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Whirling and rowmotion dynamics on the chain of V's poset

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Abstract. Given a finite poset P, we study the *whirling* action on vertex-labelings of P with the elements $\{0, 1, 2, \ldots, k\}$. When such labelings are (weakly) order-reversing, we call them *k-bounded P-partitions*. We give a general equivariant bijection between k-bounded P-partitions and order ideals of the poset $P \times [k]$ which conveys whirling to the well-studied *rowmotion* operator. As an application, we derive periodicity and homomesy results for rowmotion acting on the *chain of V's* poset $\mathsf{V} \times [k]$. We are able to generalize some of these results to the more complicated dynamics of rowmotion on $\mathsf{C}_n \times [k]$, where C_n is the **claw poset** with n unrelated elements each covering $\widehat{0}$.

Keywords: posets, chain of V's, dynamical algebraic combinatorics, homomesy, P-partitions, rowmotion, whirling.

1 Introduction

We connect the well-studied operation of *rowmotion* on the order ideals of a finite poset with the less familiar *whirling* action on P-partitions with bounded labels. One of our main results is an equivariant bijection that carries one to the other for any finite poset P. We then leverage this to study the rowmotion action on the "chain of V's" poset $\mathsf{V}_k := \mathsf{V} \times [k]$ (a 3-element V-shaped poset cross a finite chain, see Figure 2), which has surprisingly good dynamical properties. We also generalize this to the case where we replace V with a ***n*-claw**, a poset with a single minimal element covered by exactly n incomparable elements. In both cases we obtain both periodicity results and homomesy.

Let P be a finite poset, and $\mathcal{J}(P)$ be the set of order ideals of P. (For basic poset definitions, we refer the reader to Stanley [9, Ch. 3].) *Combinatorial rowmotion* is an invertible map $\rho : \mathcal{J}(P) \to \mathcal{J}(P)$ which takes each ideal $I \in \mathcal{J}(P)$ to the order ideal generated by the minimal elements of the complement of I in P. The periodicity of this map on products of chains was first studied by Brouwer and Schrijver [2], and Cameron and Fon-der-Flaass [3]. Later Striker and Williams [10] considered it as one element of the "toggle group" of a poset and related it to a kind of "promotion" operator on order

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ideals. Around the same time, Armstrong, Stump, and Thomas [1] studied rowmotion on *root posets*, relating it to "Kreweras complementation" on noncrossing partitions, and used this to prove a conjecture of Panyushev about the equality of the average cardinality of antichains for each rowmotion orbit.

Propp and Roby [7] noticed that this conjecture was merely one instance of a much broader phenomenon which they dubbed *homomesy*. Given a finite set S, a "statistic" $f : S \to \mathbb{C}$, and an invertible map φ on S, we call f *homomesic* if the average value of f is the same for every φ-orbit $\mathcal{R}$, i.e., $\frac{1}{\#\mathcal{R}} \sum_{x \in \mathcal{R}} f(x) = c$, where c is a constant not dependent on the choice of orbit $\mathcal{R}$. The confluence of all this work was the beginning of *dynamical algebraic combinatorics* as a distinct area within algebraic combinatorics (with antecedents going back to the Robinson–Schensted–Knuth correspondence and related operations on Young tableaux such as promotion, evacuation, and cyclage). In the past decade, the subfield has grown in a number of directions, and the study of rowmotion has been of continuing interest. For more background information, see the survey articles of Hopkins [4], Roby [8], and Striker [11].

Cameron and Fon-der-Flaass [3] were the first to describe rowmotion as a product of involutions called *toggles*, as detailed in Section 1.1. A natural generalization of toggling at a poset element x is "*whirling* at x," which cycles the label at x among j possible values. (Toggles are the case when $j = 2$.) Joseph, Propp, and Roby defined these and the operation of *whirling* on sets of functions between finite sets, obtaining various homomesy results for various classes of functions (injective, surjective, etc.) [6]. This is described in Section 2.

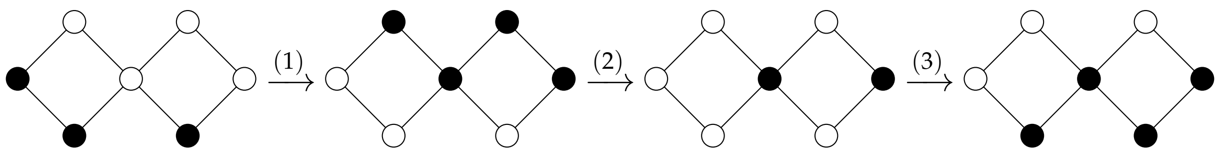
A bijective function $f : P \to [p]$ (with $\#P = p$) such that $f(x) < f(y)$ whenever $x <_P y$ is called a *linear extension*. We denote by $\mathcal{L}(P)$ the set of *all linear extensions* of P; its cardinality, $e(P)$, is an important numerical invariant of a poset. Its refinement, the *order polynomial* $\Omega_P(k)$, counts the number of k-bounded P-partitions. For some special posets P, mainly ones connected with Lie theory (root and minuscule posets) and those of partition or shifted shapes, product formulae for $\Omega_P(k)$ are known. Hopkins surveys these posets, the formulae, and gives the heuristic: *Posets with order-polynomial product formulae are the same as the posets with good dynamical behavior.* The one poset in his list whose rowmotion dynamics were relatively unexplored is $\mathsf{V} \times [k]$, a gap this paper fills. In separate work Hopkins and Rubey study the dynamics of Schützenberger promotion on linear extensions of $\mathsf{V} \times [k]$, which also exhibit unusually good behavior [5].

