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Murnaghan-Type Representations of the Elliptic Hall Algebra

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Abstract. We construct a new family of graded representations $\widetilde{W}_\lambda$ indexed by Young diagrams λ for the positive elliptic Hall algebra $\mathcal{E}^+$ which generalizes the standard $\mathcal{E}^+$ action on symmetric functions. These representations have homogeneous bases of eigenvectors for the action of the Macdonald element $P_{0,1} \in \mathcal{E}^+$ generalizing the symmetric Macdonald functions. We find an explicit combinatorial rule for the action of the multiplication operators $e_r[X]^\bullet$ generalizing the Pieri rule for symmetric Macdonald functions.

Keywords: Macdonald polynomials, elliptic Hall algebra, double affine Hecke algebra

1 Introduction

The space of symmetric functions, Λ, is a central object in algebraic combinatorics deeply connecting the fields of representation theory, geometry, and combinatorics. In his influential paper [11], Macdonald introduced a special basis $P_\lambda[X;q,t]$ for Λ over $\mathbb{Q}(q,t)$ simultaneously generalizing many other important and well-studied symmetric function bases like the Schur functions $s_\lambda[X]$. These symmetric functions $P_\lambda[X;q,t]$, called the symmetric Macdonald functions, exhibit many striking combinatorial properties and can be defined as the eigenvectors of a certain operator $\Delta : \Lambda \to \Lambda$, called the Macdonald operator, constructed using polynomial difference operators. It was discovered through the works of Bergeron, Garsia, Haiman, Tesler, and many others [10] [1] [2] that variants of the symmetric Macdonald functions called the modified Macdonald functions $\widetilde{H}_\lambda[X;q,t]$ have deep ties to the geometry of the Hilbert schemes $\mathrm{Hilb}_n(\mathbb{C}^2)$. On the side of representation theory, it was shown first in full generality by Cherednik [4] that one can recover the symmetric Macdonald functions by considering the representation theory of certain algebras called the spherical double affine Hecke algebras (DAHAs) in type GL_n.

The positive elliptic Hall algebra (EHA), $\mathcal{E}^+$, was introduced by Burban and Schiffmann [3] as the positive subalgebra of the Hall algebra of the category of coherent sheaves on an elliptic curve over a finite field. This algebra has connections to many

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areas of mathematics including, most importantly for the present paper, to Macdonald theory. In [13], Schiffmann and Vasserot realize $\mathcal{E}^+$ as a stable limit of the positive spherical DAHAs in type GL_n. They show further that there is a natural action of $\mathcal{E}^+$ on Λ aligning with the spherical DAHA representations originally considered by Cherednik. In particular, the action of $P_{0,1} \in \mathcal{E}^+$ gives the Macdonald operator Δ. The action of $\mathcal{E}^+$ on Λ can be realized as the action of certain generalized convolution operators on the torus equivariant K-theory of the schemes $\mathrm{Hilb}_n(\mathbb{C}^2)$.

Dunkl and Luque in [6] introduced symmetric and non-symmetric vector-valued (vv.) Macdonald polynomials. The term vector-valued here refers to polynomial-like objects of the form $\sum_\alpha c_\alpha X^\alpha \otimes v_\alpha$ for some scalars c_α, monomials X^α, and vectors v_α lying in some $\mathbb{Q}(q,t)$-vector space. The non-symmetric vv. Macdonald polynomials are distinguished bases for certain DAHA representations built from the irreducible representations of the finite Hecke algebras in type A. These DAHA representations are indexed by Young diagrams and exhibit interesting combinatorial properties relating to periodic Young tableaux. The symmetric vv. Macdonald polynomials are distinguished bases for the spherical (i.e. Hecke-invariant) subspaces of these DAHA representations. Naturally, the spherical DAHA acts on these spherical subspaces with the special element $Y_1 + \ldots + Y_n$ of spherical DAHA acting diagonally on the symmetric vv. Macdonald polynomials.

Dunkl and Luque in [6] (and in later work of Colmenarejo, Dunkl, and Luque [5] and Dunkl [7]) only consider the finite rank non-symmetric and symmetric vv. Macdonald polynomials. It is natural to ask if there is an infinite-rank stable-limit construction using the symmetric vv. Macdonald polynomials to give generalized symmetric Macdonald functions and associated representations of the positive elliptic Hall algebra $\mathcal{E}^+$. In this paper, we will describe such a construction (Thm. 2). We will obtain a new family of graded $\mathcal{E}^+$-representations $\widetilde{W}_\lambda$ indexed by Young diagrams λ and a natural generalization of the symmetric Macdonald functions $\mathfrak{P}_T$ indexed by certain labellings of infinite Young diagrams built as limits of the symmetric vv. Macdonald polynomials. For combinatorial reasons there is essentially a unique natural way to obtain this construction. For any λ we will consider the increasing chains of Young diagrams $\lambda^{(n)} = (n - |\lambda|, \lambda)$ for $n \geq |\lambda| + \lambda_1$ to build the representations $\widetilde{W}_\lambda$. These special sequences of Young diagrams are central to Murnaghan's Theorem [12] regarding the reduced Kronecker coefficients. As such we refer to the $\mathcal{E}^+$-representations $\widetilde{W}_\lambda$ as Murnaghan-type. For $\lambda = \emptyset$ we recover the $\mathcal{E}^+$ action on Λ and the symmetric Macdonald functions $P_\mu[X;q,t]$. We will show that these Murnaghan-type representations $\widetilde{W}_\lambda$ are mutually non-isomorphic. The existence of these representations of the elliptic Hall algebra raises many questions about possible new relations between Macdonald theory and geometry. Other authors have constructed families of $\mathcal{E}^+$-representations [8] [9]. Although there should exist a relationship between the Murnaghan-type representations $\widetilde{W}_\lambda$ and those of other authors, the construction in this paper appears to be distinct from prior $\mathcal{E}^+$-module constructions.

