k \le \min\{a,b\}$. Set $\mathfrak{r}_T$ to be the maximal integer $k \ge 0$ such that $T' \in SST(\lambda(a-k, b-k, c+k))$.

For $a \in \mathbb{Z}_+$, let

$$\mathbf{T}^{\mathfrak{g}}(a) \;=\; \{\, T \in SST_{\mathbb{N}}(\lambda(a,b,c)) \,|\, (b,c) \in \mathcal{H}^{\mathfrak{g}},\ \mathfrak{r}_T \le r^{\mathfrak{g}} \,\},$$

where

$$\mathcal{H}^{\mathfrak{g}} = \begin{cases} \{0\} \times \mathbb{Z}_+ & \text{if } \mathfrak{g} = \mathfrak{c} \\ \mathbb{Z}_+ \times \mathbb{Z}_+ & \text{if } \mathfrak{g} = \mathfrak{b} \\ 2\mathbb{Z}_+ \times 2\mathbb{Z}_+ & \text{if } \mathfrak{g} = \mathfrak{d} \end{cases}, \qquad r^{\mathfrak{g}} = \begin{cases} 0 & \text{if } \mathfrak{g} = \mathfrak{b}, \mathfrak{c} \\ 1 & \text{if } \mathfrak{g} = \mathfrak{d} \end{cases},$$

and

$$\overline{\mathbf{T}}^{\mathfrak{d}}(0) = \bigsqcup_{(b,c) \in \mathcal{H}^{\mathfrak{d}}} SST_{\mathbb{N}}(\lambda(0,b,c+1)).$$

For $(\lambda, \ell) \in \mathscr{P}(G)$, put $t = \ell(\lambda)$ and

$$\widehat{\mathbf{T}}^{\mathfrak{g}}(\lambda,\ell) = \begin{cases} \mathbf{T}^{\mathfrak{g}}(\lambda_\ell) \times \cdots \times \mathbf{T}^{\mathfrak{g}}(\lambda_1) & \text{if } t \le \ell, \\ \overline{\mathbf{T}}^{\mathfrak{d}}(0)^{t-\ell} \times \mathbf{T}^{\mathfrak{d}}(\lambda_{2\ell-t}) \times \cdots \times \mathbf{T}^{\mathfrak{d}}(\lambda_1) & \text{if } t > \ell. \end{cases}$$

Definition 3.5 ([12, 13]). *A spinor model $\mathbf{T}^{\mathfrak{g}}(\lambda,\ell)$ of shape $(\lambda,\ell) \in \mathscr{P}(G)$ is the set of $(T_\ell, \dots, T_1) \in \widehat{\mathbf{T}}^{\mathfrak{g}}(\lambda,\ell)$ such that each pair (T_{i+1}, T_i) satisfies the admissibility condition (cf. [12, Definition 6.7], [13, Definition 3.4]) for $1 \le i \le \ell - 1$.*

Theorem 3.6 ([12, Theorem 7.4], [13, Theorem 4.4]). *For $(\lambda,\ell) \in \mathscr{P}(G)$, the set $\mathbf{T}^{\mathfrak{g}}(\lambda,\ell)$ is a $\mathfrak{g}_\infty$-crystal and is isomorphic to $B(\Pi^{\mathfrak{g}}(\lambda,\ell))$ as $\mathfrak{g}_\infty$-crystals.*

Remark 3.7. *In this extended abstract, it is sufficient to describe $B(\lambda)$ for $\lambda \in P^+_{\mathrm{int}}$ (not P^+), and we intentionally omit some related notions; $\mathbf{T}^{\mathrm{sp}}$ in particular. The skipped ones can be found in [12, 13], which cover whole extremal (highest) weight crystals $B(\lambda)$ for $\lambda \in P^+$.*

The character of $\mathbf{T}^{\mathfrak{g}}(\lambda,\ell)$ is defined to be

$$\operatorname{ch}\mathbf{T}^{\mathfrak{g}}(\lambda,\ell) \;=\; t^{\ell}\sum_{(T_\ell,\ldots,T_1)\in\mathbf{T}^{\mathfrak{g}}(\lambda,\ell)}\prod_{i=1}^{\ell}\mathbf{x}^{T_i}$$

where t is a formal symbol and $\mathbf{x}^T=\prod_{i=1}^{\infty}x_i^{m_i}$, with m_i being the number of appearances of $i\geq 1$ in a semistandard tableau T. Indeed, we understand $x_i=e^{\epsilon_i}$ and $t=e^{\Pi_0^{\mathfrak{g}}}$ when we consider them as elements in the group algebra $\mathbb{Z}[P]$. An explicit formula of the character of a spinor model will be explained in Section 4.