This paper is organized as follows. In Section 1 after the introduction, we review the toggling definition of rowmotion. Section 2 describes whirling, and includes the equivariant bijection which allows us to study rowmotion on V_k as whirling on k-bounded P-partitions. Section 3 contains our main periodicity and homomesy results for rowmotion on V_k, which use decompositions of the "orbit board" of the corresponding whirling action into "whorms". Finally, Section 4 contains the periodicity and homomesy results which generalize to rowmotion on the "chain of claws" graph, $\mathsf{C}_n \times [k]$. A version of this paper with full proofs will appear soon on the arXiv.

1.1 Rowmotion as a product of toggles

Definition 1.1. We define the (order-ideal) rowmotion map, $\rho : \mathcal{J}(P) \to \mathcal{J}(P)$ as follows: For any $I \in \mathcal{J}(P)$, $\rho(I)$ is the order ideal generated by the minimal elements of the complement of I, as in the example below.

Example 1.2. Here is one iteration of ρ on an order ideal with the action broken down into its three steps: (1) complement, (2) take minimal elements, (3) saturate down.

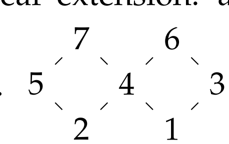


Rowmotion has an alternate definition as a composition of toggling involutions, which has proven useful for understanding and generalizing many of its properties. Cameron and Fon-der-Flaass [3] showed that for any finite poset P, rowmotion can be realized as "toggling once at each element of P along any linear extension (from top to bottom)". Other toggling orders also lead to interesting maps, such as Striker–Williams "promotion" (of order ideals) of a poset, which is toggling from left-to-right along "files" of a poset [10].

Definition 1.3. For each fixed $x \in P$ define the *(order-ideal) toggle* $\tau_x : \mathcal{J}(P) \to \mathcal{J}(P)$ by

$$\tau_x(I) = \begin{cases} I \setminus \{x\} & \text{if } x \in I \text{ and } I \setminus \{x\} \in \mathcal{J}(P) \\ I \cup \{x\} & \text{if } x \notin I \text{ and } I \cup \{x\} \in \mathcal{J}(P) \\ I & \text{otherwise.} \end{cases}$$

It is an easy exercise to show that order-ideal toggles [3, §2] are involutions, and that toggles at incomparable elements commute (a special case of Prop 2.7).

Example 1.4. We will toggle each node down the following fixed linear extension: at each step we consider whether or not to toggle the red node in or out. 

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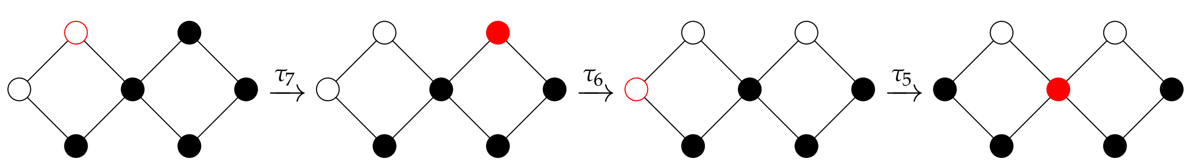
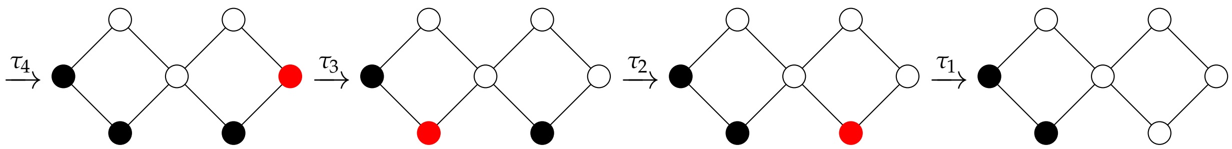
For this linear extension we toggle the elements from top-to-bottom, then left-to-right.





Proposition 1.5 ([3, Lemma 1]). *Let $x_1, x_2, \ldots, x_p$ be any linear extension (i.e., any order-preserving listing of the elements) of a finite poset P with p elements. Then the composite map $\tau_{x_1}\tau_{x_2}\cdots\tau_{x_p}$ coincides with the rowmotion operation ρ.*

2 Whirling

2.1 Whirling function between finite sets

Let $\mathcal{F} \subseteq [k]^{[n]}$ be a family of functions $f : [n] \to [k]$. For the rest of section 2.1, we use $\{1, \ldots, k\} = [k]$ to represent the congruence classes of $\mathbb{Z}/k\mathbb{Z}$, as opposed to the usual $\{0, 1, \ldots, k-1\}$. For fixed values of k and n, we represent such functions in *one-line* notation, e.g., $f = 21344$ represents the function $f \in [4]^{[5]}$ with $f(1) = 2$, $f(2) = 1$, $f(3) = 3$, $f(4) = 4$, and $f(5) = 4$.