For technical reasons regarding the misalignment of the spectrum of the Cherednik operators Y_i we will need to restate many of the results of Dunkl and Luque in [6] using a re-oriented version of the Cherednik operators θ_i. This alternative choice of conventions greatly assists during the construction of the generalized Macdonald functions $\mathfrak{P}_T$. The combinatorics underpinning the non-symmetric vv. Macdonald polynomials originally defined by Dunkl and Luque will be reversed in the conventions appearing in this paper.

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2 Definitions and Notations

2.1 Some Combinatorics

We start with a description of many of the combinatorial objects which we will need for the remainder of this paper.

Definition 1. *A **partition** is a (possibly empty) sequence of weakly decreasing positive integers. Denote by $\mathbb{Y}$ the set of all partitions. Given a partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ we set $\ell(\lambda) := r$ and $|\lambda| := \lambda_1 + \ldots + \lambda_r$. For $\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathbb{Y}$ and $n \geq n_\lambda := |\lambda| + \lambda_1$ we set $\lambda^{(n)} := (n - |\lambda|, \lambda_1, \ldots, \lambda_r)$. We will identify partitions as defined above with **Young diagrams** of the corresponding shape in English notation i.e. justified up and to the left.*

Fix a partition λ with $|\lambda| = n$. We will require each of the following combinatorial constructions for types of labellings of the Young diagram λ. If a diagram λ appears as the domain of a labelling function then we are referring to the set of boxes of λ as the domain.

- *A non-negative **reverse Young tableau** $\mathrm{RYT}_{\geq 0}(\lambda)$ is a labelling $T : \lambda \to \mathbb{Z}_{\geq 0}$ which is weakly decreasing along rows and columns.*

- *A non-negative **reverse semi-standard Young tableau** $\mathrm{RSSYT}_{\geq 0}(\lambda)$ is a labelling $T : \lambda \to \mathbb{Z}_{\geq 0}$ which is weakly decreasing across rows and strictly decreasing down columns.*

- *A **standard Young tableau** $\mathrm{SYT}(\lambda)$ is a labelling $\tau : \lambda \to \{1, \ldots, n\}$ which is strictly increasing along rows and columns.*

- *A non-negative **periodic standard Young tableau** $\mathrm{PSYT}_{\geq 0}(\lambda)$ is a labelling $\tau : \lambda \to \{jq^b : 1 \leq j \leq n, b \geq 0\}$ in which each $1 \leq j \leq n$ occurs in exactly one box of λ and where the labelling is strictly increasing along rows and columns. Here we order the formal products jq^m by $jq^m < kq^\ell$ if $m > \ell$ or in the case that $m = \ell$ we have $j < k$. Note that $SYT(\lambda) \subset \mathrm{PSYT}_{\geq 0}(\lambda)$.*

Example 1. $$\begin{array}{cccccc} 17q^7 & 15q^5 & 16q^5 & 11q^3 & 7q^1 & 2q^0 \\ 14q^6 & 12q^4 & 13q^4 & 9q^2 & 8q^0 & \\ 10q^2 & 4q^1 & 5q^1 & 6q^1 & & \\ 3q^1 & 1q^0 & & & & \end{array} \in \mathrm{PSYT}_{\geq 0}(6,5,4,2)$$

Definition 2. *Given a box, $\square$, in a Young diagram λ we define the content of $\square$ as $c(\square) := a - b$ where $\square$ is in row b and column a. Let $\tau \in \mathrm{PSYT}_{\geq 0}(\lambda)$ and $1 \leq i \leq n$. Whenever $\tau(\square) = iq^b$ for some box $\square \in \lambda$ we will write $c_\tau(i) := c(\square)$ and $w_\tau(i) := b$. Let $1 \leq j \leq n-1$ and suppose that for some boxes $\square_1, \square_2 \in \lambda$ that $\tau(\square_1) = jq^m$ and $\tau(\square_2) = (j+1)q^\ell$. Let τ' be the labelling defined by $\tau'(\square_1) = (j+1)q^m$, $\tau'(\square_2) = jq^\ell$, and $\tau'(\square) = \tau(\square)$ for $\square \in \lambda \setminus \{\square_1, \square_2\}$. If $\tau' \in \mathrm{PSYT}_{\geq 0}(\lambda)$ then we write $s_j(\tau) := \tau'$. Let $\Psi(\tau) \in \mathrm{PSYT}_{\geq 0}(\lambda)$ be the labelling defined by whenever $\tau(\square) = kq^a$ then either $\Psi(\tau)(\square) = (k-1)q^a$ when $k \geq 2$ or $\Psi(\tau)(\square) = nq^{a+1}$ when $k = 1$. We give the set $\mathrm{PSYT}_{\geq 0}(\lambda)$ a partial order defined by the following cover relations.*