3.3 Kashiwara–Nakashima tableaux

For $n\in\mathbb{Z}_+$, let $\mathcal{I}_n^{\mathfrak{g}}$ be the following ordered sets.

$$\begin{aligned}
\mathcal{I}_n^{\mathfrak{b}} &= \{\,\overline{n}<\cdots<\overline{1}<0<1<\cdots<n\,\}\\
\mathcal{I}_n^{\mathfrak{c}} &= \{\,\overline{n}<\cdots<\overline{1}<1<\cdots<n\,\}\\
\mathcal{I}_n^{\mathfrak{d}} &= \left\{\,\overline{n}<\cdots<\overline{2}<\begin{matrix}\overline{1}\\ 1\end{matrix}<2<\cdots<n\,\right\}
\end{aligned}$$

Here, $\mathcal{I}_n^{\mathfrak{d}}$ is a partially ordered set, and $(1,\overline{1})$ is the unique non-comparable pair in $\mathcal{I}_n^{\mathfrak{d}}$.

Definition 3.8 ([9]). *The ($\mathfrak{g}_n$-type) KN tableau of shape $\lambda\in\mathcal{P}$ is an $\mathcal{I}_n^{\mathfrak{g}}$-semistandard tableau T of shape λ such that each column of T is admissible and adjacent columns of T do not have certain (a,b)-configurations (cf. [9]). We denote by $\mathbf{KN}_n^{\mathfrak{g}}(\lambda)$ the set of KN tableaux of shape λ. Note that the condition for a tableau to be $\mathcal{I}_n^{\mathfrak{g}}$-semistandard is similar as the usual one with some exceptions (cf. [3, 9, 15]).*

For $\lambda\in\mathcal{P}$, we easily check that $\mathbf{KN}_n^{\mathfrak{g}}(\lambda)\subseteq\mathbf{KN}_{n+1}^{\mathfrak{g}}(\lambda)$ for $n\geq 1$. As a role of KN tableaux corresponding to the infinite rank, Lecouvey [15] introduces a tableau model, which we call a $\mathfrak{g}_\infty$-type KN tableau.

Definition 3.9 ([15]). *For $\lambda\in\mathcal{P}$, define*

$$\mathbf{KN}^{\mathfrak{g}}(\lambda) \;=\; \bigcup_{n\geq\ell(\lambda)}\mathbf{KN}_n^{\mathfrak{g}}(\lambda)$$

where the union is over $n>\ell(\lambda)$ when $\mathfrak{g}=\mathfrak{d}$. It is the set of $\mathcal{I}^{\mathfrak{g}}$-semistandard tableaux of shape λ satisfying the same (a,b)-configuration conditions as those in Definition 3.8, where $\mathcal{I}^{\mathfrak{g}}$ is the following ordered set.

$$\begin{aligned}
\mathcal{I}^{\mathfrak{b}} &= \{\cdots<\overline{n}<\cdots<\overline{1}<0<1<\cdots<n<\ldots\}\\
\mathcal{I}^{\mathfrak{c}} &= \{\cdots<\overline{n}<\cdots<\overline{1}<1<\cdots<n<\ldots\}\\
\mathcal{I}^{\mathfrak{d}} &= \left\{\cdots<\overline{n}<\cdots<\overline{2}<\begin{matrix}\overline{1}\\ 1\end{matrix}<2<\cdots<n<\ldots\right\}
\end{aligned}$$

Here, a pair $(1,\overline{1})$ in $\mathcal{I}^{\mathfrak{d}}$ is the unique non-comparable pair in $\mathcal{I}^{\mathfrak{d}}$.