Definition 2.1 ([6, Definition 2.3]). For $f \in \mathcal{F}$ we define the *whirl* $w_i : \mathcal{F} \to \mathcal{F}$ at index i as follows: repeatedly add 1 (modulo k) to the value of $f(i)$ until we get a function in $\mathcal{F}$.

Example 2.2. Let $\mathcal{F} = \{f \in [4]^{[5]} : f(1) \neq f(2)\}$. If we apply w_2 to $f = 21344$, adding 1 in the second position gives 22344, but this is not in $\mathcal{F}$. Adding 1 again in this position gives the result: $w_2(f) = 23344$.

4	1	5
6	2	1
3	4	2
5	6	3
1	2	4
3	5	6
4	1	2
5	3	4
6	5	1
2	6	3

Figure 1

We will now highlight some specific results from the paper where whirling was first introduced. Let $\mathrm{Inj}_m(n,k)$ be the set of *m-injective functions*, that is, functions $f : [n] \to [k]$ such that $\#f^{-1}(t) \leq m$ for all $t \in [k]$. Similarly, let $\mathrm{Sur}_m(n,k)$ be the set of *m-surjective functions*, that is, $f : [n] \to [k]$ such that $\#f^{-1}(t) \geq m$ for all $t \in [k]$. Note that injective functions are 1-injections and surjective functions are 1-surjections. We also define the statistic $\eta_j(f) = \#f^{-1}(\{j\})$.

Theorem 2.3. *[6, Theorem 2.11] Fix $\mathcal{F}$ to be either $\mathrm{Inj}_m(n,k)$ or $\mathrm{Sur}_1(n,k)$ for given $n, k, m \in \mathbb{P}$. Then under the action of $\mathbf{w} = w_n \circ w_{n-1} \circ \cdots \circ w_1$ on $\mathcal{F}$, η_j is $\frac{n}{k}$-mesic for any $j \in [k]$*

This result is conjectured to hold for $\mathrm{Sur}_m(n,k)$, but is still open for $m > 1$. Proof details can be found in Sections 2.2–2.4 of [6].

Example 2.4. Here is the orbit of $\mathbf{w}$ on $\mathrm{Inj}_1(3,6)$ containing $f = 415$.

$$415 \xrightarrow{\mathbf{w}} 621 \xrightarrow{\mathbf{w}} 342 \xrightarrow{\mathbf{w}} 563 \xrightarrow{\mathbf{w}} 124 \xrightarrow{\mathbf{w}} 356 \xrightarrow{\mathbf{w}} 412 \xrightarrow{\mathbf{w}} 534 \xrightarrow{\mathbf{w}} 651 \xrightarrow{\mathbf{w}} 263 \xrightarrow{\mathbf{w}}$$

Figure 1 shows the corresponding *orbit board* (a matrix whose rows are the successive orbit elements) partitioned into chunks. Notice that each value $1, 2, \ldots, 6$ appear exactly 5 times in this orbit of size 10, in accordance with the $1/2$-mesy of Theorem 2.3.

2.2 k-bounded P-partitions

Now we extend the definition of whirling to k-bounded P-partitions. Throughout the rest of the paper, P will denote a finite poset. Define $[0, k] := \{0, 1, 2, \ldots, k\}$.

A *P-partition* is a map σ from P to $\mathbb{N}$ such that if $x <_P y$, then $\sigma(x) \geq \sigma(y)$ [9, Ch. 3].

Definition 2.5. A *k-bounded P-partition* is a function $f : P \to [0, k]$ such that if $x \leq_P y$, then $f(x) \geq f(y)$. Let $\mathcal{F}_k(P)$ be the set of all such functions.

Throughout the rest of the paper we use $\{0, 1, \ldots, k\}$ to represent the congruence classes of $\mathbb{Z}/(k+1)\mathbb{Z}$, as usual.

Definition 2.6. For $f \in \mathcal{F}_k(P)$ and $x \in P$, define $w_x : \mathcal{F}_k(P) \to \mathcal{F}_k(P)$, called the *whirl at x*, as follows: repeatedly add 1 (mod $k+1$) to the value of $f(x)$ until we get a function in $\mathcal{F}_k(P)$. This new function is $w_x(f)$.

The case $k = 1$ of the above definition recovers toggling of order ideals (Def. 1.3).

Proposition 2.7. *If $x, y \in P$ are incomparable, then $w_x w_y(f) = w_y w_x(f)$.*

Definition 2.8. Let $(x_1, x_2, \ldots x_p)$ be a linear extension of P. Define $w : \mathcal{F}_k(P) \to \mathcal{F}_k(P)$ by $w := w_{x_1} w_{x_2} \ldots w_{x_p}$. The above proposition shows that this is well-defined, since one can get from any linear extension to any other by a sequence of interchanges of incomparable elements.