- *For all $\tau \in \mathrm{PSYT}_{\geq 0}(\lambda)$, $\Psi(\tau) > \tau$.*
- *If $w_\tau(i) < w_\tau(i+1)$ then $s_i(\tau) > \tau$.*
- *If $w_\tau(i) = w_\tau(i+1)$ and $c_\tau(i) - c_\tau(i+1) > 1$ then $s_i(\tau) > \tau$.*

Define the map $\mathfrak{p}_\lambda : \mathrm{PSYT}_{\geq 0}(\lambda) \to \mathrm{RYT}_{\geq 0}(\lambda)$ by $\mathfrak{p}_\lambda(\tau)(\square) = b$ whenever $\tau(\square) = iq^b$. We will write $\mathrm{PSYT}_{\geq 0}(\lambda; T)$ for the set of all $\tau \in \mathrm{PSYT}_{\geq 0}(\lambda)$ with $\mathfrak{p}_\lambda(\tau) = T \in \mathrm{RYT}_{\geq 0}(\lambda)$.

Example 2. $$\Psi\left(\begin{array}{cccccc} 1q^7 & 3q^5 & 5q^5 & 8q^2 & 12q^1 & 17q^0 \\ 2q^6 & 4q^5 & 6q^5 & 14q^0 & 16q^0 & \\ 7q^2 & 10q^1 & 11q^1 & 15q^0 & & \\ 9q^1 & 13q^0 & & & & \end{array}\right) = \begin{array}{cccccc} 17q^8 & 2q^5 & 4q^5 & 7q^2 & 11q^1 & 16q^0 \\ 1q^6 & 3q^5 & 5q^5 & 13q^0 & 15q^0 & \\ 6q^2 & 9q^1 & 10q^1 & 14q^0 & & \\ 8q^1 & 12q^0 & & & & \end{array}$$

Lemma 1. *Let $\lambda \in \mathbb{Y}$ and $T \in \mathrm{RYT}_{\geq 0}(\lambda)$. There exist $\min(T), \mathrm{top}(T) \in \mathrm{PSYT}_{\geq 0}(\lambda; T)$ such that for all $\tau \in \mathrm{PSYT}_{\geq 0}(\lambda; T)$, $\min(T) \leq \tau \leq \mathrm{top}(T)$.*

Example 3. *Given* $T = \begin{array}{cccccc} 7 & 5 & 5 & 2 & 1 & 0 \\ 6 & 5 & 5 & 0 & 0 & \\ 2 & 1 & 1 & 0 & & \\ 1 & 0 & & & & \end{array} \in \mathrm{RYT}_{\geq 0}(6,5,4,2)$ *we have that*

$$\min(T) = \begin{array}{|c|c|c|c|c|c|}\hline 17q^7 & 12q^5 & 13q^5 & 10q^2 & 6q^1 & 1q^0 \\ \hline 16q^6 & 14q^5 & 15q^5 & 2q^0 & 3q^0 \\ \cline{1-5} 11q^2 & 7q^1 & 8q^1 & 4q^0 \\ \cline{1-4} 9q^1 & 5q^0 \\ \cline{1-2} \end{array} \quad \text{and } \operatorname{top}(T) = \begin{array}{|c|c|c|c|c|c|}\hline 1q^7 & 3q^5 & 5q^5 & 8q^2 & 12q^1 & 17q^0 \\ \hline 2q^6 & 4q^5 & 6q^5 & 14q^0 & 16q^0 \\ \cline{1-5} 7q^2 & 10q^1 & 11q^1 & 15q^0 \\ \cline{1-4} 9q^1 & 13q^0 \\ \cline{1-2} \end{array} .$$

Definition 3. *Let $\lambda \in \mathbb{Y}$ with $|\lambda| = n$ and $T \in \mathrm{RYT}_{\geq 0}(\lambda)$. Define $S(T) \in \mathrm{SYT}(\lambda)$ by ordering the boxes of λ according to $\square_1 \leq \square_2$ if and only if*

- *$T(\square_1) > T(\square_2)$ or*
- *$T(\square_1) = T(\square_2)$ and $\square_1$ comes before $\square_2$ in the column-standard labelling of λ.*

Define the composition $\mu(T)$ of n so that the Young subgroup $\mathfrak{S}_{\mu(T)}$ of $\mathfrak{S}_n$ is the group generated by the $s_i \in \mathfrak{S}_n$ such that the entries iq^a and $(i+1)q^b$ occur in the same row of $\min(T)$ for some $a, b \geq 0$.

Example 4. *For $T \in \mathrm{RYT}_{\geq 0}(6,5,4,2)$ as in Example 3 we have that* $S(T) = \begin{array}{|c|c|c|c|c|c|}\hline 1 & 3 & 5 & 8 & 12 & 17 \\ \hline 2 & 4 & 6 & 14 & 16 \\ \cline{1-5} 7 & 10 & 11 & 15 \\ \cline{1-4} 9 & 13 \\ \cline{1-2} \end{array}$.