It is known that $\mathbf{KN}^{\mathfrak{g}}_n(\lambda)$ is a $\mathfrak{g}_n$-crystal and is isomorphic to $B(\varpi_\lambda)$ as $\mathfrak{g}_n$-crystals [9]. Then we can extend this $\mathfrak{g}_n$-crystal structure to $\mathbf{KN}^{\mathfrak{g}}(\lambda)$. Moreover, we show that these extended $\mathfrak{g}_n$-crystal structures on $\mathbf{KN}^{\mathfrak{g}}(\lambda)$ ranging over $n \geq \ell(\lambda)$ are compatible. From this observation, we induce a $\mathfrak{g}_\infty$-crystal structure on $\mathbf{KN}^{\mathfrak{g}}(\lambda)$. One of the main results is that this $\mathfrak{g}_\infty$-crystal $\mathbf{KN}^{\mathfrak{g}}(\lambda)$ is isomorphic to the extremal weight crystal $B(\varpi_\lambda)$.

Theorem 3.10 ([3, Theorem 4.11])**.** *For $\lambda \in \mathcal{P}$, there exists an isomorphism of $\mathfrak{g}_\infty$-crystals.*

$$\mathbf{KN}^{\mathfrak{g}}(\lambda) \cong B(\varpi_\lambda)$$

4 Jacobi–Trudi type character formulas

For $r \in \mathbb{N}$, let $e_r(\mathbf{x})$ be the r-th elementary symmetric function in $\mathbf{x} = \{x_1, x_2, \ldots, \}$, and set $e_0(\mathbf{x}) = 1$ and $e_r(\mathbf{x}) = 0$ for $r < 0$. For $r \in \mathbb{Z}$, define

$$\begin{aligned} E_r(\mathbf{x}) &= \sum_{i=0}^{\infty} e_i(\mathbf{x}) e_{r+i}(\mathbf{x}), \\ E'_r(\mathbf{x}) &= E_r(\mathbf{x}) - E_{r+2}(\mathbf{x}), \qquad E''_r(\mathbf{x}) = E_r(\mathbf{x}) + E_{r+1}(\mathbf{x}). \end{aligned}$$

We can easily check that $E_r(\mathbf{x}) = E_{-r}(\mathbf{x})$ for $r \in \mathbb{Z}$. In addition, we easily derive the following identities using $E^{\diamondsuit}_r(\mathbf{x})$ ($\diamondsuit \in \{\,\cdot\,, \prime, \prime\prime\}$).

Proposition 4.1 ([3, Proposition 5.2])**.** *For $a \in \mathbb{Z}_+$ ($a \in \mathbb{N}$ when $\mathfrak{g} = \mathfrak{d}$), the following equalities hold.*

$$\begin{aligned} &\mathrm{ch}\,\mathbf{T}^{\mathfrak{c}}(a) \;=\; tE'_a(\mathbf{x}), \qquad \mathrm{ch}\,\mathbf{T}^{\mathfrak{b}}(a) \;=\; tE''_a(\mathbf{x}), \qquad \mathrm{ch}\,\mathbf{T}^{\mathfrak{d}}(a) \;=\; tE_a(\mathbf{x}), \\ &\mathrm{ch}\,\mathbf{T}^{\mathfrak{d}}(0) + \mathrm{ch}\,\overline{\mathbf{T}}^{\mathfrak{d}}(0) \;=\; tE_0(\mathbf{x}), \quad \mathrm{ch}\,\mathbf{T}^{\mathfrak{d}}(0) - \mathrm{ch}\,\overline{\mathbf{T}}^{\mathfrak{d}}(0) \;=\; t\left(\sum_{i=0}^{\infty} e_i(\mathbf{x})\right)\left(\sum_{i=0}^{\infty}(-1)^i e_i(\mathbf{x})\right) \end{aligned}$$

In general, we explicitly write the (Jacobi–Trudi type) character formula of a spinor model in terms of $E^{\diamondsuit}_r(\mathbf{x})$ ($\diamondsuit \in \{\,\cdot\,, \prime, \prime\prime\}$).