Example 2.9. Let P be the V poset with labels $\begin{array}{ccc} \ell & & r \\ & \diagdown \; \diagup & \\ & c & \end{array}$, $k = 2$, and $w = w_c w_r w_\ell$.

$$\begin{array}{ccc} 0 & & 2 \\ & \diagdown \; \diagup & \\ & 2 & \end{array} \xrightarrow{w_\ell} \begin{array}{ccc} 1 & & 2 \\ & \diagdown \; \diagup & \\ & 2 & \end{array} \xrightarrow{w_r} \begin{array}{ccc} 1 & & 0 \\ & \diagdown \; \diagup & \\ & 2 & \end{array} \xrightarrow{w_c} \begin{array}{ccc} 1 & & 0 \\ & \diagdown \; \diagup & \\ & 1 & \end{array}$$

There is a natural bijection between order ideals of a poset P and 1-bounded P-partitions in $\mathcal{F}_1(P)$. Specifically, a 1-bounded P-partition in $\mathcal{F}_1(P)$ is simply the indicator function of an order ideal $I \in J(P)$. We extend this to an equivariant bijection $\mathcal{F}_k(P) \to \mathcal{J}(\mathcal{P} \times [k])$ which sends w to ρ, meaning the following diagram commutes.

$$\begin{tikzcd} \mathcal{F}_k(P) \arrow[r, "w"] \arrow[d, leftrightarrow] & \mathcal{F}_k(P) \arrow[d, leftrightarrow] \\ \mathcal{J}(P \times [k]) \arrow[r, "\rho"] & \mathcal{J}(P \times [k]) \end{tikzcd}$$

We will call the chains $\{(x, 1), (x, 2), \ldots, (x, k)\} \subseteq P \times [k]$, for $x \in P$, the *fibers* of $P \times [k]$, and construct an equivariant bijection that first sends w_x to order-ideal toggling down the fiber $\{(x, 1), (x, 2), \ldots, (x, k)\}$.

Lemma 2.10. *There is an equivariant bijection between $\mathcal{F}_k(P)$ and $\mathcal{J}(P \times [k])$ which sends w_x to the toggle product $\tau_{(x,1)}\tau_{(x,2)} \cdots \tau_{(x,k)}$.*

Theorem 2.11. *Fix any linear extension $(x_1, x_2, \ldots, x_p) \in \mathcal{L}(P)$. There is an equivariant bijection between $\mathcal{F}_k(P)$ and $\mathcal{J}(P \times [k])$ which sends whirling, $w = w_{x_1} w_{x_2} \cdots w_{x_p}$, to rowmotion on $\mathcal{J}(P \times [k])$.*

The following definitions will allow us to partition orbit boards of whirling into subsets called *whorms*.

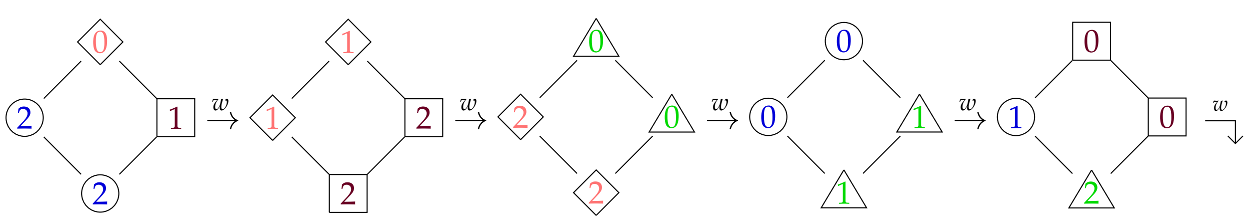
Definition 2.12. For any $x \in P$ and $f \in \mathcal{F}_k(P)$, define (x, f) to be a *whirl element*. The whirl element (y, g) is *whirl successive* of (x, f) if either:

1. $y = x$ and $g(y) = w(f)(x) = f(x) + 1$, or
2. x covers y, $f = g$, and $f(x) = g(y)$.

We consider whirl-successive elements to be whirl elements which are one step away from each other, either by moving one covering relation down the poset or by whirling the function at the element, and ending one label greater. *While we must consider the entire P-partitions f and g to check whether two whirl elements are whorm connected, we think of whirl elements as being simply $(x, f(x))$, the location and its label, and indicate them in this way in the examples that follow.*

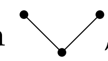
Definition 2.13. Two whirl elements (x, f) and (y, g) are *whorm-connected* if there exists a sequence of whirl-successive elements $\{(x, f) = (x_0, f_0), (x_1, f_1), \ldots, (x_p, f_p) = (y, g)\}$. A *whorm* is a maximal set of whorm-connected whirl elements, that is, if (x, f) is in a whorm and (x, f) is whorm-connected to (y, g), then (y, g) is in the whorm.

Example 2.14. An orbit of whirling P-partitions (for $P = [2] \times [2]$) with its four whorms indicated by the same color and (redundantly) node-shape.