Definition 4. *Let $\lambda \in \mathbb{Y}$, with $|\lambda| = n$ and $\tau \in \mathrm{PSYT}_{\geq 0}(\lambda; T)$. An ordered pair of boxes $(\square_1, \square_2) \in \lambda \times \lambda$ is called an* ***inversion pair*** *of τ if $S(T)(\square_1) < S(T)(\square_2)$ and $i > j$ where $\tau(\square_1) = iq^a$, $\tau(\square_2) = jq^b$ for some $a, b \geq 0$. The set of all inversion pairs of τ will be denoted by $\mathrm{Inv}(\tau)$.*

Example 5. *In the labelling* $\begin{array}{|c|c|c|c|c|c|}\hline 17q^7 & 12q^5 & 13q^5 & 10q^2 & 6q^1 & 1q^0 \\ \hline 16q^6 & 14q^5 & 15q^5 & 2q^0 & 3q^0 \\ \cline{1-5} 11q^2 & 7q^1 & 8q^1 & 4q^0 \\ \cline{1-4} 9q^1 & 5q^0 \\ \cline{1-2} \end{array}$ *the pairs $(17q^7, 12q^5)$, $(14q^5, 13q^5)$, and $(5q^0, 4q^0)$ are examples of inversions. Here we have referred to boxes according to their labels.*

2.2 Positive Double Affine Hecke Algebra

Here we describe the positive double affine Hecke algebras in type GL_n.

Definition 5. *Define the* ***positive double affine Hecke algebra*** *$\mathcal{D}_n$ to be the $\mathbb{Q}(q,t)$-algebra generated by $T_1, \ldots, T_{n-1}$, $\theta_1, \ldots, \theta_n$, and $X_1, \ldots, X_n$ subject to the relations*

- $(T_i - 1)(T_i + t) = 0$
- $T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}$
- $T_iT_j = T_jT_i$, $|i-j| > 1$
- $\theta_i\theta_j = \theta_j\theta_i$
- $\theta_{i+1} = tT_i^{-1}\theta_iT_i^{-1}$
- $T_i\theta_j = \theta_jT_i$, $j \notin \{i, i+1\}$
- $X_iX_j = X_jX_i$
- $X_{i+1} = tT_i^{-1}X_iT_i^{-1}$
- $T_iX_j = X_jT_i$, $j \notin \{i, i+1\}$
- $\pi_nX_i = X_{i+1}\pi_n$
- $\pi_nX_n = qX_1\pi_n$.

where in the above $\pi_n := t^{n-1}\theta_1T_1^{-1}\cdots T_{n-1}^{-1}$. The ***finite Hecke algebra*** *$\mathcal{H}_n$ is the subalgebra of $\mathcal{D}_n$ generated by the elements $T_1, \ldots, T_{n-1}$ and the* ***positive affine Hecke algebra*** *$\mathcal{A}_n$ is the subalgebra of $\mathcal{D}_n$ generated by the elements $T_1, \ldots, T_{n-1}, \theta_1, \ldots, \theta_n$.*

Remark 1. *Note that $\mathcal{D}_n$ has a $\mathbb{Z}_{\geq 0}$-grading determined by $\deg(X_i) = 1$ and $\deg(\theta_i) = \deg(T_i) = 0$. We will sometimes write $\theta_i^{(n)}$ for $\theta_i \in \mathcal{A}_n$ to differentiate between $\theta_i \in \mathcal{A}_m$.*

Definition 6. *Let $\epsilon^{(n)} \in \mathcal{H}_n$ denote the (normalized) trivial idempotent given by*

$$\epsilon^{(n)} := \frac{1}{[n]_t!}\sum_{\sigma\in\mathfrak{S}_n} t^{\binom{n}{2}-\ell(\sigma)}T_\sigma$$

where $[n]_t! := \prod_{i=1}^{n}(\frac{1-t^i}{1-t})$. We will write $[\mu]_t! := [\mu_1]_t!\cdots[\mu_r]_t!$ for any composition $\mu = (\mu_1, \ldots, \mu_r)$. The ***positive spherical double affine Hecke algebra*** *$\mathcal{D}_n^{sph}$ is the (non-unital) subalgebra of $\mathcal{D}_n$ given by $\mathcal{D}_n^{sph} := \epsilon^{(n)}\,\mathcal{D}_n\,\epsilon^{(n)}$.*

Remark 2. *Given any $\mathcal{D}_n$-module V the space $\epsilon^{(n)}(V)$ is naturally a $\mathcal{D}_n^{sph}$-module. Note that although $1 \notin \mathcal{D}_n^{sph}$ the algebra $\mathcal{D}_n^{sph}$ is unital with unit $\epsilon^{(n)}$. Further, $\mathcal{D}_n^{sph}$ has a grading inherited from $\mathcal{D}_n$.*

2.3 Positive Elliptic Hall Algebra

Here we give a very brief description of the positive elliptic Hall algebra.