Definition 4.2 ([14])**.** *For $\diamondsuit \in \{\,\cdot\,, \prime, \prime\prime\}$ and $(\lambda, \ell) \in \mathscr{P}(G)$, denote*

$$\begin{aligned} \Sigma^{\diamondsuit}_{(\lambda,\ell)}(\mathbf{x}) &= \det\left(E^{\diamondsuit}_{(\lambda_{\ell-i+1}+i-1)+(j-1)}(\mathbf{x}) + \delta(j \neq 1) E^{\diamondsuit}_{(\lambda_{\ell-i+1}+i-1)-(j-1)}(\mathbf{x})\right)_{i,j=1,\ldots,\ell} \\ &= \begin{vmatrix} E^{\diamondsuit}_{\lambda_\ell} & E^{\diamondsuit}_{\lambda_\ell+1} + E^{\diamondsuit}_{\lambda_\ell-1} & \cdots & E^{\diamondsuit}_{\lambda_\ell+(\ell-1)} + E^{\diamondsuit}_{\lambda_\ell-(\ell-1)} \\ E^{\diamondsuit}_{\lambda_{\ell-1}+1} & E^{\diamondsuit}_{(\lambda_{\ell-1}+1)+1} + E^{\diamondsuit}_{(\lambda_{\ell-1}+1)-1} & \cdots & E^{\diamondsuit}_{(\lambda_{\ell-1}+1)+(\ell-1)} + E^{\diamondsuit}_{(\lambda_{\ell-1}+1)-(\ell-1)} \\ \vdots & \vdots & \ddots & \vdots \\ E^{\diamondsuit}_{\lambda_1+\ell-1} & E^{\diamondsuit}_{(\lambda_1+\ell-1)+1} + E^{\diamondsuit}_{(\lambda_1+\ell-1)-1} & \cdots & E^{\diamondsuit}_{(\lambda_1+\ell-1)+(\ell-1)} + E^{\diamondsuit}_{(\lambda_1+\ell-1)-(\ell-1)} \end{vmatrix} \end{aligned}$$

where $\delta(P) = 0$ if a statement P is false and $\delta(P) = 1$ otherwise. Also, define $S^{\mathfrak{g}}_{(\lambda,\ell)}(\mathbf{x})$ by

$$\begin{aligned}
S^{\mathfrak{c}}_{(\lambda,\ell)}(\mathbf{x}) &= \Sigma'_{(\lambda,\ell)}(\mathbf{x}) \\
S^{\mathfrak{b}}_{(\lambda,\ell)}(\mathbf{x}) &= \Sigma''_{(\lambda,\ell)}(\mathbf{x}) \\
S^{\mathfrak{d}}_{(\lambda,\ell)}(\mathbf{x}) &= \begin{cases} \Sigma_{(\lambda,\ell)}(\mathbf{x}) & \text{if } t = \ell, \\ \frac{1}{2}\Sigma_{(\lambda,\ell)}(\mathbf{x}) + \frac{1}{2}\left(\sum_{i=0}^{\infty} e_i(\mathbf{x})\right)\left(\sum_{i=0}^{\infty}(-1)^i e_i(\mathbf{x})\right)\Sigma'_{(\lambda,\ell-1)}(\mathbf{x}) & \text{if } t < \ell, \\ \frac{1}{2}\Sigma_{(\mu,\ell)}(\mathbf{x}) - \frac{1}{2}\left(\sum_{i=0}^{\infty} e_i(\mathbf{x})\right)\left(\sum_{i=0}^{\infty}(-1)^i e_i(\mathbf{x})\right)\Sigma'_{(\mu,\ell-1)}(\mathbf{x}) & \text{if } t > \ell, \end{cases}
\end{aligned}$$

where $t = \ell(\lambda)$ and $\mu = (\lambda_1, \ldots, \lambda_{2\ell-t})$. Note that the pair (μ, ℓ) appearing when $t > \ell$ satisfies that $(\mu,\ell) \in \mathscr{P}(G)$ and $\ell(\mu) < \ell$.

Proposition 4.3 ([3, Proposition 5.4]). *For $(\lambda, \ell) \in \mathscr{P}(G)$, the following holds.*

$$\operatorname{ch} \mathbf{T}^{\mathfrak{g}}(\lambda,\ell) = t^{\ell} S^{\mathfrak{g}}_{(\lambda,\ell)}(\mathbf{x})$$