3 Periodicity and homomesy for rowmotion on $\mathsf{V} \times [k]$

In this section we consider the dynamics of rowmotion acting on the order ideals of the chain of V's poset V_k, establishing its periodicity and finding interesting examples of homomesy.

Definition 3.1. Let V be the 3-element poset with Hasse diagram , and define $\mathsf{V}_k = \mathsf{V} \times [k]$, where $[k]$ is the chain poset. We call V_k the *chain of V's poset*.

Example 3.2. Figure 2 shows the Hasse diagram of V_k with our vertex-labeling convention.

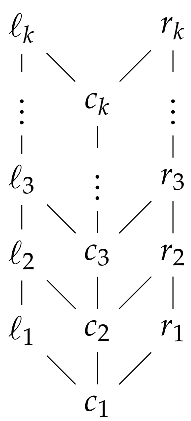


Figure 2

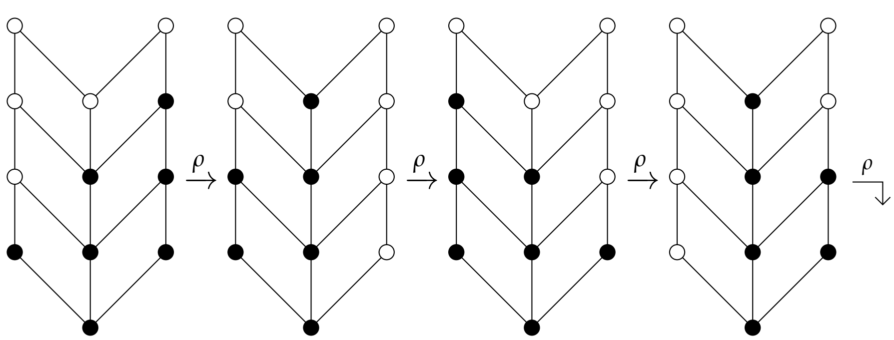
Our main goals for this section are the following theorems. We will leverage the equivariant bijection and the notion of whorms from the last section.

Theorem 3.3. *The order of rowmotion on $\mathcal{J}(\mathsf{V}_k)$ is $2(k+2)$.*

Theorem 3.4. *Let χ_s be the indicator function for $s \in \mathsf{V}_k$. We have the following homomesies for the action of ρ on $\mathcal{J}(\mathsf{V}_k)$*

1. *The statistic $\chi_{\ell_i} - \chi_{r_i}$ is 0-mesic for all $i \in [k]$.*
2. *The statistic $\chi_{\ell_1} + \chi_{r_1} - \chi_{c_k}$ is $\frac{2(k-1)}{k+2}$-mesic.*

Example 3.5. This ρ-orbit on $\mathcal{J}(\mathsf{V}_4)$ has size 4, which divides $2(4+2) = 12$. The homomesies are also easily checked, e.g., across the orbit the total number of elements at rank 1 in the side fibers is 6, minus the two at the top of the center fiber, for an average of $\frac{6-2}{4} = 1 = \frac{2(4-1)}{4+2}$, agreeing with Theorem 3.4(2).



To prove these theorems we utilize our equivariant bijection (Theorem 2.11) from $\mathcal{J}(\mathsf{V}_k)$ to $\mathcal{F}_k(\mathsf{V})$, then represent the latter by triples $f = (\ell, c, r)$ with $\ell \leq c$ and $r \leq c$. This bijection ϕ sends an order ideal I to a triple (ℓ, c, r), counting the number of elements of the order ideal in the left, center, and right fibers respectively.

Example 3.6. Here is the orbit of $\mathcal{F}_4(\mathsf{V})$ corresponding to Example 3.5.

$$(1,3,3) \xrightarrow{w} (2,4,0) \xrightarrow{w} (3,3,1) \xrightarrow{w} (0,4,2) \xrightarrow{w}$$

Proposition 3.7. *The number of order ideals of V_k is given by $|\mathcal{J}(\mathsf{V}_k)| = \frac{k(k+1)(2k+1)}{6}$.*

3.1 Center-seeking whorms

1	2	0
2	3	1
3	4	2
4	4	3
0	4	4
1	1	0
0	2	1
1	3	2
2	4	3
3	4	4
4	4	0
0	1	1

0	2	0
1	3	1
2	4	2
3	3	3
0	4	0
1	1	1

Figure 3

To show that the order of ρ on $\mathcal{J}(\mathsf{V}_k)$ is $2(k+2)$ we end up proving something stronger, namely that $\rho^{k+2}(I)$ is the reflection of I across the the center chain. Our method is to investigate the whorms that arise from repeatedly whirling a k-bounded P-partition.

Recall from Definition 2.13 that, given a whirling orbit board, $\mathcal{O} = \{f, w(f), w^2(f), \dots\}$ of w on $\mathcal{F}_k(\mathsf{V})$, a whorm ς is a maximal set of whorm-connected elements. Figure 3 shows two orbit boards of $\mathcal{F}_4(\mathsf{V})$, one with six whorms and one with two whorms. Notice that each whorm in the second orbit has two "starting" positions.