Definition 7. *For $\ell > 0$ define the special elements $P_{0,\ell}^{(n)}, P_{\ell,0}^{(n)} \in \mathcal{D}_n^{sph}$ by*

- $P_{0,\ell}^{(n)} = \epsilon^{(n)}\left(\sum_{i=1}^{n}\theta_i^{\ell}\right)\epsilon^{(n)}$
- $P_{\ell,0}^{(n)} = q^{\ell}\epsilon^{(n)}\left(\sum_{i=1}^{n}X_i^{\ell}\right)\epsilon^{(n)}$.

Theorem 1. *[13] There is a unique graded algebra surjection $\mathcal{D}_{n+1}^{sph} \to \mathcal{D}_n^{sph}$ determined for $\ell > 0$ by $P_{0,\ell}^{(n+1)} \to P_{0,\ell}^{(n)}$ and $P_{\ell,0}^{(n+1)} \to P_{\ell,0}^{(n)}$.*

The existence of the graded algebra surjections $\mathcal{D}_{n+1}^{\mathrm{sph}} \to \mathcal{D}_n^{\mathrm{sph}}$ allows for the following definition.

Definition 8. *[13] The **positive elliptic Hall algebra** $\mathcal{E}^+$ is the stable limit of the graded algebras $\mathcal{D}_n^{sph}$ with respect to the maps $\mathcal{D}_{n+1}^{sph} \to \mathcal{D}_n^{sph}$. For $\ell > 0$ define the special elements of $\mathcal{E}^+$, $P_{0,\ell} := \lim_n P_{0,\ell}^{(n)}$ and $P_{\ell,0} := \lim_n P_{\ell,0}^{(n)}$.*

Remark 3. *The elements $P_{0,\ell}^{(n)}, P_{\ell,0}^{(n)}$ for $\ell > 0$ generate $\mathcal{D}_n^{sph}$ and the elements $P_{0,\ell}, P_{\ell,0}$ for $\ell > 0$ generate $\mathcal{E}^+$ [13]. Further $\mathcal{E}^+$ has a $\mathbb{Z}_{\geq 0}$-grading determined by $\deg(P_{0,\ell}) = 0$ and $\deg(P_{\ell,0}) = \ell$.*

3 DAHA Modules from Young Diagrams

3.1 The $\mathcal{D}_n$-module V_λ

We begin by defining a collection of DAHA modules indexed by Young diagrams $\lambda \in \mathbb{Y}$. These modules are the same as those appearing in [6] but we take the approach of using induction from $\mathcal{A}_n$ to $\mathcal{D}_n$ for their definition.

Definition 9. *Let $\lambda \in \mathbb{Y}$ with $|\lambda| = n$. We will denote by S_λ the irreducible $\mathcal{H}_n$ module corresponding to the partition λ. Define the algebra homomorphism $\rho_n : \mathcal{A}_n \to \mathcal{H}_n$ by $\rho_n(T_i) = T_i$ and $\rho_n(\theta_1) = 1$. Let $\rho_n^*(S_\lambda)$ denote the $\mathcal{A}_n$ module determined by $X(v) := \rho_n(X)(v)$ for $X \in \mathcal{A}_n$ and $v \in S_\lambda$. Define the $\mathcal{D}_n$-module V_λ to be the induced module $V_\lambda := \mathrm{Ind}_{\mathcal{A}_n}^{\mathcal{D}_n} \rho_n^*(S_\lambda)$.*

We fix a distinguished basis $\{e_\tau | \tau \in \mathrm{SYT}(\lambda)\}$ for S_λ consisting of $\rho_n(\theta^{(n)})$-weight vectors uniquely normalized such that the next proposition holds. The defining relations for the $\mathcal{H}_n$ modules S_λ have been omitted but they can be inferred from the next proposition. The modules V_λ naturally have the basis given by $X^\alpha \otimes e_\tau$ where X^α is a monomial and $\tau \in \mathrm{SYT}(\lambda)$. Note that the action of π_n on V_λ is invertible so we may consider the action of π_n^{-1} although we have not formally included π_n^{-1} into the algebra $\mathcal{D}_n$.

Using the theory of intertwiners for DAHA and some combinatorics we are able to show the following structural results. The F_τ appearing below are the version of the non-symmetric vv. Macdonald polynomials following our conventions.

Proposition 1. *There exists a basis of V_λ consisting of $\theta^{(n)}$-weight vectors F_τ for $\tau \in \mathrm{PSYT}_{\geq 0}(\lambda)$ with distinct $\theta^{(n)}$-weights such that the following hold:*

- $\theta_i^{(n)}(F_\tau) = q^{w_\tau(i)} t^{c_\tau(i)} F_\tau$
- *If $\tau \in \mathrm{SYT}(\lambda)$ then $F_\tau = 1 \otimes e_\tau$.*

- *If* $s_i(\tau) > \tau$ *then* $F_{s_i(\tau)} = \left(tT_i^{-1} + \frac{(t-1)q^{w\tau(i+1)}t^{c\tau(i+1)}}{q^{w\tau(i)}t^{c\tau(i)} - q^{w\tau(i+1)}t^{c\tau(i+1)}} \right) F_\tau$.
- $F_{\Psi(\tau)} = q^{w_1(\tau)} X_n \pi_n^{-1} F_\tau$.