5 The Grothendieck ring

Let $\mathcal{C}$ be the category of $\mathfrak{g}_\infty$-crystals whose object B has connected components isomorphic to $B(\lambda^0) \otimes B(\lambda^+)$ for some $\lambda^0 \in E$ and $\lambda^+ \in P^+_{\text{int}}$ with some finiteness conditions (see [3, Section 6.1]). We show that $\mathcal{C}$ is a monoidal category under the tensor product of crystals [3, Theorem 6.1]. Let $\mathcal{K} = \mathcal{K}(\mathcal{C})$ be the Grothendieck group of $\mathcal{C}$, i.e., the additive group of isomorphism classes $[B]$ for $B \in \mathcal{C}$. Define a multiplication on $\mathcal{K}$ by

$$[B] \cdot [B'] = [B \otimes B'].$$

Then we can show $\mathcal{K}$ forms an associative $\mathbb{Z}$-algebra. Note that we can find corresponding results for type A in [11].

We explain an algebra structure of $\mathcal{K}$ using the decomposition of tensor products of underlying crystals into connected components (cf. [16, Section 4]). Let $\mathcal{K}^0$ and $\mathcal{K}^+$ be the subgroups of $\mathcal{K}$ generated by $[B(\lambda)]$ for $\lambda \in E$ and $\lambda \in P^+_{\text{int}}$, respectively. It is clear $\mathcal{K} \subseteq \mathcal{K}^0 \otimes \mathcal{K}^+$ by definition. Conversely, for given $\lambda^0 \in E$ and $\lambda^+ \in P^+$, we have $B(\lambda^0) \otimes B(\lambda^+) \cong B(\lambda)$ for $\lambda \in P$ (cf. [16, Theorem 4.4]). Thus, we have $\mathcal{K} = \mathcal{K}^0 \otimes \mathcal{K}^+$.

To explain an algebra structure of $\mathcal{K}^0$, we consider the following decomposition.

Proposition 5.1 ([15]). *For $\lambda, \mu \in E$, we have*

$$B(\lambda) \otimes B(\mu) \cong \bigoplus_{\nu \in E} B(\nu)^{\oplus \mathtt{LR}^{\nu^\dagger}_{\lambda^\dagger \mu^\dagger}},$$

where $\mathtt{LR}^{\nu^\dagger}_{\lambda^\dagger\mu^\dagger}$ is the Littlewood-Richardson coefficient for partitions $\lambda^\dagger$, $\mu^\dagger$, and $\nu^\dagger$.

As a corollary, we know that $\mathcal{K}^0$ is a subalgebra of $\mathcal{K}$ and obtain an algebra isomorphism between $\mathcal{K}^0$ and the ring Sym of symmetric functions since their structure constants coincide.

Proposition 5.2 ([3, Proposition 6.5])**.** *There exists an algebra isomorphism*

$$\Psi^0 : \mathcal{K}^0 \longrightarrow \mathrm{Sym} \tag{5.1}$$

which, for $\lambda \in \mathcal{P}$, sends $[B(\varpi_\lambda)]$ to s_λ.

On the other hand, for $k \geq 0$ and $(\lambda, \ell) \in \mathscr{P}(G)$, put

$$H_k^{\mathfrak{g}} = [B(\Pi_k^{\mathfrak{g}})], \qquad H^{\mathfrak{g}}(\lambda,\ell) = [B(\Pi^{\mathfrak{g}}(\lambda,\ell))]$$

and $\overline{H}^{\mathfrak{d}}(0) = [B(\overline{\Pi}_0^{\mathfrak{d}})]$. By the semisimplicity result in [12, 13], we deduce that $[B] = [B']$ in $\mathcal{K}^+$ if and only if $\mathrm{ch}(B) = \mathrm{ch}(B')$. Thus, we can rewrite Proposition 4.3 as follows.

Proposition 5.3 ([3, Proposition 6.2])**.** *When $\mathfrak{g} = \mathfrak{d}$ and $(\lambda, \ell) \in \mathscr{P}(G)$ with $\ell(\lambda) < \ell$, the following identity holds in $\mathcal{K}^+$.*

$$\begin{aligned} H^{\mathfrak{d}}(\lambda,\ell) &= \frac{1}{2}\det(H^{\mathfrak{d}}_{(\lambda_{\ell-i+1}+i-1)+(j-1)} + \delta(j\neq 1)H^{\mathfrak{d}}_{(\lambda_{\ell-i+1}+i-1)-(j-1)})_{i,j=1,\dots,\ell} \\ &\quad + \frac{1}{2}(H_0^{\mathfrak{d}} - \overline{H}_0^{\mathfrak{d}})H^{\mathfrak{c}}(\lambda,\ell-1) \end{aligned}$$