Each whorm in an orbit board of $\mathsf{V} \times [k]$ starts on the left, or the right, or both left and right; we call the former *one-tailed* and the later *two-tailed*. Since these whorms move down the orbit board at every step, except for one move to the center, we consider them as a sequence of function values in the orbit board which start at 0 and end at k, where one value is repeated when moving into the center. We call these *center-seeking whorms*. (Since an orbit board is actually a cylinder, we have a "can of worms" to deal with.) In the left orbit of Figure 4 we isolate one example of a left whorm: $\varsigma = \{(\ell,(0,3,3)),(\ell,(1,4,0)),(\ell,(2,2,1)),(c,(2,2,1)),(c,(0,3,2)),(c,(1,4,3))\}$, visualized within an orbit board of $\mathcal{F}_4(\mathsf{V})$. It is easy to see that an orbit board is tiled either entirely by one-tailed whorms or entirely by two-tailed whorms. (See the discussion at the start of Section 4.)

We first observe that all whorms have $k+2$ elements, since each contains the $k+1$ elements $0, \dots, k$, exactly one of which is doubled.

Define $\mathsf{b}(\varsigma) := 1 + \min\{f(c) : (c,f) \in \varsigma\}$, the number of elements in the outer columns and $\mathsf{e}(\varsigma) := k+2-\mathsf{b}(\varsigma)$, the number of elements in the center column. For the red whorm in the orbit on the left of Figure 4, $\mathsf{b}(\varsigma) = 3$ and $\mathsf{e}(\varsigma) = 3$.

Example 3.8. The right orbit board in Figure 4 is the previous example with all the whorms colored. The number of elements in the left column of the yellow, red, and orange whorms are $5, 3, 4$ respectively, and the orbit board is of length 12.

It follows that the order of whirling divides the sum of $\mathsf{b}(\varsigma)$ over all whorms $\varsigma \in S$. In the setting of $\mathcal{F}_k(\mathsf{V})$, as long as we know $\mathsf{b}(\varsigma)$ and whether $f(\ell) = 0$, $f(r) = 0$, or both, then we can recover the entire whorm.

1	2	2
2	3	0
3	4	1
4	4	2
0	3	3
1	4	0
2	2	1
0	3	2
1	4	3
2	4	4
3	3	0
0	4	1

1	2	2
2	3	0
3	4	1
4	4	2
0	3	3
1	4	0
2	2	1
0	3	2
1	4	3
2	4	4
3	3	0
0	4	1

Figure 4

Definition 3.9. We place a circular order on the whorms. Let ς_1 and ς_2 be whorms in an orbit board of $\mathcal{F}_k(\mathsf{V})$. If there exists $(c, f) \in \varsigma_1$ with $f(c) = k$ such that $(c, w(f)) \in \varsigma_2$, then we say ς_2 is *in front of* ς_1. We call a sequence of whorms *consecutive* if each is in front of the next. In a one-tailed orbit board, consecutive whorms alternate starting from the left and right.

Example 3.10. In Figure 4 the blue (horizontal lines) whorm is in front of the red (crosshatch) whorm, which is in front of the green (northwest lines) whorm.

Lemma 3.11. *Assume an orbit board $\mathcal{O}$ of w on $\mathcal{F}_k(\mathsf{V})$ has all one-tailed whorms. Let ς_1, ς_2, and ς_3 be three consecutive whorms, that is, ς_3 is in front of ς_2 which is in front of ς_1 in $\mathcal{O}$. Then,* $\mathsf{b}(\varsigma_1) + \mathsf{b}(\varsigma_2) + \mathsf{b}(\varsigma_3) = 2(k + 2)$. *Otherwise, if there are two-tailed whorms, then* $\mathsf{b}(\varsigma_1) + \mathsf{b}(\varsigma_2) = k + 2$.

In fact, the entire orbit board can be reconstructed simply from knowing the values of $\mathsf{b}(\varsigma_1)$ and $\mathsf{b}(\varsigma_2)$ for two consecutive whorms in the one-tailed case, and from a single $\mathsf{b}(\varsigma_1)$ in the two-tailed case.

Example 3.12. In Figure 4 we have $k = 4$, $\mathsf{b}(\text{green}) = 4$, $\mathsf{b}(\text{red}) = 3$, and $\mathsf{b}(\text{blue}) = 5$, which sum to $12 = 2(4 + 2)$.

Lemma 3.13. *Given an orbit board with one-tailed whorms, let ς_1, ς_2, ς_3, ς_4 be consecutive, then* $\mathsf{b}(\varsigma_4) = \mathsf{b}(\varsigma_1)$. *Furthermore, if the orbit board contains two-tailed whorms, then* $\mathsf{b}(\varsigma_1) = \mathsf{b}(\varsigma_3)$.

Notice that for orbits with one-tailed whorms, we are *not* claiming the board starts to repeat; since whorms alternate sides, ς_4 will start on the opposite side from ς_1. If we keep applying the previous Lemma to even more consecutive whorms, we see $\mathsf{b}(\varsigma_5) = \mathsf{b}(\varsigma_2)$ and $\mathsf{b}(\varsigma_6) = \mathsf{b}(\varsigma_3)$. Finally we get $\mathsf{b}(\varsigma_7) = \mathsf{b}(\varsigma_1)$ and the pattern repeats. Therefore there are at most six unique whorms in a one-tailed orbit board.