Example 6.

$$\begin{aligned} F_{\begin{array}{|c|c|}\hline 1q & 2q \\ \hline 3 \\ \cline{1-1}\end{array}} &= t^{-2} X_1 X_2 \otimes e_{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}} + t^{-2} \left(\frac{1-t}{1-qt^2} \right) X_2 X_3 \otimes e_{\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}} \\ &+ \frac{t^{-2}}{1+t} \left(\frac{1-t}{1-qt^2} \right) X_2 X_3 \otimes e_{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}} - t^{-3} \left(\frac{1-t}{1-qt^2} \right) X_1 X_3 \otimes e_{\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}} \\ &+ \frac{t^{-1}}{1+t} \left(\frac{1-t}{1-qt^2} \right) X_1 X_3 \otimes e_{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}} \end{aligned}$$

Using Mackey decomposition we obtain the following.

Proposition 2. *The* $\mathcal{D}_n$*-module* V_λ *has the following decomposition into* $\mathcal{A}_n$*-submodules:*

$$\mathrm{Res}^{\mathcal{D}_n}_{\mathcal{A}_n} V_\lambda = \bigoplus_{T \in \mathrm{RYT}_{\geq 0}(\lambda)} U_T$$

where $U_T := span_{\mathbb{Q}(q,t)}\{F_\tau : \mathfrak{p}_\lambda(\tau) = T\}$*. Further, each* $\mathcal{A}_n$*-module* U_T *is irreducible.*

3.2 Connecting Maps Between $V_{\lambda^{(n)}}$

In order to build the inverse systems which we will use to define Murnaghan-type modules for $\mathcal{E}^+$, we need to consider the following maps.

Definition 10. *Let* $\lambda \in \mathbb{Y}$*. For* $n \geq n_\lambda$ *define* $\Phi_\lambda^{(n)} : V_{\lambda^{(n+1)}} \to V_{\lambda^{(n)}}$ *as the* $\mathbb{Q}(q,t)$*-linear map determined by*

$$\Phi_\lambda^{(n)}(X^\alpha \otimes e_\tau) = \begin{cases} X_1^{\alpha_1} \cdots X_n^{\alpha_n} \otimes e_{\tau|_{\lambda^{(n)}}} & \alpha_{n+1} = 0, \tau(\square_0) = n+1 \\ 0 & \textit{otherwise} \end{cases}$$

where $\square_0 = \lambda^{(n+1)}/\lambda^{(n)}$.

The next proposition is a crucial step in proving the main theorem of this paper. Its proof relies heavily on the use of the re-oriented Cherednik operators θ_i and their spectral analysis as well as the existence of a triangular monomial expansion of the F_τ.

Proposition 3. *Let* $T \in \mathrm{RYT}_{\geq 0}(\lambda^{(n)})$ *and* $T' \in \mathrm{RYT}_{\geq 0}(\lambda^{(n+1)})$ *be such that* $T(\square) = T'(\square)$ *for* $\square \in \lambda^{(n)}$ *and* $T'(\square_0) = 0$ *for* $\square_0 = \lambda^{(n+1)}/\lambda^{(n)}$*. Then* $\Phi_\lambda^{(n)}(F_{\mathrm{top}(T')}) = F_{\mathrm{top}(T)}$.

The maps $\Phi_\lambda^{(n)}$ possess another remarkable stability property regarding the action of the elements $P_{(0,\ell)}^{(n)}$ for $\ell > 0$.

Proposition 4. *For all $\ell > 0$ and $n \geq n_\lambda$,*

$$\Phi_\lambda^{(n)}\left(P_{(0,\ell)}^{(n+1)} - \sum_{\square \in \lambda^{(n+1)}} t^{\ell c(\square)}\right) = \left(P_{(0,\ell)}^{(n)} - \sum_{\square \in \lambda^{(n)}} t^{\ell c(\square)}\right)\Phi_\lambda^{(n)}.$$

4 Positive EHA Representations from Young Diagrams

In this section we build $\mathcal{E}^+$-modules using the maps $\Phi_\lambda^{(n)}$ and the stability of the F_τ basis already described.

4.1 The $\mathcal{D}_n^{\mathbf{sph}}$-modules $W_{\lambda^{(n)}}$

Here we consider the spherical subspaces of the V_λ modules.

Definition 11. *For $\lambda \in \mathbb{Y}$ with $|\lambda| = n$ define the $\mathcal{D}_n^{sph}$-module $W_\lambda := \epsilon^{(n)}(V_\lambda)$.*

We will need the following combinatorial description of the AHA submodules of V_λ which contain a nonzero T_i-invariant vector.

Proposition 5. *For $\lambda \in \mathbb{Y}$ with $|\lambda| = n$ and $T \in \mathrm{RYT}_{\geq 0}(\lambda)$,*

$$\dim_{\mathbb{Q}(q,t)} \epsilon^{(n)}(U_T) = \begin{cases} 1 & T \in \mathrm{RSSYT}_{\geq 0}(\lambda) \\ 0 & T \notin \mathrm{RSSYT}_{\geq 0}(\lambda). \end{cases}$$

We define the symmetric vv. Macdonald polynomials in the following way. These will agree up to scalars with those in [6].