When $\mathfrak{g} = \mathfrak{d}$ and $(\lambda, \ell) \in \mathscr{P}(G)$ with $\ell(\lambda) > \ell$, the following identity holds in $\mathcal{K}^+$.

$$\begin{aligned} H^{\mathfrak{d}}(\lambda,\ell) &= \frac{1}{2}\det(H^{\mathfrak{d}}_{(\mu_{\ell-i+1}+i-1)+(j-1)} + \delta(j\neq 1)H^{\mathfrak{d}}_{(\mu_{\ell-i+1}+i-1)-(j-1)})_{i,j=1,\dots,\ell} \\ &\quad - \frac{1}{2}(H_0^{\mathfrak{d}} - \overline{H}_0^{\mathfrak{d}})H^{\mathfrak{c}}(\mu,\ell-1), \end{aligned}$$

Here, $t = \ell(\lambda)$ and $\mu = (\lambda_1, \dots, \lambda_{2\ell-t}, 0^{t-\ell})$. Otherwise, the following identity holds in $\mathcal{K}^+$.

$$H^{\mathfrak{g}}(\lambda,\ell) = \det(H^{\mathfrak{g}}_{(\lambda_{\ell-i+1}+i-1)+(j-1)} + \delta(j\neq 1)H^{\mathfrak{g}}_{(\lambda_{\ell-i+1}+i-1)-(j-1)})_{i,j\in[\ell]}$$

As a corollary, we know that $\mathcal{K}^+$ is a subalgebra of $\mathcal{K}$. Even though above two cases seem to contain different variables coming from $H^{\mathfrak{c}}$ (not $H^{\mathfrak{d}}$), the identity $E'_r = E_r - E_{r+2}$ implies that $H_k^{\mathfrak{c}}$ is a polynomial in $\{H_i^{\mathfrak{d}}\}$.

Let $\mathbf{h} = \{\mathtt{h}_k \,|\, k \in \mathbb{Z}_+\}$ ($\{\mathtt{h}_k \,|\, k \in \mathbb{Z}_+\} \cup \{\overline{\mathtt{h}}_0\}$ when $\mathfrak{g}_\infty = \mathfrak{d}_\infty$) be commuting formal variables, and $\mathbb{Z}[\![\mathbf{h}]\!]$ be the set of formal power series in $\mathbf{h}$. Using Proposition 5.3 and [3, Lemma 6.3], we can construct an isomorphism between $\mathcal{K}^+$ and $\mathbb{Z}[\![\mathbf{h}]\!]$.

Theorem 5.4 ([3, Theorem 6.4])**.** *Define $\Phi^+ : \mathbb{Z}[\![\mathbf{h}]\!] \longrightarrow \mathcal{K}^+$ by a $\mathbb{Z}$-algebra homomorphism sending $\mathtt{h}_k$ to $H_k^{\mathfrak{g}}$ (and $\overline{\mathtt{h}}_0$ to $\overline{H}_0^{\mathfrak{d}}$ when $\mathfrak{g} = \mathfrak{d}$). Then Φ^+ is an isomorphism of $\mathbb{Z}$-algebras.*

Finally, we can explicitly describe an algebra structure of $\mathcal{K}$. Based on the above results, we know that $\{[B(\varpi_i)] \,|\, i \geq 1\} \cup \{[B(\Pi^{\mathfrak{g}}_j)] \,|\, j \geq 0\}$ generates $\mathcal{K}$ as a $\mathbb{Z}$-algebra and hence it is sufficient to find a basis expansion of $[B(\Pi^{\mathfrak{g}}_a)] \cdot [B(\varpi_b)]$. A general result for the basis expansion of $[B(\Pi^{\mathfrak{g}}(\lambda,\ell))] \cdot [B(\varpi_\mu)]$ for $(\lambda,\ell) \in \mathscr{P}(G)$ and $\mu \in \mathcal{P}$ is given in [16, Section 4.3]. In particular, we obtain the basis expansion of $[B(\Pi^{\mathfrak{g}}_a)] \cdot [B(\varpi_b)]$ by applying the general result (see [3, Proposition 6.6]).