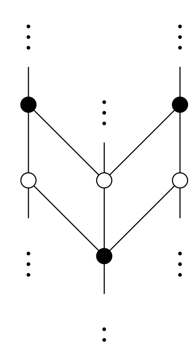


Figure 5

Lemma 3.14. *Given an orbit board with one-tailed whorms, there are at most six distinct whorms.*

Theorem 3.15. *Let $(x, y, z) \in \mathcal{F}_k(\mathsf{V})$, then $w^{k+2}(x, y, z) = (z, y, x)$.*

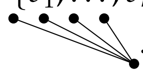
Corollary 3.16. *The order of w on $\mathcal{F}_k(\mathsf{V})$ divides* $2(k + 2)$.

Lemma 3.17. *Under the action of rowmotion on order ideals of $\mathcal{J}(\mathsf{V}_k)$, the difference of successive flux-capacitor indicator functions, $F_i - F_{i+1}$ is $\frac{3}{k+2}$-mesic for $i \in [2, k-1]$.*

This lemma can be generalized to the following theorem.

Theorem 3.18. *For $k > 1$. Let $F_i = \chi_{\ell_i} + \chi_{r_i} + \chi_{c_{i-1}}$. Under the action of rowmotion on order ideals of $\mathcal{J}(\mathsf{V}_k)$, the difference of arbitrary flux- capacitors is $F_i - F_j$ is $\frac{3(j-i)}{k+2}$-mesic.*

4 Periodicity and homomesy for rowmotion on $\mathsf{C}_n \times [k]$

We define the *claw poset* $\mathsf{C}_n = \{b_1, \ldots, b_n, \widehat{0}\}$ where each b_i covers $\widehat{0}$. For example, the Hasse diagram of C_4 would be .

Using the established equivariant bijection between $\mathcal{J}(\mathsf{C}_n \times [k])$ and k-bounded P-partitions $\mathcal{F}_k(\mathsf{C}_n)$ that sends rowmotion to whirling, we can prove similar homomesies and periodicity to that of $\mathsf{C}_2 = \mathsf{V}$. Now instead of *triples* of numbers, we will consider orbit boards of $(n+1)$-*tuples* on $[0, k]$, $\Big(f(b_1), f(b_2), \ldots, f(b_n), f(\widehat{0})\Big)$, satisfying $f(b_i) \leq f(\widehat{0})$ for each $i \in [n]$.

1	0	1	2	3
2	1	2	3	3
3	2	3	0	3
0	3	0	1	3
1	0	1	2	2
2	1	2	0	3
3	2	3	1	3
0	3	0	2	3
1	0	1	3	3
2	1	2	0	2
0	2	0	1	3
1	3	1	2	3
2	0	2	3	3
3	1	3	0	3
0	2	0	1	2

Figure 6

In Figure 6, note that if two entries are the same among the first n in a given row, then those positions (columns) remain the same throughout the *entire* orbit board. This is because the entries $b_1, \ldots, b_n$ represent the result of whirling at *incomporable* elements of the poset C_n. Furthermore, these two entries must belong to the same whorm, because each will be whorm-connected via $\widehat{0}$ exactly when their value matches the value of the last entry. These observations will allow us to generalize our peridocity and homomesy results from V to C_n.

Definition 4.1. For $A \subseteq [0, k]$, define the family of order-reversing maps $\mathcal{F}_k^A(\mathsf{C}_n) = \{f : f \in \mathcal{F}_k(\mathsf{C}_n) \text{ and } f(b_j) \in A \text{ for all } j \in [n]\}$. For any fixed A we denote $\overline{w}$ to be whirling on the non-$\widehat{0}$ elements of order-reversing maps $f \in \mathcal{F}_k^A(\mathsf{C}_n)$. Which is equivalent to incrementing each non-$\widehat{0}$ value, but only allowing values within A.

Given $f \in \mathcal{F}_k(\mathsf{C}_n)$, set $A(f) = \{a : f(b_j) = a \text{ for some } j \in [n]\}$, the set of values that the P-partition f attains on the non-$\widehat{0}$ elements of C_n. Set $\alpha = \#A$ and $\alpha(f) = \#A(f)$. For any $f, g \in \mathcal{F}_k(\mathsf{C}_n)$, if $g = w^j(f)$ for some $j \in \mathbb{N}$, then $\alpha(f) = \alpha(g)$. So we may sometimes write just α when an orbit is fixed. For this section, we impose $A = A(f)$ when computing $\overline{w} : \mathcal{F}_k(\mathsf{C}_n) \to \mathcal{F}_k(\mathsf{C}_n)$ of an order-reversing map f.