Definition 12. *Let $T \in \mathrm{RSSYT}_{\geq 0}(\lambda)$. Define $P_T \in \epsilon^{(n)}(U_T)$ to be the unique element of the form*

$$P_T = F_{\mathrm{top}(T)} + \sum_{\tau \in \mathrm{PSYT}_{\geq 0}(\lambda;T) \setminus \{\mathrm{top}(T)\}} \kappa_\tau F_\tau.$$

We can now use Prop. 1 and Prop. 3 to prove the following results for the P_T.

Proposition 6. *For all $T \in \mathrm{RSSYT}_{\geq 0}(\lambda)$,*

$$P_T = \sum_{\tau \in \mathrm{PSYT}_{\geq 0}(\lambda;T)} \prod_{(\square_1,\square_2) \in \mathrm{Inv}(\tau)} \left(\frac{q^{T(\square_1)} t^{c(\square_1)+1} - q^{T(\square_2)} t^{c(\square_2)}}{q^{T(\square_1)} t^{c(\square_1)} - q^{T(\square_2)} t^{c(\square_2)}}\right) F_\tau.$$

Example 7. $P_{\begin{smallmatrix}\boxed{1}\boxed{1}\\\boxed{0}\end{smallmatrix}} = F_{\begin{smallmatrix}\boxed{1q}\boxed{2q}\\\boxed{3}\end{smallmatrix}} + \left(\frac{qt^2-t^{-1}}{qt-t^{-1}}\right) F_{\begin{smallmatrix}\boxed{1q}\boxed{3q}\\\boxed{2}\end{smallmatrix}} + \left(\frac{qt^2-t^{-1}}{qt-t^{-1}}\right)\left(\frac{qt-t^{-1}}{q-t^{-1}}\right) F_{\begin{smallmatrix}\boxed{2q}\boxed{3q}\\\boxed{1}\end{smallmatrix}}$

Proposition 7. *The set $\{P_T : T \in \mathrm{RSSYT}_{\geq 0}(\lambda)\}$ is a $\mathbb{Q}(q,t)[\theta_1,\dots,\theta_n]^{\mathfrak{S}_n}$-weight basis for W_λ and $P_{0,\ell}^{(n)}(P_T) = \left(\sum_{\square\in\lambda} q^{\ell T(\square)} t^{\ell c(\square)}\right) P_T$.*

The following is an important stability result for the symmetric vv. Macdonald polynomials. Its proof relies on Prop. 3 and Prop. 7.

Corollary 1. *Let $T \in \mathrm{RSSYT}_{\geq 0}(\lambda^{(n)})$ and $T' \in \mathrm{RSSYT}_{\geq 0}(\lambda^{(n+1)})$ such that $T(\square) = T'(\square)$ for $\square \in \lambda^{(n)}$ and $T'(\square_0) = 0$ for $\square_0 = \lambda^{(n+1)}/\lambda^{(n)}$. Then $\Phi_\lambda^{(n)}(P_{T'}) = P_T$.*

4.2 Stable Limit of the $W_{\lambda^{(n)}}$

We now can define the stable-limit spaces $\widetilde{W}_\lambda$ and the generalized symmetric Macdonald functions.

Definition 13. *Let $\lambda \in \mathbb{Y}$. Define the infinite diagram $\lambda^{(\infty)} := \bigcup_{n\geq n_\lambda} \lambda^{(n)}$. Define $\Omega(\lambda)$ to be the set of all labellings $T : \lambda^{(\infty)} \to \mathbb{Z}_{\geq 0}$ such that $|\{\square \in \lambda^{(\infty)} : T(\square) \neq 0\}| < \infty$, T decreases weakly across rows, and T decreases strictly down columns.*

*Define the space $W_\lambda^{(\infty)}$ to be the inverse limit $\varprojlim W_{\lambda^{(n)}}$ with respect to the maps $\Phi_\lambda^{(n)}$. Let $\widetilde{W}_\lambda$ be the subspace of all bounded X-degree elements of $W_\lambda^{(\infty)}$. For any symmetric function $F \in \Lambda$ define $F[X]^\bullet$ to be the corresponding multiplication operator on $\widetilde{W}_\lambda$. Lastly, for $T \in \Omega(\lambda)$ define the **generalized symmetric Macdonald function** $\mathfrak{P}_T := \lim_n P_{T|_{\lambda^{(n)}}} \in \widetilde{W}_\lambda$.*

Remark 4. *Each $\mathfrak{P}_T$ is homogeneous of X-degree $\deg(\mathfrak{P}_T) = \sum_{\square\in\lambda^{(\infty)}} T(\square) < \infty$. The set of all $\mathfrak{P}_T$ for $T \in \Omega(\lambda)$ gives a $\mathbb{Q}(q,t)$-basis of $\widetilde{W}_\lambda$. Lastly, the multiplication operators $F[X]^\bullet$ are well-defined since $\Phi_\lambda^{(n)} X_{n+1} = 0$.*

Using Prop. 4 we can define the following operators on $\widetilde{W}_\lambda$ generalizing the Macdonald operators on the space of symmetric functions Λ.