To characterize the algebra structure of $\mathcal{K}$, we introduce a set $\mathbf{z} = \{\mathtt{z}_k \,|\, k \in \mathbb{N}\}$ of (other) commuting formal variables. Define $\mathcal{A}_0 = \mathbb{Z}[\![\mathbf{h}]\!]$, $\mathcal{A}_n = \mathcal{A}_0[\mathtt{z}_1, \ldots, \mathtt{z}_n]$ for $n \in \mathbb{N}$, and $\mathcal{A} = \sum_{n\geq 0} \mathcal{A}_n$. We inductively define a $\mathbb{Z}$-algebra structure on $\mathcal{A}$ as follows.

- The multiplication on $\mathcal{A}_0$ is the usual multiplication.
- Suppose that the multiplication on $\mathcal{A}_{n-1}$ is defined. Define $a\mathtt{z}_n = \mathtt{z}_n a + \delta_n(a)$ for $a \in \mathcal{A}_{n-1}$, where δ_n is a derivation on $\mathcal{A}_{n-1}$ such that

$$\mathfrak{c}_\infty \qquad \begin{cases} \delta_n(\mathtt{z}_k) = 0 & (1 \leq k \leq n-1) \\ \delta_n(\mathtt{h}_a) = \displaystyle\sum_{i=0}^{n-1} \sum_{j=0}^{\min\{a,n-i\}} \mathtt{z}_i \mathtt{h}_{a+n-i-2j} & (a \in \mathbb{Z}_+) \end{cases}$$

$$\mathfrak{b}_\infty \qquad \begin{cases} \delta_n(\mathtt{z}_k) = 0 & (1 \leq k \leq n-1) \\ \delta_n(\mathtt{h}_a) = \displaystyle\sum_{i=0}^{n-1} \mathtt{z}_i \left(\sum_{j=0}^{\min\{a,b-i\}} \mathtt{h}_{a+b-i-2j} + \delta(b-i>a) \sum_{k=1}^{b-i-a} \mathtt{h}_{b-i-a-k} \right) & (a \in \mathbb{Z}_+) \end{cases}$$

$$\mathfrak{d}_\infty \qquad \begin{cases} \delta_n(\mathtt{z}_k) = 0 & (1 \leq k \leq n-1) \\ \delta_n(\mathtt{h}_0) = \displaystyle\sum_{i=0}^{n-1} \bigoplus_{j=0}^{\lfloor \frac{n-i}{2} \rfloor} \mathtt{z}_i \mathtt{h}_{b-i-2j}, \quad \delta_n(\overline{\mathtt{h}}_0) = \sum_{i=0}^{n-1} \bigoplus_{j=0}^{\lfloor \frac{n-i}{2} \rfloor} \mathtt{z}_i \overline{\mathtt{h}}_{b-i-2j} & \\ \delta_n(\mathtt{h}_a) = \displaystyle\sum_{i=0}^{n-1} \mathtt{z}_i \left(\sum_{j=0}^{\min\{\lfloor \frac{a+b-i}{2} \rfloor, b-i\}} \mathtt{h}_{a+b-i-2j} + \delta(b-i \geq a) \sum_{k=0}^{\lfloor \frac{b-i-a}{2} \rfloor} \overline{\mathtt{h}}_{b-i-a-2k} \right) & (a \in \mathbb{N}) \end{cases}$$

where $\overline{\mathtt{h}}_a = \mathtt{h}_a$ for $a \geq 1$. Now, we obtain an algebra isomorphism between $\mathcal{K}$ and $\mathcal{A}$.

Theorem 5.5 ([3, Theorem 6.8])**.** *The assignment sending $[B(\Pi^{\mathfrak{g}}_a)]$ to $\mathtt{h}_a$ ($[B(\overline{\Pi}^{\mathfrak{d}}_0)]$ to $\overline{\mathtt{h}}_0$) and $[B(\varpi_b)]$ to $\mathtt{z}_b$ defines a $\mathbb{Z}$-algebra isomorphism $\Psi : \mathcal{K} \to \mathcal{A}$. Indeed, we have $\Psi = \Psi^0 \otimes \Psi^+$, where Ψ^+ is the inverse map of Φ^+ given in Theorem 5.4.*

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