Example 4.2. Consider $f = (1, 3, 3, 0, 4, 1, 6) \in \mathcal{F}_9(\mathsf{C}_6)$. We see $A(f) = \{0, 1, 3, 4\}$ so

$$\overline{w}(1, 3, 3, 0, 4, 1, 6) = (3, 4, 4, 1, 0, 3, 6).$$

The last entry remains unchanged and the earlier entries are increasing cyclically within the set $A(f) = \{0, 1, 3, 4\}$. In the special case where V ($= \mathsf{C}_2$) are set within any orbit A will have at most two elements, hence $\overline{w}$ will just toggle between those two values at the left and the right. This means that $\overline{w}$ is the same as reflecting values across the center of V, which we already saw was the effect of w^{k+2}. Our next result generalizes this to the case C_n.

Lemma 4.3. *Let $f \in \mathcal{F}_k(\mathsf{C}_n)$ and $\alpha = \alpha(f)$. If $\varsigma_1, \ldots, \varsigma_{\alpha+1}$ are $\alpha + 1$ consecutive whorms, then*

$$\mathsf{b}(\varsigma_1) + \cdots + \mathsf{b}(\varsigma_{\alpha+1}) = \alpha(k + 2).$$

Proposition 4.4. *Let w be whirling k-bounded P-partitions on $\mathcal{F}_k(\mathsf{C}_n)$. For any $f \in \mathcal{F}_k(\mathsf{C}_n)$ and $A = A(f)$, we have $w^{k+2}(f) = \overline{w}(f)$.*

The proof of this theorem can be approached with whorms. Define $\mathsf{b}(\varsigma) = 1 + \min\{f(\hat{0}) : (\hat{0}, f) \in \varsigma\}$. If there exists $(\hat{0}, f) \in \varsigma_1$ with $f(\hat{0}) = k$ such that $(\hat{0}, w(f)) \in \varsigma_2$, then we say ς_2 is *in front of* ς_1. In Figure 6, the pink snake is in front of the red snake.

Corollary 4.5. *Let $f \in \mathcal{F}_k(\mathsf{C}_n)$ and $\alpha = \alpha(f)$. If $\varsigma_1, \ldots, \varsigma_{\alpha+2}$ are consecutive whorms, then $\mathsf{b}(\varsigma_1) = \mathsf{b}(\varsigma_{\alpha+2})$.*

If $f \in \mathcal{F}^k(\mathsf{C}_n)$ satisfies $f(\widehat{0}) \notin A(f)$, then f will contain entries from $\alpha + 1$ distinct whorms. From Proposition 4.4, we will have at most $\alpha(\alpha + 1)$ whirls in an orbit board (each action of $\widehat{w}$ resulting in $\alpha + 1$ whorms potentially distinct from those previous, as in Figure 6). On the other hands, consider the orbit board of $\mathcal{F}_3(\mathsf{C}_4)$ in Figure 7 with $\alpha = 4$. Here $w(f) = w^5(f)$ so the orbit is only 4 rows long with 4 distinct whorms. In general, we can extend this to a super orbit board with $\alpha(\alpha + 1)$ whorms.

0	1	2	3	3
1	2	3	0	3
2	3	0	1	3
3	0	1	2	3

Figure 7

Theorem 4.6. *Let $m = \min(k, n)$. The order of rowmotion on $\mathcal{J}(\mathsf{C}_n \times [k])$ divides $m!(k + 2)$.*

Only the analogue of the first homomesy in Theorem 3.4 holds.

Theorem 4.7. *Let $\chi_{(i,j)}$ be the indicator function for $(i, j) \in \mathsf{C}_n \times [k]$. Then for the action of rowmotion on $\mathcal{J}(\mathsf{C}_n \times [k])$, the statistic $\chi_{(i,a)} - \chi_{(j,a)}$ is 0-mesic for all $i, j \in [n]$ and $a \in [k]$.*

Remark 4.8. The average of the statistic $\left(\sum_{i=1}^{n} \chi_{(i,1)}\right) - \chi_{(\hat{0},k)}$ (analogous to Theorem 3.4(2)) turns out to be **dependent** on $\alpha(f)$ (for any $f \in \mathcal{O}$) and can be computed as

$$\frac{n(\alpha)(k+2) - (n+\alpha)(\alpha+1)}{(\alpha)(k+2)}.$$

Consider the super-orbit with $n\alpha(k + 2)$ entries among the non-minimal elements. We know $\chi_{(i,1)}(I) = 0$ if and only if for corresponding f, $f(i) = 0$. But this is counted by the number of whorm beginnings, that is $n(\alpha+1)$. Furthermore, $\chi_{(\widehat{0},k)}(I)=1$ if and only if for corresponding f, $f(\widehat{0})=k$, which is counted by the number of whorm endings, that is $\alpha(\alpha+1)$. Therefore the average is obtained.

The "flux-capacitor" homomesy of Theorem 3.18 also generalizes to the claw-graph setting, and has a similar proof.

Theorem 4.9. *Let $B_i=\chi_{(i-1,\widehat{0})}+\sum_{j=1}^{n}\chi_{(i,j)}$. Then for the action of rowmotion on $\mathcal{J}(\mathsf{C}_n\times[k])$, B_i-B_j is $\frac{(j-i)(n+1)}{k+2}$-mesic for all $i,j\in[n]$.*

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