Definition 14. *For $\ell > 0$ define the operator $\Delta_\ell : \widetilde{W}_\lambda \to \widetilde{W}_\lambda$ to be the stable-limit*
$$\Delta_\ell := \lim_n \left(P_{0,\ell}^{(n)} - \sum_{\square\in\lambda^{(n)}} t^{\ell c(\square)}\right).$$

4.3 $\mathcal{E}^+$ Action on $\widetilde{W}_\lambda$

Finally, we are ready to state the main result of this paper. This theorem follows by applying Prop. 4, Cor. 1, and an argument of Schiffmann-Vasserot (Lemma 1.3 in [13]).

Theorem 2 (Main Theorem). *For $\lambda \in \mathbb{Y}$, $\widetilde{W}_\lambda$ is a graded $\mathcal{E}^+$-module with action determined for $\ell > 0$ by $P_{\ell,0} \to q^\ell p_\ell[X]^\bullet$ and $P_{0,\ell} \to \Delta_\ell$. Further, $\widetilde{W}_\lambda$ is spanned by a basis of eigenvectors $\{\mathfrak{P}_T\}_{T\in\Omega(\lambda)}$ with distinct eigenvalues for the operator $\Delta = \Delta_1$ which we will refer to as the* ***Macdonald operator****.*

Remark 5. *For $\lambda = \emptyset$, $\widetilde{W}_\emptyset = \Lambda$ recovers the standard representation of $\mathcal{E}^+$. In this case, $\Omega(\emptyset) \equiv \mathbb{Y}$ and $\mathfrak{P}_\mu = P_\mu[X; q^{-1}, t]$ (up to nonzero scalar).*

By considering the grading of each module $\widetilde{W}_\lambda$ and the spectral theory of the Macdonald operator Δ we can prove the following.

Proposition 8. *For $\lambda, \mu \in \mathbb{Y}$ distinct, $\widetilde{W}_\lambda \ncong \widetilde{W}_\mu$ as graded $\mathcal{E}^+$-modules.*

5 Pieri Rule

In this section we give the description of a Pieri rule for the generalized symmetric Macdonald functions $\mathfrak{P}_T$. We need to consider the following q,t-rational function.

Definition 15. *For $T \in \mathrm{RSSYT}_{\geq 0}(\lambda)$ define*

$$K_T(q,t) := \frac{[\mu(T)]_t!}{[n]_t!} \prod_{(\square_1,\square_2)\in \mathrm{Inv}(\min(T))} \left(\frac{q^{T(\square_1)}t^{c(\square_1)} - q^{T(\square_2)}t^{c(\square_2)+1}}{q^{T(\square_1)}t^{c(\square_1)} - q^{T(\square_2)}t^{c(\square_2)}} \right).$$

Using Prop. 6 and some book-keeping we obtain the following finite-rank Pieri formula.

Theorem 3. *For $T \in \mathrm{RSSYT}_{\geq 0}(\lambda)$ and $1 \leq r \leq n$ we have the expansion*

$$e_r[X_1 + \ldots + X_n]P_T = \sum_S d^{(r)}_{S,T} P_S$$

where

$$\frac{d^{(r)}_{S,T}}{t^{\binom{r}{2}} e_r(1,\ldots,t^{n-1}) K_S(q,t)} = \sum_{\substack{\tau \in \mathrm{PSYT}_{\geq 0}(\lambda;T) \\ s.t. \\ \Psi^r(\tau)\in \mathrm{PSYT}_{\geq 0}(\lambda;S)}} t^{c_\tau(1)+\ldots+c_\tau(r)} \prod_{(\square_1,\square_2)\in\mathrm{Inv}(\tau)} \left(\frac{q^{T(\square_1)}t^{c(\square_1)+1} - q^{T(\square_2)}t^{c(\square_2)}}{q^{T(\square_1)}t^{c(\square_1)} - q^{T(\square_2)}t^{c(\square_2)}} \right)$$
$$\times \prod_{(\square_1,\square_2)\in\mathrm{Inv}(\Psi^r(\tau))} \left(\frac{q^{S(\square_1)}t^{c(\square_1)} - q^{S(\square_2)}t^{c(\square_2)}}{q^{S(\square_1)}t^{c(\square_1)} - q^{S(\square_2)}t^{c(\square_2)+1}} \right)$$

and S ranges over all $S \in \mathrm{RSSYT}_{\geq 0}(\lambda)$ one can obtain from T by adding r 1's to the boxes of T with at most one 1 being added to each box.

Definition 16. *For $S, T \in \Omega(\lambda)$ and $r \geq 1$ define $\mathfrak{d}^{(r)}_{S,T}$ by $e_r[X]^{\bullet}(\mathfrak{P}_T) = \sum_{S \in \Omega(\lambda)} \mathfrak{d}^{(r)}_{S,T} \mathfrak{P}_S$. Define the **rank** $\mathrm{rk}(T)$ to be the minimal $n \geq n_\lambda$ such that $T|_{\lambda^{(\infty)}/\lambda^{(n)}} = 0$.*

We can use the stability from Cor. 1 to obtain a Pieri rule.

Corollary 2 (Pieri Rule)**.** *Let $S, T \in \Omega(\lambda)$ and $r \geq 1$. For all $n \geq \mathrm{rk}(T) + r$*

$$\mathfrak{d}^{(r)}_{S,T} = d^{(r)}_{S|_{\lambda^{(n)}}, T|_{\lambda^{(n)}}}.$$

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